$\boxed{\text{Ziyan Li}}$

Groups and Representations

Problem set 3

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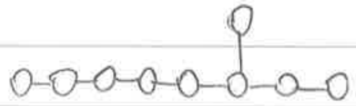
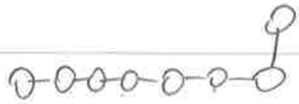
Wk 8, Fri 01-Dec 09:00-10:30

(1)

a) the extended Dynkin diagram for E_8 is

$E_8' =$ 

* Dynkin diagrams of :

$SO(16) =$  $\sim$ 

$\sim D_8$

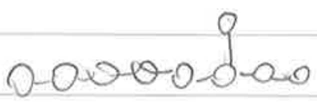
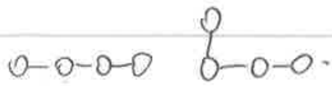
this is the last dot taken away from E_8'

$\therefore \quad E_8 \supset SO(16)$

* Dynkin diagrams of $SU(5)$

$SU(5) \sim A_{5-1} \sim A_4$: 

take the ~~upper~~ middle dot out from E_8' gives

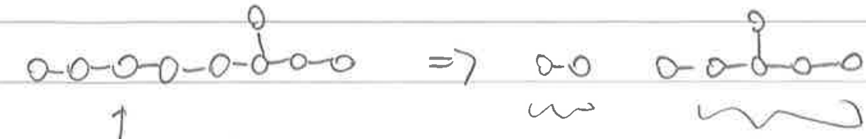
 $\Rightarrow$ 

$\Rightarrow$ 

$\sim A_4 \sim SO(5)$ $\quad$ $SU(5) \sim A_4$

$\Rightarrow \quad E_8 \supset SU(5) \times SU(5)$

* take out the third dot of E_8', we have



take out

$\sim A_2 \sim SU(3)$ $\qquad$ E_6

$$\Rightarrow \quad E_8 \supset SU(3) \times E_6$$

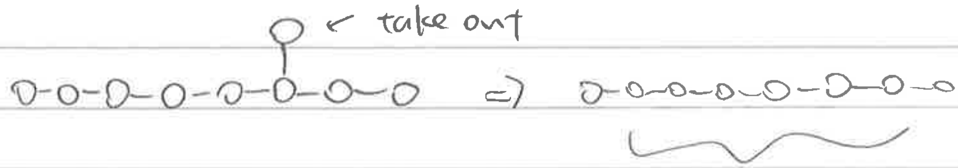
* similarly, take out the 2nd dot

take out

$\sim A_1 \sim SU(2)$ $\qquad$ E_7

$$E_8 \supset SU(2) \times E_7$$

* take ~~of~~ out the upper dot of E_8', gives

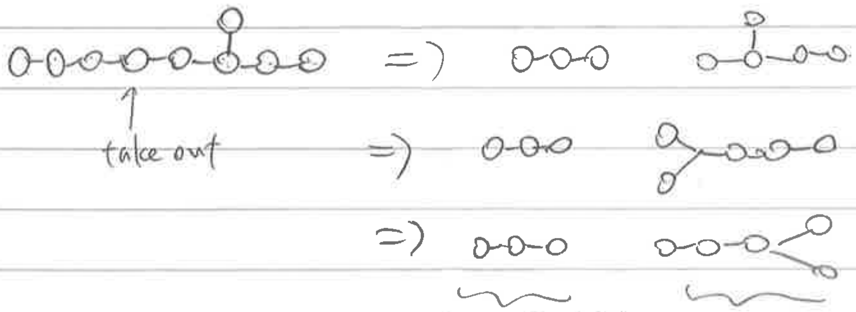


take out

$\sim A_8 \sim SU(9)$

$$\Rightarrow \quad E_8 \supset SU(9)$$

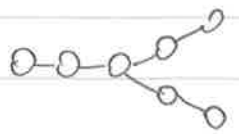
* take out the 4th dot of E_8', we



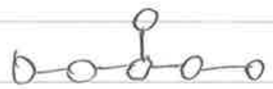
take out

$\sim A_3 \sim SU(4)$ $\qquad$ $\sim D_5 \sim SO(10)$

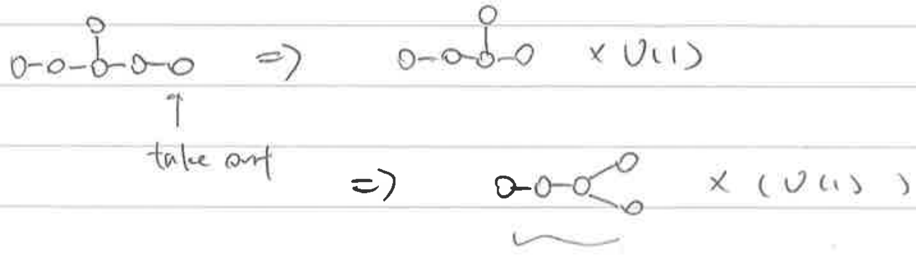
$$\Rightarrow \quad E_8 \supset SU(4) \times SO(10)$$

b) Consider the extended ~~Dynkind~~ Dynkin diagram of E_6 is $E_6' =$ 

and the Dynkin diagram of E_6 to be.

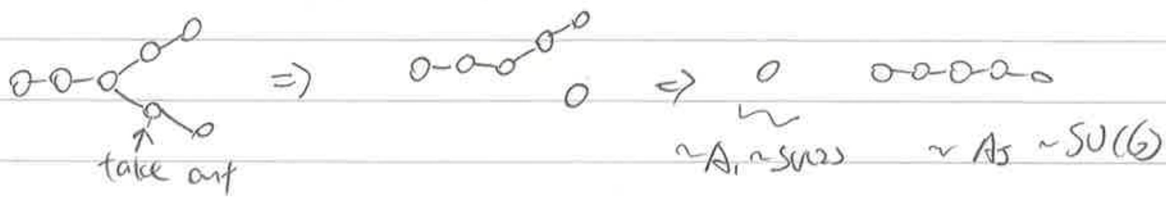


* take out the last dot from E_6, since we are doing this to E_6, we have to include a $U(1)$ factor. So



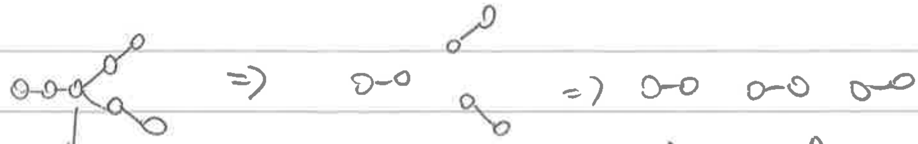
$\rightarrow$ $E_6 \supset SO(10) \times U(1)$

* take out ~~one~~ any one dot that is next to the ~~ed~~ end dots. from E_6'



$E_6 \supset SU(2) \times SU(6)$

* take out the middle dot



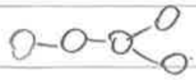
take out

$\sim A_2$ $\sim A_2$ $\sim A_2$

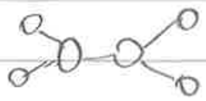
$\sim SU(3)$ $\sim SU(3)$ $\sim SU(3)$

$$E_6 \supset SU(3) \times SU(3) \times SU(3)$$

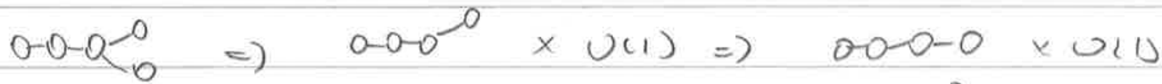
c) Consider Dynkin diagram of $SO(10)$, Which has algebra D_5

D_5 : 

and Extended Dynkin diagrams D_5'

D_5' 

* taking out one end dot of D_5 and include a $U(1)$

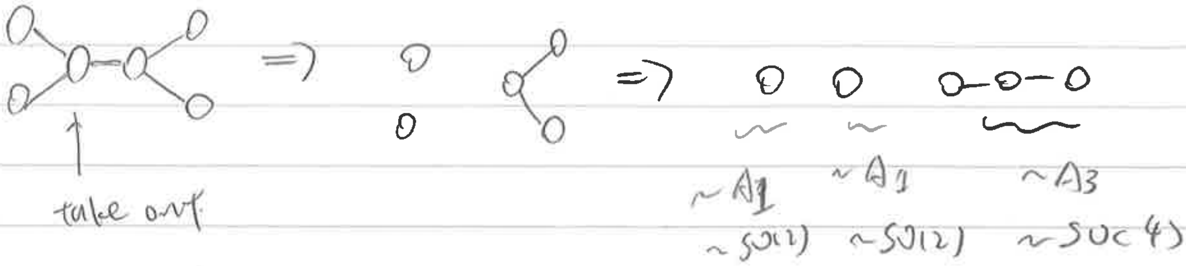


take out

$\sim A_4 \sim SU(5)$

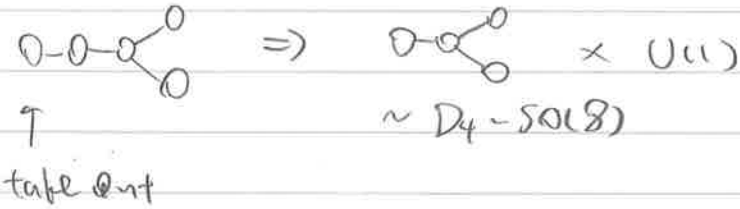
$$\Rightarrow \quad SO(10) \supset SU(5) \times U(1)$$

* taking out one of the middle dot from D_5'



$$\Rightarrow \quad SO(10) \supset SU(2) \times SU(2) \times \cancel{SO(4)}\ SU(4).$$

* take out the first dot from D_5 and include a U(1) factor :



$$\rightarrow \quad SO(10) \supset SO(8) \times U(1)$$

②

a) The Cartan matrix for $SU(5)$ is

(~~the~~ technically the complexification $SU(5)_{\mathbb{C}}$)

$$A(A_4) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

to work out the representation $\underset{\sim}{5}$'s weigh system. Note that

$\underset{\sim}{5}$ has dynkin label for highest weight to be

$M = (1\ 0\ 0\ 0)$

the Dynkin labels for the simple roots are the rows of cartan matrix

$$\alpha_1 = (2\ -1\ 0\ 0)$$
$$\alpha_2 = (-1\ 2\ -1\ 0)$$
$$\alpha_3 = (0\ -1\ 2\ -1)$$
$$\alpha_4 = (0\ 0\ -1\ 2)$$

If M is weight and a complete string of roots is $M+p\alpha, \cdots, M, \cdots, M-m\alpha$ then

$$m-p = \frac{2\langle M, \alpha\rangle}{\langle \alpha, \alpha\rangle} \quad \text{where}$$

$\langle \alpha, \beta \rangle = (h_\alpha, h_\beta) = \Gamma(ad(h_\alpha)\ ad(h_\beta))$.

then Given a weight M, and a simple root α_j, then

If $m_j = p_j + M_j > 0$ with $M_j = 2\frac{\langle M, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$

we then have $M - \alpha_j$ is also a root.

The procedure goes as follows.

~~$M = (1000)$~~

$\because$ $M = (1000)$ heighest weight

$\therefore$ $\boxed{1\ 0\ 0\ 0}$

For this, $p = (0, 0, 0, 0)$

the first pair $1 + 0 = 1 > 0$, all other pairs of M_j and p_j have $m_j + p_j = 0$

$\therefore$ $M - \alpha_1 = (1000) - (2\ {-1}\ 0\ 0)$

$= (-1\ 1\ 0\ 0)$ is a weight

$\boxed{-1\ 1\ 0\ 0}$

For this, $p = (1\ 0\ 0\ 0)$, 2^{nd} pair $1 + 0 > 0$

$\therefore$ $M - \alpha_1 - \alpha_2 = (-1\ 1\ 0\ 0) - (-1\ 2\ {-1}\ 0)$

$= (0\ {-1}\ 1\ 0)$

$\boxed{0 \ -1 \ 1 \ 0}$

~~p~~

For this, $\because$ $M-\alpha_1$ is weight, $M-\alpha_2$ is not weight $\therefore$ $M-\alpha_1-\alpha_2$ has

$p=(0\ 1\ 0\ 0)$, ~~non~~ third pair $1+0>0$.

$\therefore$ next weight ~~is~~ $M-\alpha_1-\alpha_2-\alpha_3$

$$= (0\ -1\ 1\ 0)-(0\ -1\ +2\ -1)$$

$$= (0\ 0\ -1\ 1)$$

$\boxed{0 \ 0 \ -1 \ 1}$

For this, $M-\alpha_1-\alpha_2$ is a weight but $M-\alpha_1-\alpha_3$, $M-\alpha_2-\alpha_3$ are not $\therefore$

$p=(0,0,1,0)$

Now the 4th pair $1+0>1$. next weight is

$$M-\alpha_1-\alpha_2-\alpha_3-\alpha_4 = (0\ 0\ -1\ 1)-(0\ 0\ -1\ 2)$$

$$= (0\ 0\ 0\ -1)$$

$\boxed{0 \ 0 \ 0 \ -1}$

Now for this only $M-\alpha_1-\alpha_2-\alpha_3$ is a weight if we remove one of the α's

$\therefore \quad p = (0\ 0\ 0\ 1)$

Now $0+0=0$, $0+0=0$, $0+0=0$, $1+(-1)=0$

None of $M_j + p_j > 0$

So process terminates.

the boxed $\boxed{}$ numbers are weight systems (Dynkin labels) of $\underset{\sim}{5}$ of $SU(5)$

- Now for $\underset{\sim}{\bar{5}} \sim (0\ 0\ 0\ 1)$, the Dynkin label of highest weight.

$M = \boxed{0\ 0\ 0\ 1}$

For this $p = (0\ 0\ 0\ 0)$.

the 4^{th} pair $1+0>0$, next weight is

$M - \alpha_4 = (0001) - (0\ 0\ {-1}\ 2)$

$= (0\ 0\ 1\ {-1})$

$\boxed{0\ 0\ 1\ {-1}}$

For this $p = (0\ 0\ 0\ 1)$

Only the 3rd pair $1+0 > 1$, next weight.

$M - \alpha_4 - \alpha_3 = (0\ 0\ 1\ -1) - (0\ -1\ 2\ -1)$

$= (0\ 1\ -1\ 0)$

$\boxed{0 \quad 1 \quad -1 \quad 0}$

For this, $M - \alpha_4$ is weight but $M - \alpha_3$ isn't

$\therefore\ p = (0\ 0\ 1\ 0)$

the 2^{nd} pair $1+0 > 0$

$\therefore$ next weight $M - \alpha_4 - \alpha_3 - \alpha_2 = (0\ 1\ -1\ 0) - (-1\ 2\ -1\ 0)$

$= (1\ -1\ 0\ 0)$

$\boxed{1 \quad -1 \quad 0 \quad 0}$

For this $p = (0\ 1\ 0\ 0)$, 1^{st} pair $1+0 > 0$

$\therefore$ next weight $M - \alpha_4 - \alpha_3 - \alpha_2 - \alpha_1 = (1\ -1\ 0\ 0) - (2\ -1\ 0\ 0)$

$= (-1\ 0\ 0\ 0)$

$\therefore\ \boxed{-1 \quad 0 \quad 0 \quad 0}$

For this $p = (1\ 0\ 0\ 0)$

Now no pair has $m + p > 0$

$\therefore$ terminates

the numbers in boxes $\boxed{}$ are weights of representation $\bar{\underset{\sim}{5}}$ of $SU(5)$.

b)

Now for $\underset{\sim}{10} \sim (0\,1\,0\,0)$ of $SU(5)$

~~Start~~ $M = \boxed{0\ 1\ 0\ 0}$

$p = (0\,0\,0\,0)$

same procedure :

$\downarrow$

2nd $1+0>1$

$\downarrow$

$M - \alpha_2 = (0\,1\,0\,0) - (-1\ 2\ -1\ 0)$

$= (1\ -1\ 1\ 0)$

$\boxed{1\ -1\ 1\ 0}$

$\downarrow$ $p = (0\,1\,0\,0)$

$\Downarrow$

1st $1+0>1$ & 3rd $1+0>1$.

$\Downarrow$ $\Downarrow$

$M - \alpha_2 - \alpha_1$ $M - \alpha_2 - \alpha_3$

$\boxed{-1\ 0\ 1\ 0}$ $\boxed{1\ 0\ -1\ 1}$

$\Downarrow$ $\Downarrow$

⇓

$p = (1\ 0\ 0\ 0)$

only 3rd $|+0>1$

$M - \alpha_2 - \alpha_1 - \alpha_3$

equal

$\boxed{-1\ 1\ -1\ 1}$

$p = (1\ 0\ 1\ 0)$

⇓

2nd & 4th.

$M - 2\alpha_2 - \alpha_1 - \alpha_3$

$\boxed{0\ -1\ 0\ 1}$

$p = (0\ 1\ 0\ 0)$

4th. ↓

↓

$M - 2\alpha_2 - \alpha_1 - \alpha_3 - \alpha_4$

⇓

$p = (0\ 0\ 1\ 0)$

~~first~~ 1st $|+0>1$

$M - \alpha_2 - \alpha_3 - \alpha_1$

$\cancel{\boxed{-1\ 1\ -1\ 1}}$

4th $|+0>1$.

$M - \alpha_2 - \alpha_3 - \alpha_4$

$\boxed{1\ 0\ 0\ -1}$

$p = (0\ 0\ 0\ 1)$.

1st.

$M - \alpha_2 - \alpha_1 - \alpha_3 - \alpha_4$

$\boxed{-1\ 1\ 0\ -1}$

$p = (1\ 0\ 0\ 1)$

2nd

$M - 2\alpha_2 - \alpha_1 - \alpha_3 - \alpha_4$.

$$\boxed{0 \quad -1 \quad 1 \quad -1}$$

$p = (0\ 1\ 0\ 1)$

3^{rd}

$\downarrow$

$M - 2\alpha_2 - \alpha_1 - 2\alpha_3 - \alpha_4.$

$$\boxed{0 \quad 0 \quad -1 \quad 0}$$

$p = (0\ 0\ 1\ 0)$

No $p + M > 0$, terminates.

— Now, the $\underset{\sim}{\text{IS}}$ $\sim (2\ 0\ 0\ 0)$

$M = \boxed{2\ 0\ 0\ 0}$

$p = (0\ 0\ 0\ 0)$

1^{st} $2 + 0 > 0$

$\Downarrow$

$M - \alpha_1 = \boxed{0 \quad 1 \quad 0 \quad 0}$

$p = (1\ 0\ 0\ 0)$

1^{st} and 2^{nd}.

$\Downarrow$ $\Downarrow$

$M-2\alpha_1$ $\quad$ $M-\alpha_1-\alpha_2$.

$\boxed{-2\ 2\ 0\ 0}$ $\quad$ $\boxed{1\ -1\ 1\ 0}$

$p=(2\ 0\ 0\ 0)$. $\quad$ $P=(0\ 1\ 0\ 0)$

2^{nd} $\quad$ 1^{st} & 3^{rd}.

$\Downarrow$ $\Downarrow$ $\Downarrow$.

$M-2\alpha_1-\alpha_2$ $\quad$ $M-2\alpha_1-\alpha_2$ $\quad$ $M-\alpha_1-\alpha_2-\alpha_3$

$\boxed{-1\ 0\ 1\ 0}$ $\quad$ $\boxed{1\ 0\ -1\ 1}$

$p=(1\ 1\ 0\ 0)$ $\quad$ $P=(0\ 0\ 1\ 0)$

2^{nd} & 3^{rd} $\quad$ 1^{st} & 4^{th}.

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

$M-2\alpha_1-2\alpha_2$ $\quad$ $M-2\alpha_1-\alpha_2-\alpha_3$ $\quad$ $M-2\alpha_1-\alpha_2-\alpha_3$

$M-\alpha_1-\alpha_2-\alpha_3-\alpha_4$

$\boxed{0\ -2\ 2\ 0}$ $\quad$ $\boxed{-1\ 1\ -1\ 1}$ $\quad$ $\boxed{1\ 0\ 0\ -1}$

$p=(0\ 2\ 0\ 0)$ $\quad$ $p=(1\ 0\ 1\ 0)$ $\quad$ $p=(0\ 0\ 0\ 1)$

3rd 2nd & 4th 1st.

$M-2\alpha_1-2\alpha_2-\alpha_3$

$M-2\alpha_1-\alpha_2-\alpha_3-\alpha_4$

$M-2\alpha_1-2\alpha_2-\alpha_3$

$M-2\alpha_1-\alpha_2-\alpha_3-\alpha_4$

$\boxed{0\ -1\ 0\ 1}$

$\boxed{-1\ 1\ 0\ -1}$

$p=(0\ 1\ 1\ 0).$

$p=(1\ 0\ 0\ 1).$

3rd 4th

2nd

~~$M-$~~

$M-2\alpha_1-2\alpha_2-2\alpha_3$

$M-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4$

$\boxed{0\ 0\ -2\ 2}$

$\boxed{0\ -1\ 1\ -1}$

$p=(0\ 0\ 1\ 0)$

$p=(0\ 1\ 0\ 1)$

~~3rd~~ 4th.

3rd.

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow$

$M-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$ $\qquad\qquad$ $M-2\alpha_1-2\alpha_2-2\alpha_3-\alpha_4$

$\searrow \qquad \swarrow$

$$\boxed{0 \;\; 0 \;\; -1 \;\; 0}$$

$p=(0\ 0\ 1\ 1)$

4^{th}

$\downarrow$ $\qquad$ $M-2\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$

$$\boxed{0 \;\; 0 \;\; 0 \;\; -2}$$

$p=(0\ 0\ 0\ 1)$

$\rightarrow$ terminates.

The weight systems of $\underset{\sim}{10}$ and $\underset{\sim}{5}$, represented in Dynkin labels, are boxed in $\boxed{}$

c) the weights of tenser product of representations of the same semi-simple Lie ~~gr alge group~~ algebra is the sum of the respective weights of the representations.

consider $\underset{\sim}{5} \otimes \underset{\sim}{5}$

$\underset{\sim}{5}$ has

$\boxed{1\ 0\ 0\ 0}$

$\boxed{-1\ 1\ 0\ 0}$

$\boxed{0\ -1\ 1\ 0}$

$\boxed{0\ 0\ -1\ 1}$

$\boxed{0\ 0\ 0\ -1}$

The possible sums ~~are~~ between these 5 weights (including one sums itself) are

$\boxed{2\ 0\ 0\ 0} \to \underset{\sim}{15}$ | $\boxed{0\ 1\ 0\ 0} \to \underset{\sim}{10},\ \underset{\sim}{15}$ | $\boxed{1\ -1\ 1\ 0} \to \underset{\sim}{10},\ \underset{\sim}{15}$ | $\boxed{1\ 0\ -1\ 1} \to \underset{\sim}{10},\ \underset{\sim}{15}$ | $\boxed{1\ 0\ 0\ -1} \to \underset{\sim}{10},\ \underset{\sim}{15}$

$\boxed{-2\ 2\ 0\ 0} \to \underset{\sim}{15}$ | $\boxed{-1\ 0\ 1\ 0} \to \underset{\sim}{10},\ \underset{\sim}{15}$ | $\boxed{-1\ 1\ -1\ 1} \to \underset{\sim}{10},\ \underset{\sim}{15}$ | $\boxed{-1\ 1\ 0\ -1} \to \underset{\sim}{10},\ \underset{\sim}{15}$

$\boxed{0\ -2\ 2\ 0} \to \underset{\sim}{15}$ | $\boxed{0\ -1\ 0\ 1} \to \underset{\sim}{10},\ \underset{\sim}{15}$ | $\boxed{0\ -1\ 1\ -1} \to \underset{\sim}{10},\ \underset{\sim}{15}$

$\boxed{0\ 0\ -2\ 2} \to \underset{\sim}{15}$ | $\boxed{0\ 0\ -1\ 0} \to \underset{\sim}{10},\ \underset{\sim}{15}$

$\boxed{0\ 0\ 0\ -2} \to \underset{\sim}{15}$

$\sim \underset{\sim}{15}$

$\Rightarrow$ the chart for $\underset{\sim}{5} \otimes \underset{\sim}{5}$ $\sim \underset{\sim}{10}\ \&\ \underset{\sim}{15}$

we see the chart above consists of all weights of $\underset{\sim}{15}$ and $\underset{\sim}{10}$, but no weight that belongs to neither $\underset{\sim}{15}$ or $\underset{\sim}{10}$

$\rightarrow$ this chart also ~~re~~ represents $\underset{\sim}{10} \oplus \underset{\sim}{15}$.

Hence $\rightarrow$ $\underset{\sim}{5} \otimes \underset{\sim}{5} = \underset{\sim}{10} \oplus \underset{\sim}{15}$

(3) the projection matrix

$$P = P(SU(5) \supset SU(2) \times SU(3)).$$

$$= \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

a) the representation $\underset{\sim}{5}$ has weights

$$(1000) \to P(1000) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(-1100) \to P(-1100) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$(0-110) \to P(0-110) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$(00-11) \to P(00-11) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

$$(000-1) \to P(000-1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}. = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Now the resulting weights can be grouped into

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \quad \Bigg| \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

It is known that $(1,0)$ $(-1,1)$ $(0,-1)$ are the weights of representation $\underset{\sim}{3}$ of $SU(3)$

~~(0,0) is the trivi~~

(0) is the trivial representation $\underset{\sim}{1}$ of $SU(2)$

And,

(1) (-1) are the weights of representation $\underset{\sim}{2}$ of $SU(2)$

$(0,0)$ is the trivial representation $\underset{\sim}{1}$ of $SU(3)$

$\Rightarrow$ Branching of $\underset{\sim}{5}$ of $SU(5)$ is

$$\underset{SU(5)}{\underset{\downarrow}{\underset{\sim}{5}}} = (\underset{SU(3)}{\underset{\downarrow}{\underset{\sim}{3}}}, \underset{SU(2)}{\underset{\downarrow}{\underset{\sim}{1}}}) \oplus (\underset{SU(3)}{\underset{\downarrow}{\underset{\sim}{1}}}, \underset{SU(2)}{\underset{\downarrow}{\underset{\sim}{2}}})$$

- For $\bar{\underset{\sim}{5}}$ the weights are.

(0001), $(001{-1})$, $(01{-1}0)$, $(1{-1}00)$, $(-1,0,0,0)$

projections:

$$P(0001) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$P(0\,0\,1\,{-1}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$P(0\,1\,{-1}\,0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

$$P(1\,{-1}\,0\,0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

$$P(-1\,0\,0\,0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

separate into 2 groups:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}. \quad \Bigg| \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

And observe that $(0\,1)$ $(1\,{-1})$ $(-1\,0)$ are the weight system of representation $\underset{\sim}{\bar{3}}$ of $SU(3)$

that (0) is the weight system of representation $\underset{\sim}{1}$ of $SU(2)$

And

that (1) (-1) are the weight system of representation $\underset{\sim}{2}$ of $SU(2)$

and that $(0,0)$ is the weight system of representation $\underset{\sim}{1}$ of $SU(3)$.

Hence the branching of $\bar{\underset{\sim}{5}}$ of $SU(5)$ under $SU(2) \times SU(3)$ is

$$\underset{SU(5)}{\underset{\downarrow}{\bar{\underset{\sim}{5}}}} = (\underset{SU(3)}{\underset{\downarrow}{\bar{\underset{\sim}{3}}}}, \underset{SU(2)}{\underset{\downarrow}{\underset{\sim}{1}}}) + (\underset{SU(3)}{\underset{\downarrow}{\underset{\sim}{1}}}, \underset{SU(2)}{\underset{\downarrow}{\underset{\sim}{2}}})$$

Now, the $\underset{\sim}{10}$ representation of $SU(5)$ has weights

$(0\ 1\ 0\ 0)$, $(1\ -1\ 1\ 0)$, $(1\ 0\ -1\ 1)$, $(1\ 0\ 0\ -1)$,
$(-1\ 0\ 1\ 0)$, $(-1\ 1\ -1\ 1)$, $(-1\ 1\ 0\ -1)$,
$(0\ -1\ 0\ +1)$, $(0\ -1\ 1\ -1)$,
$(0\ 0\ -1\ 0)$,

projections into $SU(2) \times SU(3)$:

$$P(0\,1\,0\,0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$P(1\,-1\,1\,0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$P(1\,0\,-1\,1) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad P(1\,0\,0\,-1) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

$$P(-1\,0\,1\,0) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad P(-1\,1\,-1\,1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$P(-1\ 1\ 0\ -1) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$P(0\ -1\ 0\ +1) = \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix}.$$

$$P(0\ -1\ 1\ -1) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

$$P(0\ 0\ -1\ 0) = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

Now, separate into 3 groups

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \boxed{1}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \boxed{2}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}. \quad \boxed{3}$$

observe that :

in $\boxed{1}$ (0) is $\underset{\sim}{1}$ of $SU(2)$

$(0\ 0)$ is $\underset{\sim}{1}$ of $SU(3)$

$\boxed{1} \to (\underset{\sim}{1}, \underline{1})$

in $\boxed{2}$ (0) is $\underset{\sim}{1}$ of $SU(2)$

$(0\ 1)$ $(1\ -1)$ $(-1\ 0)$ is $\underset{\sim}{\bar{3}}$ of $SU(3)$

$\boxed{2} \to (\underset{\sim}{\bar{3}}, \underset{\sim}{1})$

in $\boxed{3}$ (1) (-1) is $\underset{\sim}{2}$ of $SU(2)$

$(1\ 0)$ $(-1\ 1)$ $(0\ -1)$ is $\underset{\sim}{3}$ of $SU(3)$.

$\boxed{3} \to (\underset{\sim}{3}, \underset{\sim}{2})$

Hence $\because \boxed{1}, \boxed{2}, \boxed{3}$

$$\therefore \underset{\sim}{10} = (\underset{\sim}{\bar{3}}, \underset{\sim}{1}) + (\underset{\sim}{3}, \underset{\sim}{2}) + (\underset{\sim}{1}, \underset{\sim}{1})$$

$\downarrow$ $SU(5)$; $\downarrow$ $SU(3)$, $\downarrow$ $SU(2)$; $\downarrow$ $SU(3)$, $\downarrow$ $SU(2)$; $\downarrow$ $SU(3)$, $\downarrow$ $SU(2)$

b) the dual vector $Y = \frac{1}{3}(-2, 1, -1, 2)$.

– For $\underset{\sim}{5}$, weights are $(1\,0\,0\,0)$ $(-1\,1\,0\,0)$ $(0\,-1\,1\,0)$ $(0\,0\,-1\,1)$ $(0\,0\,0\,-1)$

$$Y(1\,0\,0\,0) = \frac{1}{3}(-2\ 1\ -1\ 2)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = -\frac{2}{3}$$

$$Y(-1\,1\,0\,0) = \frac{1}{3}(-2\ 1\ -1\ 2)\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} = 1$$

$$Y(0\,-1\,1\,0) = \frac{1}{3}(-2\ 1\ -1\ 2)\begin{pmatrix}0\\-1\\1\\0\end{pmatrix} = -\frac{2}{3}$$

$$Y(0\,0\,-1\,1) = \frac{1}{3}(-2\ 1\ -1\ 2)\begin{pmatrix}0\\0\\-1\\1\end{pmatrix} = 1$$

$$Y(0\,0\,0\,-1) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}0\\0\\0\\-1\end{pmatrix} = -\frac{2}{3}$$

From a) we also know that $(1\,0\,0\,0)$, $(0\,-1\,1\,0)$ and $(0\,0\,0\,-1)$ are what was projected to

$$(su(2), su(3)) = (\underset{\sim}{1}, \underset{\sim}{3})$$

and $(-1\,1\,0\,0)$ and $(0\,0\,-1\,1)$ are projected to $(\underset{\sim}{2}, \underset{\sim}{1})$

Hence $U_Y(1)$ charge for $(\underset{\sim}{1}, \underset{\sim}{3})$ is $\underline{\underline{-\frac{2}{3}}}$

and for $(\underset{\sim}{2}, \underset{\sim}{1})$ is $\underline{\underline{1}}$

- The $\bar{\underset{\sim}{5}}$ representation:

$$Y(0\,0\,0\,1) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \frac{2}{3}$$

$$Y(0\,0\,1\,-1) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}0\\0\\1\\-1\end{pmatrix} = -1$$

$$Y(0\,1\,-1\,0) = \cancel{\frac{1}{3}(0\;1\;-1\;0)}\;\frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} = \frac{2}{3}$$

$$Y(1\,-1\,0\,0) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = -1$$

$$Y(-1\,0\,0\,0) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}-1\\0\\0\\0\end{pmatrix} = \frac{2}{3}$$

From a), we observe $(0\,0\,0\,1)$, $(0\,1\,-1\,0)$, $(-1\,0\,0\,0)$ are projected to $(SU(2), SU(3)) = (\mathbf{1}, \bar{\mathbf{3}})$

$\therefore$ $U_Y(1)$ charge for $(\mathbf{1}, \bar{\mathbf{3}})$ is $\underline{\underline{\frac{2}{3}}}$

that $(0\,0\,1\,-1)$ and $(1\,-1\,0\,0)$ are projected to $(\mathbf{2}, \mathbf{1})$

$\therefore$ $U_Y(1)$ charge for $(\mathbf{2}, \mathbf{1})$ is $\underline{\underline{-1}}$

– The $\mathbf{10}$ representation

$Y(0\,1\,0\,0) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \frac{1}{3}$ for $(\mathbf{2}, \mathbf{3})$

$Y(1\,-1\,1\,0) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}1\\-1\\1\\0\end{pmatrix} = -\frac{4}{3}$ for $(\mathbf{1}, \bar{\mathbf{3}})$

$Y(1\,0\,-1\,1) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}1\\0\\-1\\1\end{pmatrix} = \frac{1}{3}$ for $(\mathbf{2}, \mathbf{3})$

$Y(1\,0\,0\,-1) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = -\frac{4}{3}$ for $(\mathbf{1}, \bar{\mathbf{3}})$

$Y(-1\,0\,1\,0) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} = \frac{1}{3}$ for $(\mathbf{2}, \mathbf{3})$

$Y(-1\,1\,-1\,1) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix} = 2$ for $(\mathbf{1}, \mathbf{1})$

$Y(-1\,1\,0\,-1) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}-1\\1\\0\\-1\end{pmatrix} = \frac{1}{3}$ for $(\mathbf{2}, \mathbf{3})$

$Y(0\,-1\,0\,1) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}0\\-1\\0\\1\end{pmatrix} = \frac{1}{3}$ for $\cancel{(\mathbf{1}, \bar{\mathbf{3}})}$ $(\mathbf{2}, \mathbf{3})$

$Y(0\,-1\,1\,-1) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}0\\-1\\1\\-1\end{pmatrix} = -\frac{4}{3}$ for $(\mathbf{1}, \bar{\mathbf{3}})$

$Y(0\,0\,-1\,0) = \frac{1}{3}(-2\;1\;-1\;2)\begin{pmatrix}0\\0\\-1\\0\end{pmatrix} = \frac{1}{3}$ for $(\mathbf{2}, \mathbf{3})$

Hence we have

$(\underset{\sim}{2}, \underset{\sim}{3}) \to U_Y(1)$ charge $= \frac{1}{3}$ //
$(\underset{\sim}{1}, \underset{\sim}{\bar{3}}) \to U_Y(1)$ charge $= -\frac{4}{3}$ //
$(\underset{\sim}{1}, \underset{\sim}{1}) \to U_Y(1)$ charge $= 2$ //

c) This representation,

$$(\underset{\sim}{2}, \underset{\sim}{3})_{1/3} \oplus (\underset{\sim}{1}, \underset{\sim}{\bar{3}})_{2/3} \oplus (\underset{\sim}{1}, \underset{\sim}{\bar{3}})_{-4/3} \oplus (\underset{\sim}{2}, \underset{\sim}{1})_{-1} \oplus (\underset{\sim}{1}, \underset{\sim}{1})_{2}$$

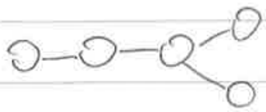
$$= \underbrace{\left[(\underset{\sim}{2}, \underset{\sim}{3})_{\frac{1}{3}} \oplus (\underset{\sim}{1}, \underset{\sim}{\bar{3}})_{-\frac{4}{3}} \oplus (\underset{\sim}{1}, \underset{\sim}{1})_{2} \right]}_{\underset{\sim}{10}} \oplus \underbrace{\left[(\underset{\sim}{1}, \underset{\sim}{\bar{3}})_{\frac{2}{3}} \oplus (\underset{\sim}{2}, \underset{\sim}{1})_{-1} \right]}_{\underset{\sim}{\bar{5}}}$$

$$= \underline{\underline{\underset{\sim}{10} \oplus \underset{\sim}{\bar{5}}}} \quad \text{of } SU(5)$$

can accommodate one standard model family.

④ The Dynkin diagram of $SO(10)$ is

D_5 ~

 and this gives a

Cartan matrix to be

$$\begin{array}{c} \left(\begin{array}{ccccc} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{array}\right) \end{array} \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{array} \Bigg\} \text{ simple roots.}$$

a) the representation with highest weight $M = (0\,0\,0\,0\,1)$

① ~~initial~~ $\boxed{0\,0\,0\,0\,1}$

$p = (0\,0\,0\,0\,0)$

5^{th}. $1+0>1$

$\downarrow$

$M - \alpha_5 = \boxed{0\;0\;1\;0\;-1}$

$p = (0\,0\,0\,0\,1)$

~~4th~~ 3^{rd}

$M - \alpha_5 - \alpha_3 = \boxed{0\;1\;-1\;1\;0}$

$p = (0\,0\,1\,0\,0)$

2nd

$M-\alpha_2-\alpha_3-\alpha_5$

$\boxed{1\ -1\ 0\ 1\ 0}$

$p=(0\,1\,0\,0\,0)$

4th.

$M-\alpha_3-\alpha_4-\alpha_5$

$\boxed{0\ 1\ 0\ -1\ 0}$

$p=(0\,0\,0\,1\,0)$

1st 4th

$M-\alpha_1-\alpha_2-\alpha_3-\alpha_5$

$\boxed{-1\ 0\ 0\ 1\ 0}$

$p=(1\,0\,0\,0\,0)$

2nd.

$M-\alpha_2-\alpha_3-\alpha_4-\alpha_5$

$\boxed{1\ -1\ 1\ -1\ 0}$

$p=(0\,1\,0\,1\,0)$

4th.

$M-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5$

$\boxed{-1\ 0\ 1\ -1\ 0}$

$P=(1\,0\,0\,1\,0)$

1st 3rd.

$M-\alpha_2-2\alpha_3-\alpha_4-\alpha_5.$

$\boxed{1\ 0\ -1\ 0\ 1}$

$P=(0\,0\,1\,0\,0)$

3rd

$M-\alpha_1-\alpha_2-2\alpha_3-\alpha_4-\alpha_5$

$\boxed{-1\ 1\ -1\ 0\ 1}$

$p=(1\,0\,1\,0\,0)$

1st 5th.

$M-\alpha_2-2\alpha_3-\alpha_4-2\alpha_5$

$\boxed{1\ 0\ 0\ 0\ -1}$

$P=(0\,0\,0\,0\,1)$

2nd $\quad$ 5th $\quad$ 1st.

$\downarrow$

$M-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5$

$\downarrow$

$\boxed{0 \; -1 \; 0 \; 0 \; 1}$

$p=(0\;1\;0\;0\;0)$

$M-\alpha_1-\alpha_2-2\alpha_3-\alpha_4-2\alpha_5.$

$\downarrow$

$\boxed{-1 \; 1 \; 0 \; 0 \; -1}$

$p=(1\;0\;0\;0\;1)$

5th. $\quad$ 2nd.

$M-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4-2\alpha_5.$

$\downarrow$

$\boxed{0 \; -1 \; 1 \; 0 \; -1}$

$p=(0\;1\;0\;0\;1)$

3rd

$\downarrow$

$M-\alpha_1-2\alpha_2-3\alpha_3-\alpha_4-2\alpha_5$

$\downarrow$

$\boxed{0 \; 0 \; -1 \; 1 \; 0}$

$p=(0\;0\;1\;0\;0)$

4th

$\downarrow$

$M-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4-2\alpha_5$

$\downarrow$

$\boxed{0 \; 0 \; 0 \; -1 \; 0}$

$$\downarrow$$

$$P = (0\ 0\ 0\ 1\ 0)$$

No pair of $P + M > 0 \Rightarrow$ terminates.

The boxed $\boxed{}$ numbers are the weights required.

- The Weyl's dimensional formula for given ~~pos~~ representation is:

The dimension is a product of factors, one for each positive root of the algebra. Each factor has a denominator which is the number of simple roots which compose the positive root. The numerator is a sum over the simple roots in the positive root, with each simple root ~~to~~ contributing unity plus the value of the Dynkin coefficient corresponding to the simple root. If the simple roots are not all the same size, each contribution to the numerator and to the denominator must be weighted by $\langle \alpha_i, \alpha_i \rangle$ (Cahn pp. 112).

For this D_5 algebra, all roots same size.

We label $(1^{k_1} 2^{k_2} 3^{k_3} 4^{k_4} 5^{k_5}) = \sum_{i=1}^{5} k_i \alpha_i$ a ~~sim~~

~~as~~ positive root ~~that consists of~~ as a sum of simple roots $\alpha_1, \cdots, \alpha_5$. $(k_i \geq 0 \in \mathbb{Z})$

There are 20 positive roots for D_5 $(SO(10))$

namely,

(1) (2) (3) (4) (5) (12) (23) (34) (35) (123) (234)
(235) (345) (1234) (2345) (1235) (12345) $(23^2 45)$
$(123^2 45)$ $(12^2 3^2 45)$

~~Dyn~~ Dynkin label of highest weight is (00001)

By Weyl's dimension formula, only the positive roots including α_5 will give a contribution differ ~~by~~ from 1 because all other positions in Dynkin label are zero except the 5th.

$(5) \to \left(\frac{1+1}{1}\right) = 2.$

$(35) \to \left(\frac{1+2}{2}\right) = \frac{3}{2}$

$(235) \to \left(\frac{1+3}{3}\right) = \frac{4}{3}$, $(345) \to \frac{4}{3}.$

$(2345) \to \left(\frac{1+4}{4}\right) = \frac{5}{4}$, $(1235) \to \frac{5}{4}.$

$(12345) \to \left(\frac{1+5}{5}\right) = \frac{6}{5}$, $(23^2 45) \to \left(\frac{1+5}{5}\right) = \frac{6}{5}.$

$123^2 45 \to \left(\frac{1+6}{6}\right) = \frac{7}{6}$

$(12^2 3^2 45) \to \left(\frac{1+7}{7}\right) = \frac{8}{7}$

$$\text{Dimension} = (2)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)^2\left(\frac{5}{4}\right)^2\left(\frac{6}{5}\right)^2\left(\frac{7}{6}\right)\left(\frac{8}{7}\right)$$

$$= \underline{\underline{16}} \qquad \therefore \boxed{(00001) \text{ is } 16}$$

b) projection $P(SO(10) \supset SU(5)) = P$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Branching can be seen by projecting the Dynkin labels of weights of $\underset{\sim}{16}$ of $SO(10)$ using P.

$$\rightarrow \quad P(00001) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = (0100) \sim \underset{\sim}{10}$$

$$P(0010\,{-1}) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = (0001) \sim \underset{\sim}{\bar{5}}$$

$P(0\ 1\ -1\ 1\ 0) = (1\ -1\ 1\ 0) \sim \cancel{16}\ \underset{\sim}{10}$

$P(1\ -1\ 0\ 1\ 0) = (0\ 0\ 1\ -1) \sim \underset{\sim}{\bar{5}}$

$P(0\ 1\ 0\ -1\ 0) = (1\ 0\ -1\ 1) \sim \underset{\sim}{10}$

$P(-1\ 0\ 0\ 1\ 0) = (-1\ 0\ 1\ 0) \sim \underset{\sim}{10}$

~~P root~~

$P(1\ -1\ 1\ -1\ 0) = (0\ 1\ -1\ 0) \sim \underset{\sim}{\bar{5}}$

$P(-1\ 0\ 1\ -1\ 0) = (-1\ 1\ -1\ 1) \sim \underset{\sim}{10}$

$P(1\ 0\ -1\ 0\ 1) = (1\ 0\ 0\ -1) \sim \underset{\sim}{10}$

$P(-1\ 1\ -1\ 0\ 1) = (0\ 0\ 0\ 0) \sim \underset{\sim}{1}$

$P(1\ 0\ 0\ 0\ -1) = (1\ -1\ 0\ 0) \sim \underset{\sim}{\bar{5}}$

$P(0\ -1\ 0\ 0\ 1) = (-1\ 1\ 0\ -1) \sim \underset{\sim}{10}$

$P(-1\ 1\ 0\ 0\ -1) = (0\ -1\ 0\ 1) \sim \underset{\sim}{10}$

$P(0\ -1\ 1\ 0\ -1) = (-1\ 0\ 0\ 0) \sim \underset{\sim}{\bar{5}}$

$P(0\ 0\ -1\ 1\ 0) = (0\ -1\ 1\ -1) \sim \underset{\sim}{10}$

$P(0\ 0\ 0\ -1\ 0) = (0\ 0\ -1\ 0) \sim \underset{\sim}{10}$

$\therefore$ the projected space consists of all weights of $\underset{\sim}{\bar{5}}$, all weights of $\underset{\sim}{10}$ and $\underset{\sim}{1}$ of $SU(5)$

$$\Rightarrow \quad \underset{\sim}{16}_{SO(10)} \to \left(\underset{\sim}{1} \oplus \underset{\sim}{\bar{5}} \oplus \underset{\sim}{10}\right)_{SU(5)}$$

c) From 3) c) we see that one standard model family fits into the $SU(5)$ representation $\underset{\sim}{\bar{5}} \oplus \underset{\sim}{10}$

Now the $\underset{\sim}{16}$ of $SO(10)$ consists of $\underset{\sim}{1} \oplus \underset{\sim}{\bar{5}} \oplus \underset{\sim}{10}$ of $SU(5)$, and thus consists of one standard model family plus an additional $SU(5)$ singlet represented by $\underset{\sim}{1}$.

$SO(10)$ thus incorporates the one standard model family of matter fields into a single irreducible representation $\underset{\sim}{16} \sim (00001)$, but with an extra singlet.

(5) The Quadratic Casimir value of ~~Ez~~ a weight ~~tz~~ $\lambda = (a_1, a_2, \cdots, a_n)$

is $C(\lambda) = \sum_{ij} G_{ij} a_i a_j + \sum_{\alpha \in \Delta_+} H_\alpha$

$= [(\lambda,\lambda) + \sum_{\alpha \in \Delta_+} (\lambda, \alpha)]$

$= (\lambda, \lambda + 2\delta)$

Where δ has Dynkin label $(\underbrace{1,1,1 \cdots, 1}_{n.})$

and $G_{ij} = (A^{-1})_{ij} \dfrac{(\alpha_j, \alpha_j)}{2}$

A is the Cartan matrix.

For the $SU(n)$ group, $(\alpha_j, \alpha_j) = 2$

$\rightarrow \dfrac{(\alpha_j, \alpha_j)}{2} = 1$ (This normalisation is consistent with the notes)

Cartan matrix $A(n) = A(A_{n-1})$

$$= \underbrace{\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & & \\ & & & & \ddots & \\ & & & & & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}}_{n-1} \Big\} n-1$$

and G in $SU(n)$ simply equals to the inverse of cartan matrix A^{-1}

$$G = A^{-1}$$

The inverse of ~~A~~ $A(n) = A(A_{n-1})$, the $n-1 \times n-1$ dimensional Cartan matrix representing $SU(n)$ is

$$G(A_{n-1}) = A^{-1}(A_{n-1}) =$$

$$\frac{1}{n}\begin{pmatrix} (n-1) & (n-2) & \cdots & & 1 \\ (n-2) & (n-1)+(n-3) & (n-1)+(n-3)-2 & & 2 \\ \vdots & (n-1)+(n-3)-2 & (n-1)+(n-3)+(n-5) & & \vdots \\ & & \vdots & \ddots & \\ & & & (n-1)+(n-3) & (n-2) \\ 1 & 2 & \cdots & (n-2) & (n-1) \end{pmatrix}$$

$$= \frac{1}{n}\begin{pmatrix} (n-1) & (n-2) & \cdots & & & 1 \\ (n-2) & 2(n-2) & & & & 2 \\ & 2(n-3) & 3(n-3) & 3(n-5) & & \\ \vdots & \vdots & 3(n-5) & 4(n-4) & & \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ & 4 & 6 & & 2(n-2) & (n-2) \\ 1 & 2 & 3 & \cdots & (n-2) & (n-1) \end{pmatrix}$$

For example :

$SU(2)$:

$$G(A_1) = \frac{1}{2}$$

$SU(3)$:

$$G(A_2) = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$SU(4)$:

$$G(A_3) = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$SU(5)$:

$$G(A_4) = \frac{1}{5}\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Those can be verified by ~~direct~~ direct multiplication to show

$$G \cdot A = \mathbb{1}_{(n-1)}$$

To compute the Casimir of $SU(n)$, ~~set~~

$$C(\Lambda) = (\Lambda, \Lambda + 2\delta)$$

$$\Rightarrow C(\Lambda) = (a_1, a_2 \cdots a_n) \begin{pmatrix} \\ G = A^{-1} \\ \\ \end{pmatrix} \begin{pmatrix} a_1+2 \\ a_2+2 \\ \vdots \\ a_n+2 \end{pmatrix}.$$

— Now, for $\underset{\sim}{n} \sim (1,0,0,\dots,0) = \Lambda$ of $SU(n)$

Casimir $C\,(1,0,\dots,0)$

$$= (1,0,\dots,0)\begin{pmatrix} (n-1) & (n-2) & \cdots & 1 \\ (n-2) & & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & \cdot \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \times \frac{1}{n}$$

$$+ \frac{2}{n}(1,0,\dots,0)\begin{pmatrix} (n-1) & (n-2) & \cdots & 1 \\ (n-2) & & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & \cdot \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \frac{(n-1)}{n} + \frac{\cancel{2}\cdot\frac{n(n-1)}{\cancel{2}}}{n} = \frac{(n+1)(n-1)}{n}$$

$$= \frac{n^2-1}{n}$$

— For $\underset{\sim}{\bar{n}} \sim (0,0,\dots,0,1) = \Lambda$ of $SU(n)$

Casimir $C\,(0,\dots,0,1) = \frac{1}{n}(0,\dots,0,1)\begin{pmatrix} (n-1) & (n-2) & \cdots & 1 \\ (n-2) & & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & \cdot \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

$$+ \frac{2}{n}(0,\dots,0,1)\begin{pmatrix} (n-1) & (n-2) & \cdots & 1 \\ (n-2) & & & 2 \\ \vdots & & & \vdots \\ 1 & 2 & \cdots & (n-1) \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \frac{n-1}{n} + \frac{2}{n}\left(\frac{n(n-1)}{2}\right) = \frac{(n+1)(n-1)}{n} = \frac{n^2-1}{n}$$

– For $\underbrace{n^2-1}$ ~ $(1,0,\cdots,0,1)$ of $SU(n)$:

Casimir $C(1,0,\cdots,0,1) = \frac{1}{n}(1,0,\cdots,0,1)\begin{pmatrix} n-1 & n-2 & \cdots & 2 & 1 \\ n-2 & & & & 2 \\ \vdots & & & & \vdots \\ 2 & & & & n-2 \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\\0\\1\end{pmatrix}$

$$+ \frac{2}{n}(1,0,\cdots,0,1)\begin{pmatrix} n-1 & n-2 & \cdots & 2 & 1 \\ n-2 & & & & 2 \\ \vdots & & & & \vdots \\ 2 & & & & n-2 \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}\begin{pmatrix}1\\1\\\vdots\\1\\1\end{pmatrix}.$$

$$= \frac{1}{n}(1 0 \cdots 0 1)\begin{pmatrix}n\\n\\\vdots\\n\end{pmatrix} + \frac{2}{n}(n \; n \; n \cdots n)\begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix}.$$

$$\cancel{= \frac{1}{n}\cdot 2n + \frac{2}{n}\cdot n^2}$$

$$\cancel{= \frac{2n^2+2n}{n} = 2(n+1)}$$

$$= \frac{1}{n}2n + \frac{2}{n}n(n-1)$$

$$= 2 + 2(n-1) = \underline{\underline{2n}}$$

b) the index $c(\Lambda)$ is related to the ~~cas~~ Casimir $C(\Lambda)$ by

$$c(\Lambda) = \frac{N(\Lambda)}{N(adj)}C(\Lambda)$$

→ dimension of representation of highest weight Λ.

→ Casimir

index

→ dimension of the adjoint representation of this ~~group~~ Lie algebra.

$$c(1,0,\cdots,0) = \frac{n}{n^2-1}\,\frac{n^2-1}{n} = 1$$

$$c(0,\cdots,0,1) = \frac{n}{n^2-1}\,\frac{n^2-1}{n} = 1$$

$$c(1,0,\cdots,0,1) = \frac{n^2-1}{n^2-1}\cdot 2n = 2n$$

The β function for an $SU(n)$ Yang-Mills theory ~~of~~ with N_f Dirac fermions in repr. $\underset{\sim}{n}$ at one loop:

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}\,c(\text{vector}) - \frac{4}{3}\,c(\text{Dirac fermions}) - \frac{1}{6}\,c(\text{spinless})\right]$$

$c(\text{vector}) = c(1,0,\cdots,0,1) = 2n$

$c(\text{Dirac fermion}) = N_f\, c(1,0,\cdots,0) = N_f$

$c(\text{spinless}) = N_s\, c(1,0,\cdots,0) = N_s = 0$

(N_s = number of spinless real scalar Bosons)

$$\therefore\ \beta(g) = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}\cdot 2n - \frac{4}{3}N_f\right]$$

$$= -\frac{g^3}{48\pi^2}\left(22n - 4N_f\right)$$

$$= -\frac{g^3}{24\pi^2}\left(11n - 2N_f\right)$$

when $N_f = 6$,

$$\beta(g) = -\frac{g^3}{24\pi^2}(11n - 12)$$

~~so for all non-trivial representations.~~

$\because \quad n \geq 2 \in \mathbb{Z}$ for $SU(n)$

$\therefore \quad \cancel{\beta(g)} \; 11n - 12 \geq 22 - \cancel{}12 = 10 > 0$

$\Rightarrow \quad \beta(g) < 0$

Hence the coupling decreases with increasing energy scale, this is the asymptotic freedom.