Groups and ~~R~~Representations

$\boxed{\text{Ziyan Li}}$

problem set 1

Tutor~~??~~ : Elmi

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- correction page:

page 27. line 2

" $P = R_1 \oplus R_1$ " should be

" $R = R_1 \oplus R_1$ "

Groups and Representations

Problem Sheet 1

Date: TBA, Deadline: TBA

1) (Normal sub-groups and group homomorphisms)

a) Consider a group G, a normal sub-group H of G and the quotient G/H together with the multiplication defined by $(gH)(\tilde{g}H) := g\tilde{g}H$. Show that this multiplication is well-defined (that is, the definition is independent on the choice of representative) and that G/H together with this multiplication forms a group. [5]

b) Let G and $\tilde{G}$ be two groups and $F : G \to \tilde{G}$ a group homomorphism. Show that $\mathrm{Im}(F)$ is a sub-group of $\tilde{G}$ and that $H := \mathrm{Ker}(F)$ is a normal sub-group of G. [7]

c) For the situation described in b) define a map $f : G/H \to \mathrm{Im}(F)$ by $f(gH) := F(g)$. Show that this map is well-defined and that it is a group isomorphism. [8]

2) (Characters of $\mathbb{Z}_n$) Consider the group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with multiplication defined by $kl := (k + l) \bmod n$. What are the conjugacy classes of $\mathbb{Z}_n$?

a) Write down the irreducible complex representations R_q (where $q = 0, 1, \ldots, n-1$) of Z_n and compute their characters χ_q. [5]

b) Show explicitly that the characters obtained in a) are ortho-normal, that is, show that $(\chi_q, \chi_p) = \delta_{qp}$. [7]

c) Focus on $\mathbb{Z}_3$. For the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

define a map $R : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{C}^3)$ by $R(k) := M^k$. Show that R is a representation of $\mathbb{Z}_3$ and determine which irreducible representations it contains. [8]

3) (Permutation groups) Denote by S_n the group of permutations of n objects, that is $S_n = \{\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \,|\, \sigma \text{ bijective}\}$. It is often useful to denote a particular permutation σ by the symbol

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix} .$$

a) Verify that S_n forms a group for all n which is non-Abelian for $n > 2$. [3]

b) Focus on S_3. Determine its conjugacy classes and show that the complete set of its complex irreducible representations consists of one two-dimensional and two one-dimensional representations. [3]

c) Find the character table of S_3. [6]

d) Consider the regular representation of S_3 and write down the projectors which correspond to the various irreducible representations. [4]

e) Find the irreducible representation content of $R_2 \otimes R_2$ where R_2 is the irreducible two-dimensional representation. [4]

4) (Generalisation of Schur's Lemma) Consider a group G and a complex representation R which can be written as $R = n_1 R_1 \oplus \cdots \oplus n_r R_r$ where R_i, $i = 1, \cdots, r$ are irreducible representations of dimensions d_i and the integers n_i indicate how often R_i appears in R.

a) Convince yourself that the representation matrices $R(g)$ can then be written as $R(g) = \mathbf{1}_{n_1} \times R_1(g) \oplus \cdots \oplus \mathbf{1}_{n_r} \times R_r(g)$. (Here, the tensor product $A \times B$ of two matrices A and B denotes the matrix obtained when every entry of A is replaced by this entry times the matrix B). [4]

b) Show that a matrix P with $[P, R(g)] = 0$ for all $g \in G$ has the general form $P = P_1 \times \mathbf{1}_{d_1} \oplus \cdots \oplus P_r \times \mathbf{1}_{d_r}$ where P_i are $n_i \times n_i$ matrices. [10]

c) Consider the representation of $U(1)$ defined by $R(e^{i\alpha}) := \mathrm{diag}(e^{i\alpha}, e^{i\alpha}, e^{-2i\alpha})$. What is the most general form of complex 3×3 matrices which commute with all representation matrices $R(e^{i\alpha})$? [6]

5) (Dihedral group D_4) The dihedral group D_4 is generated by the two matrices

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

a) Show that D_4 is of order eight, that it is non-Abelian and that it has five conjugacy classes. [6]

b) Find the complex irreducible representations and the character table of D_4. [10]

c) Work out which irreducible representations are contained in the tensor product of the two-dimensional irreducible representation with itself. [4]

(Q1) a) By the definition of a left coset,

$[g] = gH$ is the equivalence class of the

equivalence relation $g_1 \sim g_2 :\Leftrightarrow g_1^{-1} g_2 \in H$

for $H \subset G$ being a subgroup of group G

Equivalence classes are identical if the representatives are related (so they are not disjoint)

i.e. $gH = fH$ iff $g^{-1} f \in H$ $(f, \tilde{f} \in G)$

similarly $\tilde{g}H = \tilde{f}H$ iff $\tilde{g}^{-1}\tilde{f} \in H$

If $gH = fH$ $\tilde{g}H = \tilde{f}H$, then

$$(f\tilde{f})^{-1}(g\tilde{g}) = \tilde{f}^{-1} f^{-1} g \tilde{g}$$

$$= \tilde{f}^{-1} h \tilde{g} = (\tilde{f}^{-1}\tilde{g})(\tilde{g}^{-1} h \tilde{g})$$

$h \equiv f^{-1} g \in H$
$\because gH = fH$

H is a normal subgroup of G

$\therefore gH = Hg \rightarrow H = g^{-1} H g$

$\rightarrow \forall h \in H, g \in G, g^{-1} h g \in H$

$\rightarrow \tilde{g}^{-1} h \tilde{g} \in H$

$\because \tilde{g}^{-1}\tilde{f} \in H$ $\therefore$ its inverse $\tilde{f}^{-1}g \in H$

$\therefore \tilde{f}^{-1} h \tilde{g} \in H$

$\therefore (f\tilde{f})^{-1}(g\tilde{g}) \in H$

$\rightarrow (g\tilde{g})H = (f\tilde{f})H \iff gH = fH$ and
$\tilde{g}H = \tilde{f}H$

$\rightarrow$ The multiplication $(gH)(\tilde{g}H) = (g\tilde{g})H$ is independent of coset representatives

$\rightarrow$ it is well defined.

Now, check the group axioms for G/H

- $(gH)(\tilde{g}H) = g\tilde{g}H$ $\because g, \tilde{g} \in G$
$\therefore g\tilde{g} \in G$ (G is a group)
$\therefore g\tilde{g}H$ is a left coset
$\therefore g\tilde{g}H \in G/H$ (closure)

- $[(g_1H)(g_2H)](g_3H) = (g_1g_2H)(g_3H)$

$= g_1 g_2 g_3 H = (g_1 H)(g_2 g_3 H) = (g_1 H)[(g_2 H)(g_3 H)]$

$g_1, g_2, g_3 \in G$ is a group

(associativity)

- If $e \in G$ is the identity element in G, then

$$(H)(gH) = (eH)(gH) = (eg)H = gH$$

$\therefore$ the subgroup H itself is the identity element in G/H $\rightarrow$ (identity)

- let $g^{-1}g = e$, $g^{-1}, g \in G$

then $(g^{-1}H)(gH) = (g^{-1}g)H = eH = H$

identity in G/H

$\therefore$ $g^{-1}H$ is the inverse of gH in G/H

(inverse)

b) $F: G \rightarrow \tilde{G}$ a group homomorphism

$\therefore$ $\forall\, g_1, g_2 \in G$, $F(g_1 g_2) = F(g_1)F(g_2)$

By definition $Im(F) \in \tilde{G}$

- $eg = g \rightarrow e$ is identity in G

$F(e)F(g) = F(eg) = F(g)$

$\rightarrow F(e)$ is the identity in $Im(F)$

- $F(g^{-1})F(g) = F(g^{-1}g) = F(e) =$ identity in $Im(F)$

$\therefore F(g^{-1}) = F(g)^{-1}$ ① $=$ inverse of $F(g)$ in $Im(F)$

- $F(g_1)F(g_2) = F(g_1 g_2) \in Im(F) \because g_1 g_2 \in G$ $\forall g_1, g_2 \in G$

$\therefore$ closure

$\rightarrow Im(F)$ is a subgroup of $\tilde{G}$

$H = ker(F) \subset G$, $\forall h \in ker(F) = H$,

$F(h) = F(e)$ by definition.

$\because F(e) = F(e)$ trivially $\therefore e \in H$

$\forall h_1, h_2 \in H$, $F(h_1 h_2) = F(h_1)F(h_2)$

$= F(e)F(e) = F(e)$

$\rightarrow h_1 h_2 \in H \rightarrow$ <u>closure</u>

$\forall h \in H, \quad \underbrace{F(h^{-1}) = (F(h))^{-1}}_{\text{by } ①} = (F(e))^{-1} = F(e)$

$\rightarrow$ <u>$h^{-1} \in H$</u>

$\Rightarrow H$ is a subgroup of G

$\forall h \in H, \; g \in G$

Consider $F(ghg^{-1}) = F(g) F(h) F(g^{-1})$

$= F(g) F(e) F(g^{-1}) = F(g) F(g^{-1})$

$= F(gg^{-1}) = F(e)$

$\rightarrow ghg^{-1} \in H$

$\rightarrow \forall g \in G, \quad gHg^{-1} \subseteq H$ ②

$\Leftrightarrow \forall g^{-1} \in G, \quad g^{-1}H(g^{-1})^{-1} = g^{-1}Hg \subseteq H$

$\Leftrightarrow \forall g \in G, \quad g^{-1}Hg \subseteq H$ ③

② $\Rightarrow \quad gH = gHg^{-1}g \subseteq Hg$ ④

③ $\Rightarrow \quad Hg = gg^{-1}Hg \subseteq gH$ ⑤

④, ⑤ $\Rightarrow$ $gH = Hg$ $\rightarrow$ H is a normal subgroup

c) $F: G \rightarrow Im(F)$, $f(gH) = F(g)$

for f to be well defined, we require that $F(\tilde{g}) = F(g)$ $\forall g, \tilde{g} \in G$ such that $g^{-1}\tilde{g} \in H$

This is because $g^{-1}\tilde{g} \in H \rightarrow gH = \tilde{g}H$

- start from $g^{-1}\tilde{g} \in H = \ker(F)$

$\therefore F(e) = F(g^{-1}\tilde{g}) = F(g^{-1})F(\tilde{g}) = F(g)^{-1}F(\tilde{g})$,

left mutiply $F(g)$ on both sides:

$F(g) = F(\tilde{g})$ as required

$\rightarrow$ f is well-defined (independent of coset representative.)

- from a) $(gH)(\tilde{g}H) = (g\tilde{g})H$ $\forall g, \tilde{g} \in G$ is the multiplication in G/H

$f(gH)f(\tilde{g}H) = F(g)F(\tilde{g}) = F(g\tilde{g})$

$= f(g\tilde{g}H) = f((gH)(\tilde{g}H))$

$\Rightarrow$ f is a group homomerphism (1)

If $g_1, g_2 \in G$ and $g_1H \neq g_2H$, then

$g_1^{-1}g_2 \notin H = \ker(F)$

$\Rightarrow F^{-1}(g_1) F(g_2) = F(g_1^{-1}) F(g_2) = F(g_1^{-1}g_2)$

$\neq F(e) \Rightarrow F(g_2) \neq F(g_1)$

$\Rightarrow$ f is injective

$\because f(gH) = F(g) \Rightarrow Im(f) = Im(F)$

$\therefore$ image of f = the codomain of f

$\therefore$ f is surjective

togather f is bijective (2)

$\because$ (1), (2) $\Rightarrow$ f is a group isomorphism

$G/H \cong Im(F)$

② $Z_n = \{0, 1, \cdots, n-1\}$

$kl = k + l \mod n \quad \forall\, k, l \in Z_n$

Consider ~~g_1~~ $g_1, g_2 \in Z_n$ and if g_1, g_2 belong to the same conjugacy class,

$$g_2 = g g_1 g^{-1} \quad \text{for some } g \in Z_n$$

then $$g_2 g = g g_1 \underbrace{g^{-1} g}_{e} = g g_1$$

$\because Z_n$ is clearly abelian $\therefore g_2 g = g g_2$

$\therefore g g_2 = g g_1 \quad \therefore \underbrace{g^{-1} g}_{e} g_2 = \underbrace{g^{-1} g}_{e} g_1$

$\rightarrow g_2 = g_1$

Clearly each element in group Z_n is its own conjugacy class. Each conjugacy class has exactly 1 ~~one~~ element.

a) ~~Irreducid~~ Irreducible Complex representations ~~of~~ R_q of Z_n $(q = 0, 1, 2, \cdots, n-1)$ are

$$R_q(g) = e^{\frac{2\pi i q g}{n}} \quad \forall\, g \in Z_n$$
$$q \in \{0, 1, 2, \cdots, n-1\}$$

They are all 1 dimensional, so the characters (traces) are just themselves.

$$\chi_q(g) = R_q(g) = e^{\frac{2\pi i q g}{n}}$$

b) Scalar product of characters are defined by

$$(\chi_q, \chi_p) = \frac{1}{|G|} \sum_{g \in G} \chi_q^*(g)\, \chi_p(g)$$

$$= \frac{1}{n} \sum_{g=0}^{n-1} e^{\frac{-2\pi i q g}{n}} e^{\frac{2\pi i p g}{n}}$$

$$= \frac{1}{n} \sum_{g=0}^{n-1} e^{2\pi i (p-q) g / n}$$

$$= \frac{1}{n}\left[1 + \left(e^{\frac{2\pi i (p-q)}{n}}\right) + \left(e^{\frac{2\pi i (p-q)}{n}}\right)^2 + \cdots + \left(e^{\frac{2\pi i (p-q)}{n}}\right)^{n-1} \right]$$

(we used sum of geometric series)

$$= \begin{cases} \dfrac{1}{n}(1)\dfrac{\left(1-\overbrace{e^{2\pi i (p-q)}}^{=1 \ \because\ p, q \text{ are integers.}}\right)}{1-e^{\frac{2\pi i (p-q)}{n}}} = \dfrac{1}{n}(1)\dfrac{(1-1)}{1-e^{\frac{2\pi i (p-q)}{n}}} = 0 & \text{if } p \neq q \\[2ex] \dfrac{1}{n}(\underbrace{1+1+\cdots+1}_{n}) = \dfrac{n}{n} = 1 & \text{if } p = q \end{cases}$$

$$\Rightarrow (\chi_q, \chi_p) = \delta_{qp}$$

c) ~~Foc~~ Focus on $Z_3 = \{0, 1, 2\}$

$kl = k+l \mod 3 \quad \forall\, k, l \in Z_3$

$M^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad M^2 = M^1 \cdot M^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$M^3 = M \cdot M^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I = M^0$

~~$M^4 = M \cdot I = M$ ...~~

so $R(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad R(1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad R(2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

All group ~~mu~~ multiplications

Z_3 is abelian, so we don't care the ~~order~~ order.

$0 \cdot 1 = 0+1 \mod 3 = 1$

$0 \cdot 2 = 0+2 \mod 3 = 2$

$1 \cdot 2 = 1+2 \mod 3 = 0$

corresponding $R(k)$ operations.

$R(0)R(1) = I\,R(1) = R(1) = R(1)R(0) = R(1)I$

$R(0)R(2) = I\,R(2) = R(2) = R(2)R(0) = R(2)I$

$R(1)R(2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = R(0)$

~~We observe~~ $= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = R(2)R(1)$

we observe that $R(g_1)R(g_2) = R(g_1 g_2) \quad \forall\, g_1, g_2 \in Z_3$

so R is a group homomorphism

And $R: Z_3 \to \mathrm{Aut}(\mathbb{C}^3)$ $\therefore$ R is a representation.

The multiplicity of irreducible representation R_q

R is given by $(\chi_q, \chi_R) = m_q$

and $R = R_1^{\oplus m_1} \oplus R_2^{\oplus m_2} \oplus R_0^{\oplus m_0}$

~~$m_1 = (\chi_1, \chi_R)$~~ $\vec{\chi_R} = (tr(R(0)), tr(R(1)), tr(R(2)))$
$= (3, 0, 0)$

$\vec{\chi_1} = (\chi_1(0), \chi_1(1), \chi_1(2)) = (e^{\frac{2\pi i(0)}{3}}, e^{\frac{2\pi i(1)}{3}}, e^{\frac{2\pi i(2)}{3}}$
$= (1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}})$

$m_1 = (\chi_1, \chi_R) = \frac{1}{3} \vec{\chi_1}^\dagger \vec{\chi_R} = \frac{1}{3} \cdot 3 = \underline{\underline{1}}$

$\vec{\chi_2} = (e^{\frac{4\pi i}{3}(0)}, e^{\frac{4\pi i}{3}(1)}, e^{\frac{4\pi i(2)}{3}}) = (1, e^{\frac{4\pi i}{3}}, e^{\frac{2\pi i}{3}})$

$m_2 = (\chi_2, \chi_R) = \frac{1}{3} \cdot 3 = \underline{\underline{1}}$

$\vec{\chi_0} = (e^0, e^0, e^0) = (1, 1, 1)$

$m_0 = (\chi_0, \chi_R) = \frac{1}{3} \vec{\chi_0}^\dagger \vec{\chi_R} = \frac{1}{3}(3) = \underline{\underline{1}}$

$\therefore \underline{\underline{R = R_0 \oplus R_1 \oplus R_2}}$

③ $S_n = \{\sigma : \{1, \dots, n\} \to \{1, \dots, n\} \mid \sigma \text{ bijective}\}$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

a) In words a permutation σ is a bijection $\sigma : X \to X$, where $X = \{1, 2, \dots, n\}$

$\because$ σ is a bijection $\therefore$ σ has an inverse σ^{-1}

(inverse)

~~If σ_1, σ_2 are permutations, then~~

~~consider $\sigma_1 \sigma_2$~~

~~$\sigma_2^{-1}\sigma_1^{-1}\sigma_1\sigma_2$~~

~~$\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1}$~~

~~$= id$~~

~~$= \sigma_2^{-1}$~~

Also, $id = \begin{pmatrix} 1, & 2, & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$ is clearly an identity

(identity)

Consider $\sigma_1, \sigma_2 \in S_n$

$$\sigma_2^{-1}\underbrace{\sigma_1^{-1}\sigma_1}_{id}\sigma_2 = \sigma_2^{-1}\sigma_2 = id$$

$\therefore$ $\sigma_2^{-1}\sigma_1^{-1}$ is the inverse of $\sigma_1\sigma_2$

$\because$ $\sigma_1\sigma_2$ has an inverse, so it must also be a bijection from X to X

By definition $\sigma_1\sigma_2$ is also a permutation

$\rightarrow$ $\sigma_1\sigma_2 \in S_n$ (closure)

Consider an element $x \in X = \{1,2,3,\dots,n\}$

$$[(\sigma_3 \circ \sigma_2) \circ \sigma_1](x) = [\sigma_3 \circ \sigma_2](\sigma_1(x))$$

$$= \sigma_3(\sigma_2(\sigma_1(x))) = \sigma_3([\sigma_2 \circ \sigma_1](x))$$

$$= [\sigma_3 \circ (\sigma_2 \circ \sigma_1)](x) \quad \forall x \in X$$

$$\Rightarrow \quad (\sigma_3 \circ \sigma_2) \circ \sigma_1 = \sigma_3 \circ (\sigma_2 \circ \sigma_1)$$

(Associativity)

$\Rightarrow$ S_n is a group

For $n > 2$, consider $n = 3$ and two permutations

$\sigma_1, \sigma_2 \in S_3$ where $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

then $\sigma_1 \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$\sigma_2 \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ clearly $\sigma_1 \sigma_2 \neq \sigma_2 \sigma_1$

Now for general n, consider

$\sigma_1, \sigma_2 \in S_n$, $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n \\ 1 & 3 & 2 & 4 & 5 & 6 & \cdots & n \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n \\ 3 & 1 & 2 & 4 & 5 & 6 & \cdots & n \end{pmatrix}$

then $\sigma_1 \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n \\ 2 & 1 & 3 & 4 & 5 & 6 & \cdots & n \end{pmatrix}$

$\sigma_2 \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n \\ 3 & 2 & 1 & 4 & 5 & 6 & \cdots & n \end{pmatrix}$ $\therefore \sigma_1 \sigma_2 \neq \sigma_2 \sigma_1 \quad \forall n \in \mathbb{N}, n > 2$

$\Rightarrow$ S_n is non-Abelian

b) The conjugacy class of S_n is in 1:1 ~~correspondence~~ correspondence with its ~~cy~~ cycle structure.

→ This is because If we take ~~$\sigma(i) = j$~~ and $\alpha, \beta, \sigma \in S_n$ such that $\beta = \sigma\alpha\sigma^{-1}$

then ~~$\beta(\alpha(i)) = \sigma\alpha\sigma^{-1}\sigma$~~ and $\alpha(i) = j$

$$\beta(\sigma(i)) = \sigma\alpha\sigma^{-1}\sigma(i) = \sigma\alpha(i) = \cancel{\sigma(j)}\ \sigma(j)$$

$\because$ $\alpha(i) = j$, $\beta(\sigma(i)) = \sigma(j)$

Imp, σ can be seen as simply a index relabeling on both the domain and image of ~~β~~ α

$\therefore$ α, β should have the same ~~s~~ cycle structure.

→ Conversely, If α, β have same same cycle structure, then we can write $\alpha = (A_1 A_2 \cdots)(B_1 \cdots)(C_1 \cdots) \cdots$

$$\cancel{\sigma = \begin{pmatrix} \alpha(1) & \alpha(2) & \cdots & \alpha(n) \\ \beta(1) & \beta(2) & \cdots & \beta(n) \end{pmatrix}}$$ and ~~$\beta = (B_1 B_2 \cdots)$~~

~~and $\sigma\alpha\sigma^{-1} = \beta$~~

$\beta = (A'_1 A'_2 \cdots)(B'_1 \cdots)(C'_1 \cdots) \cdots$

and write $$\sigma = \begin{pmatrix} \cancel{A_1 A_2}\ A_1 & A_2 & \cdots & B_1 & \cdots & C_1 & \cdots \\ A'_1 & A'_2 & \cdots & B'_1 & \cdots & C'_1 & \cdots \end{pmatrix}$$

then $\sigma\alpha\sigma^{-1}$ changes primed ~~the integers~~ numbers to unprimed ones, do the ~~same per~~ same permutation as β and change back from ~~up~~ unprimed to primed.

This is the same as just applying β $\therefore \beta = \sigma\alpha\sigma^{-1}$

□

S_3 has 1 identity, 3 two-cycles and 2 three cycles

$\therefore$ Conjugacy classes are :

identity	$(1)(2)(3)$
two cycle	$(12)(3)$, $(13)(2)$, $(1)(23)$
three cycle	(123), (132)

There are $\underline{\underline{3}}$ conjugacy classes

$\rightarrow$ Number of irreducible representations = Number of conjugacy classes $= 3$

$\rightarrow$ Dimension of i^{th} irreducible representations dim (R_i) satisfy $\underline{\sum_{i=1}^{3}(\dim(R_i))^2 = |S_3|}$

$|S_3| = 3! = 6$

$\therefore \sum_{i=1}^{3}(\dim(R_i))^2 = 6$

3 integers, their square sum to 6

we must have $(2,1,1)$

$\therefore \dim(R_2) = 2$ (without loss of generality)

$\dim(R_t) = \dim(R_1) = 1$

$\rightarrow$ $\left.\begin{matrix}\text{One } 2D \\ \text{two } 1D\end{matrix}\right\}$ complex irreducible representations.
R_t, R_1, R_2

c) the character table:

one of R_t, R_1 should be the trivial representation, since they are 1D. Let's say R_t is.

χ_i = character of R_i

We now know

character \ conjugacy class	(1)(2)(3)	(12)(3) (1)(23) (13)(2)	(123) (132)
χ_t	1	1	1
χ_1	1	m_1	m_2
χ_2	2	m_3	m_4

→ character of all group elements for the trivial representation is identically 1

→ character of (1)(2)(3) for any representation is the dimension of representation.

→ m_1, m_2, m_3, m_4 are unknowns.

characters of different conjugacy classes are orthogonal with respect to

$$(\chi_i, \chi_j) = \frac{1}{|S_3|} \sum_{g \in S_3} \chi_i^*(g)\, \chi_j(g)$$

take care of multiplicity, we have

$$(\chi_t, \chi_1) = \cancel{\frac{1}{6}(1\times1 + 3(1\times m_1) + 1\times m_2) = 0}$$

$$= \frac{1}{6}\left((1)[1\times1] + 3[1\times m_1] + 2[1\times m_2]\right) = 0$$

$$\therefore \quad 1 + 3m_1 + 2m_2 = 0 \qquad ①$$

$$(\chi_1, \chi_1) = \frac{1}{6}\left(1 + 3|m_1|^2 + 2|m_2|^2\right) = 1$$

$$\therefore \quad 1 + 3|m_1|^2 + 2|m_2|^2 = 6 \qquad ②$$

$$(\chi_t, \chi_2) = \frac{1}{6}(2 + 3m_3 + 2m_4) = 0$$

$$\rightarrow \quad 2 + 3m_3 + 2m_4 = 0 \qquad ③$$

$$(\chi_2, \chi_2) = \frac{1}{6}\left(4 + 3|m_3|^2 + 2|m_4|^2\right) = 1$$

$$\rightarrow \quad 4 + 3|m_3|^2 + 2|m_4|^2 = 6 \qquad ④$$

$$(\chi_1, \chi_2) = \frac{1}{6}(2 + m_1^* m_3 + m_2^* m_4) = 0$$

$$\rightarrow \quad 2 + 3m_1^* m_3 + 2m_2^* m_4 = 0 \qquad ⑤$$

-! We have 10 equations for 8 unknowns

real and imaginary parts of ①, ..., ⑤ — real and imaginary parts of $m_1, \ldots, m_4$

$\therefore$ system is completely determinstic.

→ we assume $m_1, \ldots, m_4$ are all real and verify that a posterior

$\therefore$ ① $\rightarrow$ $m_2 = -\dfrac{(1+3m_1)}{2}$

sub into ② $\Rightarrow$ $1+3m_1^2+\frac{1}{2}(1+3m_1)^2 = 6$

$\therefore\ 9m_1^2+6m_1+1+2+6m_1^2 = 12$

$\therefore\ 15m_1^2+6m_1-9 = 0$

$\therefore\ 5m_1^2+2m_1-3 = 0$

$\therefore\ (m_1+1)(5m_1-3) = 0$

$\therefore\ \begin{cases} m_1 = -1 \\ m_2 = 1 \end{cases}$ or $\begin{cases} m_1 = \frac{3}{5} \\ m_2 = -\frac{7}{5} \end{cases}$

~~$m_1 m_2 = -1$ or $-\frac{21}{25}$~~

Similarly ③ $\rightarrow$ $m_4 = -\dfrac{2+3m_3}{2}$

sub $\Rightarrow$ ④ $\Rightarrow$ $4+3m_3^2+\frac{1}{2}\left(2+3m_3\right)^2 = 6$

$\therefore\ 8+6m_3^2+9m_3^2+12m_3+4 = 12$

$\therefore\ 15m_3^2+12m_3 = 0$

$\therefore$ ~~$m_3 =$~~ $m_3(15m_3+12) = 0$

$\therefore\ \begin{cases} m_3 = 0 \\ m_4 = -1 \end{cases}$ or $\begin{cases} m_3 = -\frac{4}{5} \\ m_4 = \frac{1}{5} \end{cases}$

~~$\therefore\ m_3 m_4 = 0$ or $\frac{-4}{25}$~~

$m_1 m_3 = 0,\ \frac{4}{5},\ -\frac{12}{25}$

$m_2 m_4 = -1,\ \frac{1}{5},\ \frac{7}{5},\ -\frac{7}{25}$

we need to satisfy ⑤

We have 2 possible solutions.

$$(m_1, m_2, m_3, m_4) = (-1, 1, 0, -1)$$

$$\left(\frac{3}{5}, -\frac{7}{5}, -\frac{4}{5}, \frac{1}{5}\right)$$

Now, we have 2 solutions for the system of equations, and this system is quadratic, which has at most 2 solutions. So these are ~~the~~ all solutions.

No more complex solutions.

Now check the value of m_1

$\because$ $g = (12)(3)$ ~~ac~~ acts on itself gives the identity

$\therefore$ $R_1(g) R_1(g) = id_1 = 1$

$\because$ $R_1(g)$ is 1d (just a number)

$\therefore$ $R_1(g) = \pm 1$

And $\because$ $R_1(g)$ is 1d $\therefore$ $R_1(g) = \chi_1(g) = m_1$

$\therefore$ $m_1 = \pm 1$

that elimates the second solution

$\therefore$ $(m_1, \cdots, m_4) = (-1, 1, 0, -1)$

Full character table :

character \ conjugacy class	C_1 (1)(2)(3)	C_2 (12)(3) (1)(23) (13)(2)	C_3 (123) (132)
χ_t	1	1	1
χ_1	1	−1	1
χ_2	2	0	−1

d) The regular representation of S_3 :

Group Algebra $A_{S_3} = \{ \sum_{g \in S_3} v(g)\, g \mid v(g) \in K \}$

Regular representation ~~$R_{reg}(g)\, g = V$~~

$$R_{reg}(g)\, v = g v \quad \forall \; v \in A_{S_3}, \; g \in S_3$$

$$R_{reg} : S_3 \longrightarrow \mathrm{Aut}(A_{S_3})$$

$$R_{reg} = R_1^{\oplus \dim(R_1)} \oplus R_2^{\dim(R_2)} \oplus \cdots \oplus R_k^{\oplus \dim R_k}$$

for irreducible complex representations R_i

$i \in \{1, 2, \cdots, k\}$ of S_3

$\therefore$ $\underline{R_{reg} = R_t \oplus R_1 \oplus R_2^{\oplus 2}}$

$$A_{S_3} = V_t \oplus V_1 \oplus V_2^{\oplus 2}$$

V_i is the corresponding vector space of R_i

The projector

$$P_i = \frac{\dim(R_i)}{|S_3|} \sum_{g \in S_3} \chi_i(g)^* g \quad : A_{S_3} \rightarrow A_{S_3}$$

is a projector onto $V_i^{\oplus \dim(V_i)}$, which correspond to the irreducible representation R_i

$\therefore$ Projector of

R_t : $P_t = \frac{1}{6}[(1)(2)(3) + (12)(3) + (1)(23) + (13)(2) + (123) + (132)$

R_1 : $P_1 = \frac{1}{6}[(1)(2)(3) - (12)(3) - (1)(23) - (13)(2) + (123) + (132)]$

$R_2^{\oplus 2}$: $P_2 = \frac{1}{3}[2(1)(2)(3) - (123) - (132)]$.

e)

$\cancel{\chi_{R_2 \otimes R_2} = \chi}$

$$\chi_{2\otimes 2} = \chi_{R_2 \otimes R_2}(g) = \chi_{R_2}(g)^2 \qquad \forall\, g \in S_3$$
$$= \chi_2^{\,2}(g)$$

∵

	c_1	c_2	c_3
χ_2	2	0	-1

∴

	c_1	c_2	c_3
$\chi_{2\otimes 2}$	4	0	1

The multiplicity of R_i in $R_2 \otimes R_2$ is given by $(\chi_i, \chi_{2\otimes 2}) = m_i$

i.e. $$m_i = \frac{1}{6} \sum_{g \in S_3} \chi_i(g)^* \chi_{2\otimes 2}(g)$$

$$m_t = \frac{1}{6}\big((1)(1\times 4) + 3(1\times 0) + 2(1\times 1)\big)$$
$$= 1$$

$$m_1 = \frac{1}{6}\big((1)(1\times 4) + 3(-1\times 0) + 2(1\times 1)\big) = 1$$

$$m_2 = \frac{1}{6}\big((1)(2\times 4) + 3(0\times 0) + 2(-1\times 1)\big) = 1$$

$$\Rightarrow \quad R_2 \otimes R_2 = \underline{\underline{R_t \oplus R_1 \oplus R_2}}$$

(4)

$$R = n_1 R_1 \oplus \cdots \oplus n_r R_r$$
$$= R_1^{\oplus n_1} \oplus \cdots \oplus R_r^{\oplus n_r}$$

a) $$1_{n_i} = \underbrace{\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}}_{n_i \text{ } 1\text{'s}}$$

$$1_{n_i} \otimes R_i(g) = \underbrace{\begin{pmatrix} R_i(g) & & & \\ & R_i(g) & & \\ & & \ddots & \\ & & & R_i(g) \end{pmatrix}}_{n_i \text{ } R_i(g)\text{'s}}$$

$$1_{n_1} \otimes R_1(g) \oplus \cdots \oplus 1_{n_r} \otimes R_r(g)$$

$$= \begin{pmatrix} \underbrace{R_1(g) \ \ddots \ R_1(g)}_{n_1 \text{ } R_1(g)\text{'s}} & & & \\ & \underbrace{R_2(g) \ \ddots \ R_2(g)}_{n_2 \text{ } R_2(g)\text{'s}} & & \\ & & \ddots & \\ & & & \underbrace{R_r(g) \ \ddots \ R_r(g)}_{n_r \text{ } R_r(g)\text{'s}} \end{pmatrix}$$

$$= R_1^{\oplus n_1} \oplus \cdots \oplus R_r^{\oplus n_r}$$

$\square$

b) consider some simple cases first:

→ If $R = R_1$, $[P, R_1(g)] = 0 \Rightarrow P R_1(g) = R_1(g) P$

R_1 irreducible, $n_1 = 1$, $n_i = 0$ for $i \neq 1$

then by schur's Lemma, $P = \lambda \mathbb{1}_{d_1} = P_1 \times \mathbb{1}_{d_1}$

where $P_1 = \lambda$ is a 1×1 matrix $(n_1 \times n_1)$

which is the required form.

→ Now consider $R = R_1 \oplus R_2$ with R_1, R_2 inequivalent.

then $R(g) = \begin{pmatrix} R_1(g) & 0 \\ 0 & R_2(g) \end{pmatrix}$ $\quad R_1(g) \to n_1 \times n_1$, $R_2(g) \to n_2 \times n_2$

let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ where $\left\{ \begin{array}{l} P_{11} \to n_1 \times n_1 \\ P_{12} \to n_1 \times n_2 \\ P_{21} \to n_2 \times n_1 \\ P_{22} \to n_2 \times n_2 \end{array} \right\}$ matrices

~~let~~ we have ~~PR(g) = RP~~ $PR(g) = R(g)P$

$\rightarrow \begin{pmatrix} P_{11} R_1(g) & P_{12} R_2(g) \\ P_{21} R_1(g) & P_{22} R_2(g) \end{pmatrix} = \begin{pmatrix} R_1(g) P_{11} & R_1(g) P_{12} \\ R_2(g) P_{21} & R_2(g) P_{22} \end{pmatrix}$

$\rightarrow P_{11} R_1(g) = R_1(g) P_{11} \qquad P_{22} R_2(g) = R_2(g) P_{22}$

By schur's lemma, ~~$P_{11} = \lambda_{11} \mathrm{id}$~~ $P_{11} = \lambda_{11} \mathbb{1}_{n_1}$, $P_{22} = \lambda_{22} \mathbb{1}_{n_2}$ $(\lambda_{11}, \lambda_{22} \in K)$

Also $P_{12} R_2(g) = R_1(g) P_{12}$, $P_{21} R_1(g) = R_2(g) P_{21}$

This, by Schur's Lemma, gives

P_{12}, P_{21} are either isomorphism or 0

If $n_1 \neq n_2$, P_{12}, P_{21} have no inverse $\Leftarrow$ they are not bijection $\Rightarrow$ they are not isomorphism

$\Rightarrow$ $\underline{P_{12} = P_{21} = 0}$

If $n_1 = n_2$, P_{21}, P_{12} not invertible, $\Rightarrow$ not bijection

$\Rightarrow$ $\underline{P_{12} = P_{21} = 0}$

If $n_1 = n_2$, P_{21}, P_{12} invertible, then $P_{12} R_2(g) = R_1(g) P_{12}$

$\cancel{P_{12}^{-1} P_{12}}$ $P_{12} R_2(g) P_{12}^{-1} = R_1(g) P_{12} P_{12}^{-1} = R_1(g)$

$\rightarrow$ $R_1(g), R_2(g)$ are equivalent $\Rightarrow$ $\underline{\text{contradiction}}$

$\therefore$ $P_{12} = P_{21} = 0$ if R_1, R_2 inequivalent.

$\therefore$ P has the form $\cancel{P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{12} \end{pmatrix} = \begin{pmatrix} \lambda_{11} id_{n_1} & \\ & \lambda_{22} id_{n_2} \end{pmatrix}}$

$$P = \begin{pmatrix} \lambda_{11} 1_{n_1} & 0 \\ 0 & \lambda_{22} 1_{n_2} \end{pmatrix}$$

$$= P_1 \times 1_{d_2} \oplus P_2 \times 1_{d_2}$$ where P_1, P_2

are ~~scalar~~ $\lambda_{11}, \lambda_{22}$ respectively.
1×1 matrices

This is the correct form

If $R_1 \cong R_2$, we can always find a basis in which $R_1(g) = R_2(g)$, and $\cancel{P = R_1(g) \oplus R_1(g)}$ $\cancel{= 2R_1}$

$$P = R_1 \oplus R_1 = 2R_1 = R_1^{\oplus 2}$$

$$\cancel{R = \begin{pmatrix} R_1(g) & 0 \\ 0 & R_1(g) \end{pmatrix}} \qquad R = \begin{pmatrix} R_1(g) & 0 \\ 0 & R_1(g) \end{pmatrix}$$

~~In~~ and $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ $\quad \cancel{PR(g) = RP}$

$$PR(g) = R(g)P$$

in this case we also find $P_{11} = \lambda_{11} 1_{d_1}$, $P_{22} = \lambda_{22} 1_{d_1}$

But also, $P_{12} R_1(g) = R_1(g) P_{12}$ $\quad P_{21} R_1(g) = R_1(g) P_{21}$

By Schur's Lemma, $P_{12} = \lambda_{12} 1_{d_1}$, $P_{21} = \lambda_{21} 1_{d_1}$

and
$$P = \begin{pmatrix} \lambda_{11} 1_{d_1} & \lambda_{12} 1_{d_1} \\ \lambda_{21} 1_{d_1} & \lambda_{22} 1_{d_1} \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}}_{P_1} \times 1_{d_1}$$

$$P = P_1 \times 1_{d_1} \quad \Rightarrow \text{ correct form}$$

Now we move to the more general case:

Assume $\cancel{R(g) =}$ $R' = R_1^{\oplus n_1} \oplus \cdots \oplus R_r^{\oplus n_r} \oplus \tilde{R}$ (All $R_1, \cdots, R_r$ and $\tilde{R}$ irreducible)

$\dim(\tilde{R}) = \tilde{d}$ $\quad$ and $R = R_1^{\oplus n_1} \oplus \cdots \oplus R_r^{\oplus n_r}$

and $[P, R(g)] = 0$, and P is in the form

$$P = P_1 \times 1_{d_1} \oplus \cdots \oplus P_r \times 1_{d_r}$$

consider the P' that satisfies $[P', R'(g)] = 0$

(1) If $\tilde{R}$ is inequivalent to $R_1, R_2, \cdots, R_r$

$$R'(g) = \left(\begin{array}{c|c} R(g) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & \tilde{R}(g) \end{array}\right) \qquad P' = \left(\begin{array}{c|c} P & \begin{matrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{matrix} \\ \hline X_1 \; X_2 \cdots X_k & \tilde{P} \end{array}\right)$$

where $\underline{k = n_1 d_1 + n_2 d_2 + \cdots + n_r d_r}$

then
$$R'(g) P' = \left(\begin{array}{c|c} RP & \begin{matrix} R_1(g) Y_1 \\ R_1(g) Y_2 \\ \vdots \\ R_r(g) Y_{k-1} \\ R_r(g) Y_k \end{matrix} \\ \hline \tilde{R}(g) X_1 \cdots \tilde{R}(g) X_k & \tilde{R}(g) \tilde{P} \end{array}\right)$$

$$P' R'(g) = \left(\begin{array}{c|c} PR & \begin{matrix} Y_1 \tilde{R}(g) \\ Y_2 \tilde{R}(g) \\ \vdots \\ Y_k \tilde{R}(g) \end{matrix} \\ \hline X_1 R_1(g) \; X_2 R_1(g) \cdots X_k R_r(g) & \tilde{P} \tilde{R}(g) \end{array}\right)$$

Impose $\quad P' R'(g) = R'(g) P'$

$\rightarrow \quad \tilde{R}(g) \tilde{P} = \tilde{P} \tilde{R}(g) \rightarrow$ By schur's lemma
$$\tilde{P} = \tilde{\lambda} 1_{\tilde{n}}$$

the upper left automatically satisfied by $PR = RP$

(inductive assumption)

Also need
$$X_1 R_1(g) = \tilde{R}(g) X_1$$
$$X_2 R_1(g) = \tilde{R}(g) X_2$$
$$\vdots$$
$$X_{n_1} R_1(g) = \tilde{R}(g) X_{n_1}$$
$$\vdots$$
$$X_k R_r(g) = \tilde{R}(g) X_k$$

$\because$ $\tilde{R}$ inequivalent to $R_1, \cdots, R_r$

$\therefore$ By schur's lemma $\rightarrow$ $X_1 = X_2 = \cdots = X_k = 0$

~~(subtlety subtlety~~.

(subtlety, X_1 is $\tilde{d} \times d_1$ matrix, so are $X_2, \cdots, X_{n_1}$. $X_{n_1+1}, X_{n_1+2}, \cdots, X_{n_2}$ are $\tilde{d} \times d_2$ matrices, and so on)

Also need
$$Y_1 \tilde{R}(g) = R_1(g) Y_1$$
$$Y_2 \tilde{R}(g) = R_1(g) Y_2$$
$$\vdots$$
$$Y_{n_1} \tilde{R}(g) = R_1(g) Y_{n_1}$$
$$\vdots$$
$$Y_k \tilde{R}(g) = R_r(g) Y_k$$

By schur's lemma

$Y_1 = Y_2 = \cdots = Y_k = 0$

(Y_1 is $d_1 \times \tilde{d}$ matrix, Y_{n_1+1} is $d_2 \times \tilde{d}$ matrix, and so on).

So P' looks like

$$P' = \left(\begin{array}{c|c} P & \begin{matrix}0\\0\\\vdots\\0\end{matrix} \\ \hline 0\,0\cdots 0 & \tilde{P} \end{array}\right) = \left(\begin{array}{c|c} P & \begin{matrix}0\\0\\\vdots\\0\end{matrix} \\ \hline 0\,0\cdots 0 & \tilde{\lambda} 1_{\tilde{d}} \end{array}\right)$$

$$P' = P \oplus \tilde{\lambda} 1_{\tilde{n}}$$

$$= P_1 \times 1_{d_1} \oplus \cdots \oplus P_n \times 1_{d_r} \oplus \tilde{\lambda} 1_{\tilde{d}}$$

in the correct form

(2) if $\tilde{R}$ = one of $R_1, R_2, \ldots, R_r$

then without loss of generality, assume

$\tilde{R} = R_r$

So $R' = R_1^{\oplus n_1} \oplus \cdots \oplus R_{r-1}^{\oplus n_{r-1}} \oplus R_r^{\oplus n_r + 1}$

then similar to (1), when we impose $P'R' = R'P'$

$$X_1 = X_2 = \cdots X_{k-n_r} = 0 \qquad Y_1 = Y_2 = \cdots = Y_{k-n_r} = 0$$

But

$$X_{k-n_r+1} R_r(g) = \underset{\tilde{R} = R_r}{\underset{\downarrow}{R_r(g)}} X_{k-n_r+1}$$

$$\Rightarrow \quad X_{k-n_r+1} = \lambda_1 1_{d_r}$$

similarly

$\cancel{X_{k-n_r+1} = X_{k-n_r+2} = \cdots = X_k =}$

$$X_{k-n_r+2} = \lambda_2 1_{d_r} \qquad Y_{k-n_r+1} = \rho_1 1_{d_r}$$
$$X_{k-n_r+3} = \lambda_3 1_{d_r} \qquad Y_{k-n_r+2} = \rho_2 1_{d_r}$$
$$\vdots \qquad\qquad \vdots$$
$$X_k = \lambda_{n_r} 1_{d_r} \qquad Y_k = \rho_{n_r} 1_{d_r}$$

we also have $\tilde{P} = \tilde{\lambda} 1_{\tilde{d}} = \tilde{\lambda} 1_{d_r}$

$\because d_r = \tilde{d}$ $(\tilde{R} = R_r)$

$$\therefore P' = \left(\begin{array}{c|c} P & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \begin{matrix} & P_r \times 1_{d_r} \end{matrix} & \begin{matrix} \rho_1 1_{d_r} \\ \rho_2 1_{d_r} \\ \vdots \\ \rho_{n_r} 1_{d_r} \end{matrix} \\ \hline 0 \cdots 0 \quad \lambda_1 1_{d_r} \cdots \lambda_{n_r} 1_{d_r} & \tilde{\lambda} 1_{d_r} \end{array}\right)$$

the matrix in $\boxed{}$ is $\cancel{\left(\begin{array}{c|c} P_r & \begin{matrix}\rho_1 \\ \vdots \\ \rho_{d_r}\end{matrix} \\ \hline \lambda_1 \cdots \lambda_{d_r} & \end{array}\right)}$ $\underline{\underline{P_r' \times 1_{d_r+1}}}$

where $\left(\begin{array}{c|c} P_r & \begin{matrix}\rho_1 \\ \vdots \\ \rho_{n_r}\end{matrix} \\ \hline \lambda_1 \cdots \lambda_{n_r} & \tilde{\lambda} \end{array}\right) = P_r'$ is a $(n_r+1) \times (n_r+1)$

matrix

so $P' = P_1 \times 1_{d_1} \oplus \cdots \oplus P_{r-1} \times 1_{d_{r-1}} \oplus P_r' \times 1_{d_r+1}$

(in <u>correct form</u>)

- where we've shown that If P has the required general form for representation R, then $\oplus$ any extra irreducible representation $\tilde{R}$ to R perserve the ~~form of~~ required form in the new P' that commutes with R'

- we also know that for ~~any~~ the base cases the ~~proposition~~ is true
proposition

$\therefore$ the proposition is true in general by induction principle.

c) $$R(e^{i\alpha}) = \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{-2i\alpha} \end{pmatrix}$$

$$\Rightarrow \quad R = R_1^{\oplus 2} \oplus R_{-2}$$

$$n_1 = 2 \qquad n_2 = 1$$

$$d_1 = d_2 = 1$$

$\therefore$ General form of P such that $[P, R(e^{i\alpha})] = 0$

is $P = P_1 \times \mathbb{1}_1 \oplus P_2 \times \mathbb{1}_1$ $\qquad$ $P_1 \to 2\times 2$ matrix.
$P_2 \to 1\times 1$ matrix.

$$\therefore \quad P = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} \quad \text{commutes with } R$$

5

a) $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e$

$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \tau^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -e$

$\tau^3 = -e\tau = -\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

$\tau^4 = (-e)(-e) = e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma\tau^2 = \sigma(-e) = -\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \tau^2\sigma$

$\sigma\tau^3 = \sigma(-\tau) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$

$\tau\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \sigma\tau^3$

$\tau^3\sigma = -\tau\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma\tau$

$\because$ $\sigma^2 = e$ $\quad \tau^4 = e$ $\quad \therefore$ No need to try products of σ and τ that has σ^n, $n > 2$ and τ^m, $m > 4$.

Non-repeated elements generated above are

$\{e, \sigma, \tau, \tau^2, \tau^3, \sigma\tau, \sigma\tau^2, \sigma\tau^3\} = D_4$

eight elements $\quad \therefore$ $\underline{|D_4| = 8}$

$\tau\sigma = \sigma\tau^3 \neq \sigma\tau \Rightarrow$ Non-abelian

For $g \in G$, its conjugacy class is the set

$\{xgx^{-1} \mid x \in G\}$

(1) For e, $\cancel{xgx^{-1}}$ $xex^{-1} = xx^{-1} = e \quad \forall x \in G$

$\therefore$ conjugacy class $\underline{\underline{\{e\}}}$

(2) For t^2, $xt^2x^{-1} = x(-e)x^{-1} = -ee = -e = t^2 \ \forall x \in G$

$\therefore$ conjugacy class $\underline{\underline{\{t^2\}}}$

$e^{-1} = \underline{e}$ $\quad \sigma^{-1} = \underline{\sigma}$ $\quad t^{-1} = \underline{t^3}$ $\quad (\sigma t)^{-1} = t^{-1}\sigma^{-1} = t^3\sigma = \underline{\sigma t}$

$(t^2)^{-1} = \underline{t^2}$ $\quad (\sigma t^2)^{-1} = (t^2)^{-1}\sigma^{-1} = t^2\sigma = \underline{\sigma t^2}$ $\quad (t^3)^{-1} = \underline{t}$

$(\sigma t^3)^{-1} = (t^3)^{-1}\sigma^{-1} = t\sigma = \underline{\sigma t^3}$

~~For~~

(3) For t :

$\cancel{ete^{-1} = t}$, $\sigma t \sigma^{-1} = \sigma t \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = t$

$ete^{-1} = t$

$ttt^{-1} = t$ $\qquad t^2 t (t^2)^{-1} = t$

$t^3 t (t^3)^{-1} = t^{-1} t t = t$ $\qquad (\sigma t) t (\sigma t)^{-1} = \sigma t \cancel{t t^{-1}} \sigma^{-1} = \sigma t \sigma = t$

$(\sigma t^2) t (\sigma t^2)^{-1} = \sigma t^2 t t^2 \sigma = \sigma \underbrace{t^4}_{e} t \sigma = \sigma t \sigma = t$

$(\sigma t^3) t (\sigma t^3)^{-1} = \cancel{\sigma t^3 t \sigma t^3 = \sigma}$

$\sigma \underbrace{t^3 t}_{e} \sigma t^3 = \underbrace{\sigma^2}_{e} t^3 = \underline{\underline{t^3}}$

$\Rightarrow$ conjugacy class $\underline{\underline{\{t, t^3\}}}$

(4)

For σ :

$e\sigma e^{-1} = \sigma \qquad \sigma\sigma\sigma^{-1} = \sigma^3 = \underbrace{\sigma^2}_{e}\sigma = \sigma$

$\tau\sigma\tau^{-1} = \tau\sigma\tau^3 = \sigma\tau^3\tau^3 = \sigma\underbrace{\tau^4}_{e}\tau^2 = \underline{\underline{\sigma\tau^2}}$

$\tau^2\sigma(\tau^2)^{-1} = \sigma \qquad \tau^3\sigma(\tau^3)^{-1} = \tau^3\sigma\tau = \tau^3\tau^3\sigma = \tau^2\sigma = \underline{\underline{\sigma\tau^2}}$

$(\sigma\tau)\sigma(\sigma\tau)^{-1} = \cancel{\tau\sigma^{-1}}\ \sigma\underbrace{\tau\sigma}_{\sigma\tau^2}\tau^{-1}\sigma^{-1} = \underbrace{\sigma\sigma}_{e}\tau^2\sigma^{-1}$

$= \tau^2\sigma = \underline{\underline{\sigma\tau^2}}$

$(\sigma\tau^2)\sigma(\sigma\tau^2)^{-1} = \sigma\tau^2\sigma(\tau^2)^{-1}\sigma^{-1} = \underbrace{\sigma\sigma}_{e}\sigma^{-1} = \sigma^{-1} = \sigma$

$(\sigma\tau)^3\sigma(\sigma\tau^3)^{-1} = \sigma\tau^3\sigma(\tau^3)^{-1}\sigma^{-1} = \sigma\tau^3\sigma\tau\sigma$

$= \sigma\tau^3\underbrace{\sigma\sigma}_{e}\tau^3 = \sigma\tau^6 = \underline{\underline{\sigma\tau^2}}$

$\therefore$ conjugacy class $\underline{\underline{\{\sigma, \sigma\tau^2\}}}$

(5) For $\sigma\tau$

$e(\sigma\tau)e^{-1} = \sigma\tau \qquad \sigma(\sigma\tau)\sigma^{-1} = \sigma^2\tau\sigma = \tau\sigma$

$= \underline{\underline{\sigma\tau^3}}$

$\tau(\sigma\tau)\tau^{-1} = \tau\sigma = \underline{\underline{\sigma\tau^3}}$

$\tau^2(\sigma\tau)(\tau^2)^{-1} = \sigma\tau$

$\tau^3(\sigma\tau)(\tau^3)^{-1} = \cancel{\sigma^3\sigma\tau^2}\ \tau^3\sigma\tau^2 = \tau^3\tau^2\sigma = \tau\sigma = \underline{\underline{\sigma\tau^3}}$

$(\sigma\tau)(\sigma\tau)(\sigma\tau)^{-1} = \sigma\tau$

$(\sigma\tau^2)(\sigma\tau)(\sigma\tau^2)^{-1} = \sigma\tau^2\sigma\tau(\tau^2)^{-1}\sigma^{-1}$

$= \underbrace{\sigma\sigma}_{e}\tau\sigma = \tau\sigma = \underline{\underline{\sigma\tau^3}}$

$$(\sigma t^3)(\sigma t)(\sigma t^3)^{-1} = \sigma t^3 \, \sigma t \, (t^3)^{-1} \sigma^{-1}$$
$$= \sigma t^3 \, \sigma t \, t \sigma$$
$$= \sigma t \, \underbrace{t^2 \sigma t^2}_{\sigma} \sigma = \sigma t \underbrace{\sigma^2}_{e} = \sigma t$$

$\Rightarrow$ conjugacy class

$$\{\sigma t, \sigma t^3\}$$

So there are $\underline{5}$ ~~co~~ conjugacy classes.

b) let R_i be complex irreducible representations of D_4

~~ΣR_i~~ $\quad \Sigma |\dim(R_i)|^2 = |D_4| = 8$

$\because$ there are 5 conjugacy classes,

$\therefore$ there are 5 complex irreducible representations.

$\dim(R_i)$ are all positive integers.

the only way to satisfy the above conditions is

$\dim(R_i) = \underline{1, 1, 1, 1, 2}$ for 5 representations.

Let's define R_1 to be the trivial representation.

R_2 to be the representation with $\dim(R_2) = 2$

the other 3 to be R_i, R_j, R_k and

$\dim(R_1) = \dim(R_i) = \dim(R_j) = \dim(R_k) = \underline{1}$

The multiplication table of D_4.

	e	t	t^2	t^3	σ	σt	σt^2	σt^3
e	e	t	t^2	t^3	σ	σt	σt^2	σt^3
t	t	t^2	t^3	e	σt^3	σ	σt	σt^2
t^2	t^2	t^3	e	t	σt^2	σt^3	σ	σt
t^3	t^3	e	t	t^2	σt	σt^2	σt^3	σ
σ	σ	σt	σt^2	σt^3	e	t	t^2	t^3
σt	σt	σt^2	σt^3	σ	t^3	e	t	t^2
σt^2	σt^2	σt^3	σ	σt	t^2	t^3	e	t
σt^3	σt^3	σ	σt	σt^2	t	t^2	t^3	e

This table shows that D_4 is indeed a group of order 8, if argument in a) is not satisfying

- The trivial representation R_1 has $\dim(R_1) = 1$ and $R_1(g) = 1 \quad \forall g \in D_4$
- For other 3 1-Dimensional representations R_i, R_k, R_j

→ we still have $R_i(e) = R_j(e) = R_k(e) = 1$ by definition of identity and representation.

→ Observe that

abbreviate $R = R_i$ or R_j or R_k

$$R_{i,j,k}(t^2) = R_{i,j,k}(\sigma \cdot \sigma t^2) = R(\sigma \cdot \sigma t^2).$$

$$= R(\sigma) R(\sigma) R(t) R(t) = \cancel{R(\sigma) R(\sigma)}$$

by def. of rep.

$= R(\sigma) R(t) R(\sigma) R(t)$

$= R(\sigma t \cdot \sigma t) = R(e) = 1$

$\Rightarrow R(t^2) = 1$ for $R = R_i, R_j, R_k$.

then $R(t^3) = R(t) \underbrace{R(t^2)}_{1} = R(t)$

$R(\sigma t^2) = R(\sigma) \underbrace{R(t^2)}_{1} = R(\sigma)$

and $R(\sigma t^3) = R(\sigma t \cdot t^2) = R(\sigma t) \underbrace{R(t^2)}_{1} = R(\sigma t)$

Also observe that

$R(\sigma t) = R(\sigma) R(t)$ $\boxed{1}$

$R(\sigma) = R(\sigma t^2) = R(\sigma t) R(t)$ $\boxed{2}$

$R(t) = \underbrace{R(\sigma^2 t)}_{\sigma^2 = e} = R(\sigma) R(\sigma t)$ $\boxed{3}$

And also $R(t^2) = 1 = R(t) R(t) = R(t)^2$

$\therefore R(t)^2 = 1 \Rightarrow R(t) = \pm 1$ $\boxed{4}$

$\rightarrow$ $\boxed{4}$: if $R(t) = 1$, then $\because$ $\boxed{1}, \boxed{2}, \boxed{3}$

$R(\sigma) = R(\sigma t)$, $R(\sigma) R(\sigma t) = 1$

~~∴ R~~ $\because$ R is not the trivial representation

$\therefore$ we ~~can~~ cannot have $R(\sigma) = R(t) = R(\sigma t) = 1$

Hence $R(\sigma) = R(\sigma t) = -1$

$R(t) = 1$

$\rightarrow$ $\boxed{4}$: If $R(t) = -1$, $\boxed{1}, \boxed{2}, \boxed{3}$ then gives

~~then~~ $R(\sigma t) = -R(\sigma)$

and $R(\sigma)\, R(\sigma t) = -1$

this gives $R(\sigma) = 1$, $R(\sigma t) = -1$

or $R(\sigma) = -1$, $R(\sigma t) = +1$

$\Rightarrow$ overall, ~~we~~ without loss of generality we can assign

$\rightarrow$ $R_i(1, t^2) = 1$ $\quad R_i(t, t^3) = 1$

$R_i(\sigma, \sigma t^2) = -1$ $\quad R_i(\sigma t, \sigma t^3) = -1$

$\rightarrow$ $R_j(1, t^2) = 1$ $\quad R_j(t, t^3) = -1$

$R_j(\sigma, \sigma t^2) = 1$ $\quad R_j(\sigma t, \sigma t^3) = -1$

$\rightarrow$ $R_k(1, t^2) = 1$ $\quad R_k(t, t^3) = -1$

$R_k(\sigma, \sigma t^2) = -1$ $\quad R_k(\sigma t, \sigma t^3) = 1$

$\Rightarrow$ The one left is the 2-dimensional representation, But this is precisely what was given by the question.

$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad t^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad t^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \sigma t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma t^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \sigma t^3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

the character table now looks like

character \ conjugate class	(e)	(t^2)	(t, t^3)	$(\sigma, \sigma t^2)$	$(\sigma t, \sigma t^3)$
χ_1	1	1	1	1	1
χ_i	1	1	1	-1	-1
χ_j	1	1	-1	1	-1
χ_k	1	1	-1	-1	1
χ_2	2	-2	0	0	0

(characters are simply obtained by taking the trace of representation matrices) ~~$\chi_a = tr(R_g$~~

$\chi_a(g) = tr(R_a(g))$

* Note $(\chi_q, \chi_p) = \delta_{qp}$

c)

$R_4 = R_2 \otimes R_2$, ~~the~~ c

~~χ_4~~ character $\chi_4(g) = \chi_{2\otimes 2}(g) = \chi_2(g) \times \chi_2(g)$

$= \chi_2(g)^2$

$\therefore$

	(e)	(t^2)	(t, t^3)	$(\sigma, \sigma t^2)$	$(\sigma t, \sigma t^3)$
χ_4	4	4	0	0	0

The multiplicity of ~~R~~ R_a in R_4 is given by

$$m_a = (\chi_a, \chi_4) = \frac{1}{|D_4|} \sum_{g \in D_4} \chi_a^*(g)\, \chi_4(g)$$

$\therefore m_1 = \frac{1}{8}(4+4+0+0+0) = \underline{1}$

$m_i = \frac{1}{8}(4+4+0+0+0) = 1$

$m_j = \frac{1}{8}(4+4+0+0+0) = \underline{1}$

$m_k = \frac{1}{8}(4+4+0+0+0) = \underline{1}$

$m_2 = \frac{1}{8}(2\times4 - 2\times4 + 0+0+0) = 0$

$$\therefore \quad \underline{\underline{R_2 \otimes R_2 = R_1 \oplus R_i \oplus R_j \oplus R_k}}$$