$\boxed{\text{Ziyan Li}}$

Groups and Representations

Problem Set 2

Tutor : Elmi

Fri Nov 17

①

a) $SO(4) = \{ M \in Aut(\mathbb{R}^4) \mid M^T M = 1$ and $\det(M) = 1 \}$

Expand group element M around identity :

$$M = \mathbb{1} + T + \dots \qquad (T \text{ is the generator})$$

impose conditions

$$\det(M^T M) = 1$$

$$\Rightarrow \quad 1 = \det(\mathbb{1} + T) = 1 + tr(T) + \dots$$

$$\Rightarrow \quad \underline{\underline{tr(T) = 0}}$$

$$M^T M = \mathbb{1} \Rightarrow (\mathbb{1} + T)^T (1 + T) = \mathbb{1}$$

$$\rightarrow \quad \mathbb{1} = \mathbb{1} + T + T^T + \dots$$

$$\Rightarrow \quad \underline{\underline{T = -T^T}}$$

There are 6 linearly ~~All~~ independent (traceless) antisymmetric ~~matrices~~ 4x4 matrices. We try the basis : $\{A_i, B_j\}$ $i,j = 1,2,3$

$$\cancel{A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \qquad \cancel{A_2 = \begin{pmatrix} 0 & 0 & 1 \\ & & \\ & & \end{pmatrix}}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where A_i $(i=1,2,3)$ contains the Lie Algebra of SO(3) group on the top left 3x3 corner.

Hence automatically we have

$$[A_i, A_j] = \varepsilon_{ij}{}^{k} A_k \qquad \text{(commutator is matrix commutator)}$$

<u>And</u>

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$[B_1, B_2] = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= A_3$$

~~Sim~~ Similar for other commutators, we have

$$[B_1, B_2] = A_3 \qquad [B_2, B_3] = A_1 \qquad [B_3, B_1] = A_2$$

$$\therefore \quad [B_i, B_j] = \varepsilon_{ij}{}^{k} A_k$$

$$[A_1, B_1] = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} = 0$$

similarly $[A_2, B_2] = [A_3, B_3] = 0$

$$[A_1, B_2] = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&-1&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{pmatrix} = B_3$$

similarly $[A_2, B_3] = B_1$
$[A_3, B_1] = B_2$

$\therefore \quad [A_i, B_j] = \varepsilon_{ij}{}^{k} B_k$

The full commutation relation

$$[A_i, A_j] = \varepsilon_{ij}{}^{k} A_k = [B_i, B_j] \quad , [A_i, B_j] = \varepsilon_{ij}{}^{k} B_k$$

~~Hence~~ The commutation relation is closed.

Hence $\{A_i, B_i\}_{i,j=1,2,3}$ generates a Lie Algebra of $SO(4)$, which is

$$so(4) = \mathcal{L}(SO(4)) = \text{span}\{A_i, B_j\}|_{i,j=1,2,3}.$$

$\square$

Consider $SU(2)$. its Lie algebra is known to be

$$su(2) = \mathcal{L}(SU(2)) = \text{span}\left\{T_i = -i\frac{\sigma_i}{2}\right\}_{i=1,2,3}$$

where σ_i is the i^{th} pauli matrix.

Lie group $SU(2) \times SU(2)$ has Lie algebra

$$su(2) \oplus su(2) = \text{span}\{(T_i, T_j)\}_{i,j=1,2,3}$$

$$= \text{span}\{(0, T_i), (T_j, 0)\}_{i,j=1,2,3}$$

where the last equal sign follows from the fact that only 6 out of 9 pairs of (T_i, T_j) are linearly independent and the basis can be chosen to be $(0, T_i)$, $(T_j, 0)$

If we denote $C_i = (0, T_i)$, $D_j = (T_j, 0)$

then we have
$$\left.\begin{aligned} [C_i, C_j] &= \varepsilon_{ij}{}^{k} C_k \\ [D_i, D_j] &= \varepsilon_{ij}{}^{k} D_k \\ [C_i, D_j] &= 0 \end{aligned}\right\} (*)$$

because the generators of $SU(2)$ satisfies $[T_i, T_j] = \varepsilon_{ij}{}^{k} T_k$.

Lie algebra of $SU(2) \times SU(2)$ is

$$\text{span}\{(0, T_i), (T_j, 0)\}\Big|_{i,j=1,2,3} \quad T_i = -i\frac{\sigma_i}{2}$$

$\square$

To show that $SO(4) \cong SU(2) \times SU(2)$ we need to show that a bijection between two Lie groups perserves the commutation structure.

Consider the generators of SO(4):

Let $E_i = \frac{1}{2}(A_i + B_i)$, $F_i = \frac{1}{2}(A_i - B_i)$

then $[E_i, E_j] = \frac{1}{4}\Big(\underbrace{[A_i, A_j]}_{\varepsilon_{ij}{}^k A_k} + [B_i, B_j] + [A_i, B_j] + [B_i, A_j]\Big)$

$$= \frac{1}{4}(2\varepsilon_{ij}{}^k A_k + \varepsilon_{ij}{}^k B_k - \varepsilon_{ji}{}^k B_k)$$

$$= \frac{1}{2}\varepsilon_{ij}{}^k (A_k + B_k) = \varepsilon_{ij}{}^k E_k \quad ①$$

$$[F_i, F_j] = \frac{1}{4}\big([A_i, A_j] + [B_i, B_j] - [A_i, B_j] - [B_i, A_j]\big)$$

$$= \frac{1}{4}(2\varepsilon_{ij}{}^k A_k - 2\varepsilon_{ij}{}^k B_k)$$

$$= \frac{1}{2}\varepsilon_{ij}{}^k (A_k - B_k) = \varepsilon_{ij}{}^k F_k \quad ②$$

$$[E_i, F_j] = \frac{1}{4}\big([A_i, A_j] - [B_i, B_j] + [B_i, A_j] - [A_i, B_j]\big)$$

$$= 0 \quad ③$$

~~$[E_i, E_j] =$~~ ①, ②, ③ perserves the commutation structure of (*)

Hence the bijection $M: SO(4) \mapsto SU(2) \times SU(2)$ such that

$$M\Big(\underbrace{e^{\sum_{i,j} \alpha_i E_i + \beta_j F_j}}_{\in SO(4)}\Big) = \underbrace{e^{\sum_{i,j} \alpha_i C_i + \beta_j D_j}}_{\in SU(2) \times SU(2)}$$

~~N~~ makes the two Lie groups isomorphic.

$(\alpha, \beta \in \mathbb{R})$

$\square$

b) We do the same ~~pro~~ procedure for $SO(6)$ gives again the antisymmetry for generators.

$\therefore so(6) = \mathcal{L}(SO(6)) = \text{span}\{ A_k \mid k = 1, 2, \cdots 15$ with

A_k has a 1 on the upper right ~~$(A_k)_{ij} = \delta_{ij} = (A_k)_{ji}$~~ corner, a -1 on the opposite place, and other entries are $0 \}$ ~~for $i<j$ and 0 for other entries).~~

For $SU(4)$, let generators be X_i

~~$1 + X$ $SU(4) = \{ Aut(\mathbb{C}^4)$~~

$SU(4) = \{ 4\times 4$ matrices $U^\dagger U = \mathbb{1} \mid \det(U) = 1 \}$.

$\therefore (1+X)^\dagger (1+X) = \mathbb{1}$ $\quad \mathbb{1} + X + X^\dagger = \mathbb{1}$

$X = -X^\dagger$ $\quad \det(1+X) = 1 + tr(X) + \cdots = 1$

$tr(X) = 0$

So for the generators, we want 15

linearly independent anti-Hermitian traceless 4×4 matrices.

We thus have

$$X_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad X_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad X_4 = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_5 = -\frac{i}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad X_6 = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_7 = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad X_8 = -\frac{i}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_9 = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad X_{10} = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$X_{11} = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad X_{12} = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$

$$X_{13} = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad X_{14} = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

~~$X_{15} =$~~ $X_{15} = -\frac{i}{2\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$

$\therefore$ $su(4) = \mathcal{L}(SU(4)) = \mathrm{span}\{X_i\} \big|_{i=1,2,\cdots,15}.$

The number $\underline{\underline{15}}$ comes from :

For $SO(6)$, the free parameters are in the upper (~~b~~ or lower) ~~cor~~ corner excluding diagonal

$\therefore$ ~~there~~ the dimension $= \frac{(6-1)(6)}{2} = \underline{\underline{15}}$

For $SU(4)$, the free paramceers are ~~the~~

$2 \times (\text{upper corner}) + \text{diagonal} - 1.$

since at corner we have each entry to be complex and gives 2 parameters, but along diagonal the entry has to be purely imaginary to ~~meet~~ satisfy anti Hermiticity. (the -1 comes from traceless constraint)

$\therefore$ dimension $=$ ~~$2 \times \frac{5\times(5-1)}{2} + 4 - 1$~~ $2 \times \frac{3 \times 4}{2} + 4 - 1$

$=$ ~~2×20~~ ~~$2 \times \frac{20}{2} + 3$~~ $= \underline{\underline{15}}$

To show $\mathcal{L}(SU(4)) \cong \mathcal{L}(SO(6))$, need:

$\boxed{1}$ ~~At~~ An expression of the commutation relation of $SO(6)$ in basis $A_i \big|_{i=1,\cdots,15}$

$\boxed{2}$ Find a different basis (not $X_i \big|_{i=1,\cdots,15}$) such that the commutation structure in $\boxed{1}$ is perserved.

Let's do $\boxed{1}$ first:

Consider the following relabelling of matrices

$$A_k \Big|_{k=1,2,\cdots,15} \longrightarrow M_{ij} \Big|_{i,j=1,2,\cdots,6}$$

such that

$$\cancel{M_{ij}=} \quad (M_{ij})_{ab} = \delta_{ia}\delta_{jb} - \delta_{ja}\delta_{ib} = \varepsilon_{xij}\varepsilon_{xab}$$

~~It can be shown~~ and $\underline{M_{ij} = -M_{ji}}$

For example

$$M_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -M_{21}$$

It can be shown explicitly that

$$(**) \quad [M_{ij}, M_{jk}] = M_{ik}, \quad [M_{ij}, M_{kl}] = 0 \quad (i,j,k,l \text{ distinct})$$

~~Now~~ And since M_{ij}'s are simple A_k's so M_{ij} are generators ~~of~~ (basis) of $SO(6)$

Now $\boxed{2}$:

Find a new basis for generators of $SU(4)$

consider γ matrices in 6-Dimensions
(Euclidean, so matric = diag$(1,1,1,1,1,1) = \mathbb{1}_6$
$\gamma_1, \cdots, \gamma_6$, each 8×8 Hermitian matrix
satisfying

$$\gamma_i^\dagger = \gamma_i \quad \text{, and} \quad (1)$$

$$\{\gamma_i, \gamma_j\} = \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}\mathbb{1}_6 \quad (2)$$

which implies $\gamma_i\gamma_j = -\gamma_j\gamma_i$ if $i \neq j$ (3)
and $\gamma_i\gamma_i = \mathbb{1}_6$ (4)

In 6-D there are 6 <u>linearly independent</u> γ matrices $\gamma_1, \cdots, \gamma_6$.

Now consider ~~σ_{ij}~~ 8×8 matrices

$$\sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j] \qquad (i \neq j)$$

~~Lemma 1 (L1): σ_{ij}'s are linearly independent.~~

~~Consider equation $2\sum_{i,j} \alpha_{ij}\sigma_{ij} = 0$ where~~

~~$\alpha_{ij} \in \mathbb{C}$~~

~~$\Rightarrow 2\sum_{i,j} \alpha_{ij} \frac{1}{4}[\gamma_i, \gamma_j] = 0$~~

~~$\Rightarrow \frac{1}{2}\sum_{i,j} \alpha_{ij}(\gamma_i\gamma_j - \gamma_j\gamma_i) = 0$~~

We examine the commutation relation of

$$[\sigma_{ij}, \sigma_{jk}]_{i \neq k}$$

$$= \frac{1}{16}\left[(\gamma_i\gamma_j - \gamma_j\gamma_i)(\gamma_j\gamma_k - \gamma_k\gamma_j) - (\gamma_j\gamma_k - \gamma_k\gamma_j)(\gamma_i\gamma_j - \gamma_j\gamma_i)\right]$$

$$= \frac{1}{16}\Big[\gamma_i\underbrace{\gamma_j\gamma_j}_{\mathbb{1}}\gamma_k - \gamma_i\gamma_j\gamma_k\gamma_j - \gamma_j\gamma_i\gamma_j\gamma_k + \gamma_j\gamma_i\gamma_k\gamma_j$$
$$- \gamma_j\gamma_k\gamma_i\gamma_j + \gamma_j\gamma_k\gamma_j\gamma_i + \gamma_k\gamma_j\gamma_i\gamma_j - \gamma_k\gamma_j\gamma_j\gamma_i\Big]$$

We have:

$$\gamma_i\underbrace{\gamma_j\gamma_j}_{\gamma_j\gamma_j = \mathbb{1}}\gamma_k = \gamma_i\gamma_k \quad ; \quad \gamma_i\gamma_j\underbrace{\gamma_k\gamma_j}_{\{\gamma_k,\gamma_j\}=0} = -\gamma_i\underbrace{\gamma_j\gamma_j}_{\gamma_j^2 = \mathbb{1}}\gamma_k = -\gamma_i\gamma_k$$

$$\gamma_j\gamma_i\gamma_j\gamma_k = -\gamma_j\gamma_j\gamma_i\gamma_k = -\gamma_i\gamma_k \, ; \quad \gamma_j\gamma_i\gamma_k\gamma_j = +\gamma_i\gamma_j\gamma_j\gamma_k$$
$$= \gamma_i\gamma_k$$

$$\gamma_j\gamma_k\gamma_i\gamma_j = -\gamma_j\gamma_k\gamma_j\gamma_i = \gamma_j\gamma_j\gamma_k\gamma_i$$
$$= \gamma_k\gamma_i$$

$$\gamma_j\gamma_k\gamma_j\gamma_i = -\gamma_i\gamma_j\gamma_k\gamma_i = -\gamma_k\gamma_i$$
$$\gamma_k\gamma_j\gamma_i\gamma_j = -\gamma_k\gamma_i\gamma_j\gamma_j = -\gamma_k\gamma_i$$
$$\gamma_k\gamma_j\gamma_j\gamma_i = \gamma_k\gamma_i$$

$$\therefore [\sigma_{ij}, \sigma_{jk}] = \frac{1}{16}\left(4(\gamma_i\gamma_k) - 4(\gamma_k\gamma_i)\right) = \frac{1}{4}[\gamma_i, \gamma_k]$$

$$= \sigma_{ik}$$

$$\Rightarrow \quad [\sigma_{ij}, \sigma_{jk}] = \sigma_{ik} \quad ⑤$$

$$[\sigma_{ij}, \sigma_{kl}] \Big|_{i,j,k,l \text{ distinct}}$$

$$= \frac{1}{16}\left[(\gamma_i\gamma_j - \gamma_j\gamma_i)(\gamma_k\gamma_l - \gamma_l\gamma_k) - (\gamma_k\gamma_l - \gamma_l\gamma_k)(\gamma_i\gamma_j - \gamma_j\gamma_i) \right]$$

$$= \frac{1}{16}\Big[\gamma_i\gamma_j\gamma_k\gamma_l - \gamma_i\gamma_j\gamma_l\gamma_k - \gamma_j\gamma_i\gamma_k\gamma_l + \gamma_j\gamma_i\gamma_l\gamma_k$$

$$- \gamma_k\gamma_l\gamma_i\gamma_j + \gamma_k\gamma_l\gamma_j\gamma_i + \gamma_l\gamma_k\gamma_i\gamma_j - \gamma_l\gamma_k\gamma_j\gamma_i \Big]$$

$$\left\{ \begin{array}{l} \gamma_i\gamma_j\gamma_k\gamma_l = \gamma_i\gamma_j\gamma_k\gamma_l \ ; \ \gamma_i\gamma_j\gamma_l\gamma_k = -\gamma_i\gamma_j\gamma_k\gamma_l . \\ \gamma_j\gamma_i\gamma_k\gamma_l = -\gamma_i\gamma_j\gamma_k\gamma_l \ ; \ \gamma_j\gamma_i\gamma_l\gamma_k = -(-\gamma_i\gamma_j\gamma_k\gamma_l) \\ \qquad\qquad\qquad\qquad\qquad\qquad = \gamma_i\gamma_j\gamma_k\gamma_l \\ \\ \gamma_k\gamma_l\gamma_i\gamma_j = -\gamma_k\gamma_i\gamma_l\gamma_j = +\gamma_k\gamma_i\gamma_j\gamma_l = \mp \gamma_i\gamma_k\gamma_j\gamma_l . \\ \qquad = + \gamma_i\gamma_j\gamma_k\gamma_l \\ \\ \gamma_k\gamma_l\gamma_j\gamma_i = -\gamma_k\gamma_l\gamma_i\gamma_j = -\gamma_i\gamma_j\gamma_k\gamma_l \\ \gamma_l\gamma_k\gamma_i\gamma_j = -\gamma_k\gamma_l\gamma_i\gamma_j = -\gamma_i\gamma_j\gamma_k\gamma_l \\ \gamma_l\gamma_k\gamma_j\gamma_i = -(-\gamma_k\gamma_l\gamma_i\gamma_j) = \gamma_k\gamma_l\gamma_i\gamma_j = \gamma_i\gamma_j\gamma_k\gamma_l \end{array} \right.$$

$$\Rightarrow \quad [\sigma_{ij}, \sigma_{kl}] = \frac{1}{16}\left(4(\cancel{\gamma_i\gamma_j\gamma_k\gamma_l}) - 4(\cancel{\gamma_i\gamma_j\gamma_k\gamma_l}) \right]$$

$$= 0 \qquad (6)$$

(5), (6) together reproduces the commutation structure of SO(6) as expressed in (**).

the commutators of 6-D Euclidean γ matrices are also known as the ~~generators of~~ 8 dimensional unitary representation of spin(6)

and when they are projected onto the ~~Wel~~ Weyl spinor representation gives the generators of SU(4).

~~And since σ's have the same commutation structure as~~

(~~The~~ $\because \sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$

$$\therefore \sigma_{ij}{}^\dagger = \frac{1}{4}[\gamma_j, \gamma_i]^\dagger = \frac{1}{4}[\gamma_0, \gamma_i] = -\sigma_{ij}$$

$$tr(\sigma_{ij}) = 0 \quad \because \text{trace of } [\ ] = 0 \ )$$

And since σ's have the same commutation structure as M's, we conclude that

$$\mathcal{L}(SO(6)) \cong \mathcal{L}(SU(4))$$

$$\text{i.e. } so(6) \cong su(4)$$

$\square$

c) $M^T \eta M = \eta$ where $\eta = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$.

$Sp(2n) = \{ M \in Aut(\mathbb{R}) \mid M^T \eta M = \eta \}$

Closure:

$\forall\ M_1, M_2 \in Sp(2n)$

$(M_1 M_2)^T \eta (M_1 M_2) = M_2^T (M_1^T \eta M_1) M_2$

$= M_2^T \eta M_2 = \eta$ $\square$

Associativity:

Directly follows from the ~~ss~~ associativity of matric ~~mu~~ multiplications.

Identity:

$\mathbb{1}_{2n}^T \eta \mathbb{1}_{2n} = \eta \rightarrow \mathbb{1}_{2n} \in Sp(2n)$

$\mathbb{1}_{2n}$ is the identity. $\square$

Inverse: Consider $N = \eta^T M^T \eta$

$NM = \eta^T \underbrace{M^T \eta M}_{\eta} = \eta^T \eta = \begin{pmatrix} 0 & -\mathbb{1}_n \\ +\mathbb{1}_n & 0 \end{pmatrix} \begin{pmatrix} 0 & +\mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} = (\mathbb{1}_{2n})$.

$= \mathbb{1}_{2n} \Rightarrow N = M^{-1} \Rightarrow M$ has inverse

$\therefore N = M^{-1}$ $\cancel{M^{-1} =}$

Consider $M \in Sp(2n)$ $\because$ $M^T \eta M = \eta$

$$\therefore \underbrace{(M^T)^{-1} M^T \eta M M^{-1}}_{\eta} = (M^T)^{-1} \eta M^{-1} = \eta.$$

$$\therefore \eta = (M^T)^{-1} \eta M^{-1} = (M^{-1})^T \eta (M^{-1})$$

$\Rightarrow M^{-1} \in Sp(2n)$ $\square$

The Lie algebra, Expand around the identity

$M = \mathbb{1} + X + \cdots$

$M^T \eta M = \eta \Rightarrow (\mathbb{1} + X + \cdots)^T \eta (\mathbb{1} + X + \cdots) = \eta.$

$\Rightarrow \cancel{(\mathbb{1} + X^T + \cdots)} \Rightarrow \cancel{\eta \mathbb{1}} + X^T \eta + \eta X + \cdots = \cancel{\eta}$

$\Rightarrow \quad X^T \eta + \eta X = 0$

Hence $\quad sp(2n) = \mathcal{L}(Sp(2n))$

$= \text{span}\{X_i \mid X_i^T \eta + \eta X_i\}$

– The Cartan Subalgebra:

$\because X^T \eta + \eta X = 0, \quad \eta^T = -\eta = \eta^{-1}$

$\therefore \quad \eta X = -X^T \eta = X^T(-\eta) = X^T \eta^T = (\eta X)^T$

$\therefore \quad \eta X$ is symmetric.

let $\quad X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

ηX symmetric $\Rightarrow \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ symmetric

$\Rightarrow \begin{pmatrix} C & D \\ -A & -B \end{pmatrix}$ symmetric

Hence: B, C must be symmetric

and $D = -A^T$

$\therefore \quad X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$ with B, C symmetric.

consider matrizes $E_{i,j}$ = ~~n×n m~~ $2n \times 2n$ matrix such that the component on ith row and jth column has entry 1 and other ~~entry are~~ entries are 0.

~~then a choice of Cartan subalgebra.~~

The basis of generators X is

$\boxed{H_i = E_{i,i} - E_{n+i,n+i}}$

eg $n=2$

$H_1 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

~~$X_{i,j} = E_{i,j} - E_{n+j,n+i}$~~

$X_{i,j} = E_{i,j} - E_{n+j,n+i}$

$(i \neq j)$

eg $n=2 \quad i=1 \quad j=2$

$X_{12} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$

$Y_{i,j} = E_{i,n+j} + E_{j,n+i}$

$(i \neq j)$

eg $n=2 \quad i=1 \quad j=2$

$Y_{12} = \left(\begin{array}{c|cc} 0 & 0 & 1 \\ & 1 & 0 \\ \hline 0 & & 0 \end{array}\right)$

$Z_{i,j} = E_{n+i,j}$ ~~$+ E_{j,n+i}$~~ $+ E_{n+j,i}$

$Z_{12} = \left(\begin{array}{cc|c} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 1 & 0 & \end{array}\right)$

~~tec~~ (1.C)

$U_i = E_{i,n+i}$ ~~=~~ e.g. $i=1$. $n=2$

$$U_1 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$V_i = E_{n+i,i}$ e.g. $i=1$ $n=2$

$$V_1 = \left(\begin{array}{c|c} 0 & 0 \\ \hline \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & 0 \end{array}\right) \ldots$$

$\{H_i, X_i, Y_i, Z_i, U_i, V_i\}$ togather gives the basis of generators X of ~~sp~~ $sp(2n)$ since $X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$ with $B=B^T$, $C=C^T$

The ~~Cat~~ Cartan subalgebra of $sp(2n)$ is the maximally diagonalizable subalgebra of $sp(2n)$. This is $\underline{\underline{\{H_i\}}}$

~~$\because E_{ij}E_{jk} = E_{ik}$ $\therefore$ $[H_i, H_j] = E_{ii}E_{n+i}E_{n+i}$~~

$$\underline{\underline{[H_i, H_j] = 0}} \quad \text{for} \quad i \neq j$$

~~$[H_i, H_i] =$~~ since $H_i H_j = 0$ ~~for~~ $= H_j H_i$

for $i \neq j$

$\because E_{ij} E_{jk} = E_{ik}$ $\therefore$ $[H_i, H_j] = [E_{ii} + E_{n+i\,n+i}, E_{jj}, E_{n+j}E_{n+j}]$

$$= \underbrace{[E_{ii}, E_{jj}]}_{0} + \underbrace{[E_{n+i\,n+i}, E_{jj}]}_{0} + \underbrace{[E_{ii}, E_{n+j}E_{n+j}]}_{0} + \underbrace{[E_{n+i\,n+i}, E_{n+j\,n+j}]}_{0}$$

$= 0$ for $i \neq j$ $\square$

$\cancel{E_{ij}E_{jl}}$ $E_{ij}E_{kl} = \delta_{jk} E_{il}$

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$$

$$[U_i, U_j] = [E_{i,n+i}, E_{j,n+j}]$$

$$= \delta_{n+i,j} E_{i,n+j} - \delta_{i,n+j} E_{j,n+i}$$

$$= 0 \quad \forall \; i,j \in \{1,2,\dots,n\} \quad \cancel{\text{for } i \neq j}$$

similarly $[V_i, V_j] = 0$

But $V_iV_i = U_iU_i = 0 \Rightarrow U, V$ can't be Cartan subalgebra

$$H_iH_i = (E_{ii} - E_{n+i\,n+i})(E_{ii} - E_{n+i\,n+i})$$

$$= E_{ii} + E_{n+i\,n+i} \neq 0$$

$\mathrm{tr}(H_iH_i) = 1 + 1 = 2$ for all $i = 1, \dots, n$.

$\therefore \mathrm{tr}(H_iH_j) = 2\delta_{ij}$ $\quad [H_i, H_j] = 0$ for $i \neq j$ $\quad (*)$

If there is no larger set of mutually commuting elements of $sp(2n)$ then $\{H_i\}$ is the Cartan subalgebra due to $(*)$

$$[X_{ij}, X_{kl}] = [E_{ij} - E_{n+j\,n+i}, E_{kl} - E_{n+l\,n+k}]$$

$$= [E_{ij}, E_{kl}] + [E_{n+j\,n+i}, E_{n+l\,n+k}]$$

$$= \delta_{jk}(E_{il} + E_{n+l\,n+i}) - \delta_{li}(E_{kj} + E_{n+j\,n+k}) \neq 0$$

$\therefore$ X_{ij} cannot be Cartan subalgebra.

Similarly $[Y_{ij}, Y_{kl}] \neq 0$ $\quad [Z_{ij}, Z_{kl}] \neq 0$

$$[H_i, X_{jk}] = [E_{ii} - E_{n+i\,n+i}, E_{jk} - E_{n+k\,n+j}]$$

$$= [E_{ii}, E_{jk}] + [E_{n+i\,n+i}, E_{n+k\,n+j}]$$

$$= \delta_{ij} E_{ik} - \delta_{ik} E_{ji} + \delta_{ik} E_{n+i\,n+j} - \delta_{ij} E_{n+k\,n+i}$$

$$\neq 0$$

Similarly ~~for~~ commutator of H with ~~other~~ Y, Z, U, V. $\therefore \{H_i\}$ cannot be expanded any further to have one more element commuting with $\forall H_i \in \{H_i\}$.

So $\because (*) \Rightarrow \{H_i\}$ is the Cartan subalgebra

$\dim(H) = n$ $\quad \therefore$ $\underline{\underline{rk(sp(2n)) = n}}$ $\square$

$$\dim(sp(2n)) = \dim(H) + \dim(X) + \dim(Y) + \dim(Z) + \dim(U) + \dim(V)$$

$$= n + n(n-1) + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} + n + n$$

$$= n^2 + n^2 - n + n + n = 2n^2 + n$$

$$= \underline{\underline{n(2n+1)}} \quad \square$$

2

a) $f : SU(n) \times U(1) \to U(n)$

$f((U, z)) = zU$, $U \in SU(n)$, $z \in U(1)$

consider $U_1, U_2 \in SU(n)$. $z_1, z_2 \in U(1)$

$(U_1, z_1) \circ (U_2, z_2) = (U_1 U_2 , z_1 z_2)$

$\therefore f((U_1, z_1) \circ (U_2, z_2)) = f((U_1 U_2, z_1 z_2))$

$= z_1 z_2 U_1 U_2$ ①

$f((U_1 z_1)) f((U_2 z_2)) = z_1 U_1 z_2 U_2$

$= z_1 z_2 U_1 U_2$ ②

$\because z_2$ is a complex number

$\therefore [z_1, U_1] = 0$

$\because$ ① , ② $\therefore$ f is a group homomorphism.

b) If $(z_0, U_0) \in \ker(f)$ for $z_0 \in U(1)$
$U_0 \in SU(n)$

then $f((z_0, U_0)) = id_{U(n)}$

$= \mathbb{1}_n$

$\therefore z_0 U_0 = \mathbb{1}_n$ ③

taking determinant of ③

$$Z_0^n \det(U_0) = 1 \qquad \because U_0 \in SU(n) \quad \therefore \det(U_0) = 1$$

$$\Rightarrow \quad Z_0 = e^{2\pi i q/n} \quad , \quad q \in \mathbb{Z}_n = \{1, 2, 3, \cdots, n\}$$

$$\therefore U_0 = e^{-2\pi i q/n} \mathbb{1}_n$$

$$\therefore \ker(f) = (Z_0, U_0) = \left(e^{\frac{2\pi i q}{n}}, e^{-\frac{2\pi i q}{n}} \mathbb{1}_n\right)$$

~~Ker(f)~~ Define map $\phi : \ker(f) \to \mathbb{Z}_n$ such that

$$\phi\left[\cancel{\phi}\left(e^{\frac{2\pi i q}{n}}, e^{-\frac{2\pi i q}{n}} \mathbb{1}_n\right)\right] = q \in \mathbb{Z}_n .$$

clearly ϕ is bijective since there is a 1 to 1 correspondence between all elements in two finite groups.

$$\phi\left[\left(e^{\frac{2\pi i q_1}{n}}, e^{-\frac{2\pi i q_1}{n}} \mathbb{1}_n\right) \circ \left(e^{\frac{2\pi i q_2}{n}}, e^{-\frac{2\pi i q_2}{n}} \mathbb{1}_n\right)\right]$$

$$= \cancel{\mathbb{Z}_2} = \phi\left[\left(e^{\frac{2\pi i (q_1+q_2) \bmod n}{n}}, e^{\frac{-2\pi i (q_1+q_2) \bmod n}{n}} \mathbb{1}_n\right)\right]$$

$$= (q_1 + q_2) \bmod n \qquad \text{for } q_1, q_2 \in \mathbb{Z}_n .$$

$$\phi\left[\left(e^{2\pi i q_1/n}, e^{-\frac{2\pi i q_1}{n}} \mathbb{1}_n\right)\right] \circ \phi\left[\left(e^{2\pi i q_2/n}, e^{-\frac{2\pi i q_2}{n}} \mathbb{1}_n\right)\right]$$

$$= q_1 \circ q_2 = (q_1 + q_2) \bmod n$$

$\Rightarrow$ ϕ is a group homomorphism.

Hence $\ker(f) \cong \mathbb{Z}_n$ $\square$

$\because$ ~~f(z,U)~~ $f((z,U)) = zU \in Im(f)$

$\forall\, zU \in Im(f)$, $(zU)^\dagger (zU) = \cancel{z} U^\dagger \underbrace{z^* z}_{1} U$

$= \underbrace{U^\dagger U}_{\mathbb{1}_n} = \mathbb{1}_n$

$\rightarrow zU \in U(n)$

$\Rightarrow Im(f) \subseteq U(n)$

If X is a $n \times n$ matrix such that $X^\dagger X = \mathbb{1}$
so $X \in U(n)$, then

$1 = \det(X^\dagger X) = \det(X^\dagger)\det(X) = \det{}^*(X)\det(X)$

$= |\det(X)|^2 \Rightarrow \underline{\det(X) = e^{i\alpha n} \quad \text{with } \alpha \in [0, \tfrac{2\pi}{n}]}$

then consider $n \times n$ matrix $Y = e^{-i\alpha} X$

clearly $Y^\dagger Y = X^\dagger \underbrace{e^{i\alpha} e^{-i\alpha}}_{1} X = X^\dagger X = \mathbb{1}_n \rightarrow Y \in U(n)$

in addition, $\det(Y) = \det(e^{-i\alpha} X)$

$= e^{-i\alpha n} \det(X) = e^{-i\alpha n} e^{i\alpha n} = 1$

$\rightarrow Y \in SU(n)$

$\because$ $X = e^{i\alpha} Y$ and $e^{i\alpha} \in U(1)$, $Y \in SU(n)$

$\therefore$ ~~$\forall$~~ $\forall\, X \in U(n)$, $X \in Im(f)$

$\Rightarrow U(n) \subseteq Im(f)$

Hence $U(n) \subseteq Im(f)$ $Im(f) \subseteq U(n)$

$\Rightarrow$ we have $\underline{\underline{U(n) = Im(f)}}$ $\square$

c) $Ker(f) \cong \mathbb{Z}_n$.

$Im(f) = U(n)$

and f is a bijective group homomorphism between $SU(n) \times U(1)$ and $U(n)$

$$\Rightarrow \quad \frac{SU(n) \times U(1)}{\mathbb{Z}_n} = U(n) \qquad \square$$

(3)

a) $R_L(M) = e^{\frac{1}{2}(-s^i - it^i)\sigma_i}$

$R_R(M) = e^{\frac{1}{2}(s^i - it^i)\sigma_i}$

$$R_D(M) = \begin{pmatrix} R_L(M) & 0 \\ 0 & R_R(M) \end{pmatrix}.$$

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ $\qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta^{\mu\nu}\mathbb{1}_4$

$\therefore$ If $\mu \neq \nu$ $\qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0$

$$\therefore \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = \frac{i}{2}(\gamma_\mu\gamma_\nu + \gamma_\mu\gamma_\nu)$$

$$= i\gamma_\mu\gamma_\nu$$

Consider the matrix ~~$\Omega = \exp(T^{ab}$~~

~~$\Omega = \exp(iT^{ab}$~~ $\quad \Omega = \exp(\frac{1}{4} i T^{\mu\nu}\sigma_{\mu\nu})$ where $T^{\mu\nu}$ is just a number (summation implied.)

then:

We known $\sigma_{\mu\nu} = 0$ if $\mu = \nu$

For $\mu \neq \nu$.

If $\mu, \nu = i, j \in \{1,2,3\}$ $i \neq j$

then $\sigma_{ij} = i\gamma_\mu \gamma_\nu = i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$

$= i \begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix}$

$= -i\,\varepsilon^{ijk} \begin{pmatrix} i\sigma_k & 0 \\ 0 & i\sigma_k \end{pmatrix}$ $\quad \boxed{\sigma_i\sigma_j = \cancel{\delta_{ij}} \\ = \delta_{ij}\mathbb{1}_2 + i\varepsilon^{ijk}\sigma^k}$

$\therefore$ let $T_{ij} = -\varepsilon_{ijk} t^k = -\varepsilon_{ijr} t^r$

If $\{\mu, \nu\} = \{0, \rho \in \{1,2,3\}\}$, then

$\sigma^{0\rho} = i \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}}_{\gamma_0} \underbrace{\begin{pmatrix} 0 & \sigma^\rho \\ -\sigma^\rho & 0 \end{pmatrix}}_{\gamma_\rho} = i \begin{pmatrix} -\sigma^\rho & 0 \\ 0 & \sigma^\rho \end{pmatrix}$ $\quad (\rho = 1,2,3)$

$\sigma^{\rho 0} = i\gamma_\rho\gamma_0 = -i\gamma_0\gamma_\rho = -i \begin{pmatrix} -\sigma^\rho & 0 \\ 0 & \sigma^\rho \end{pmatrix}$

let $T_{\rho 0} = -T_{0\rho} = s^\rho$

then the matrix

$$\Omega = \exp\left(\tfrac{1}{4} i T^{\mu\nu}\sigma_{\mu\nu}\right) = \cancel{\exp\left(\tfrac{i}{4}\left[T^{0\rho}\sigma^{0\rho} + T^{\rho 0}\right.\right.}$$

$$= \exp\left(\tfrac{i}{4}\left[T_{0\rho}\sigma^{0\rho} + \underbrace{T_{\rho 0}\sigma^{\rho 0}}_{= -T_{0\rho}(-\sigma^{0\rho}) = T_{0\rho}\sigma^{0\rho}} + T_{ij}\sigma^{ij}\right]\right)$$

$$= \exp\left(\tfrac{i}{4}\left[2\underbrace{T_{0\rho}}_{-s^\rho}\sigma^{0\rho} + \varepsilon^{ijk} t^r \sigma_{ij}\right]\right)$$

$$= \exp\left(\tfrac{-i}{2} s^\rho\sigma^{0\rho} + \tfrac{-i}{4}\varepsilon^{ijk} t^r \sigma_{ij}\right)$$

$$= \exp\left(-\tfrac{i}{2} s^\rho i \begin{pmatrix} -\sigma^\rho & 0 \\ 0 & \sigma^\rho \end{pmatrix} - \tfrac{1}{4}\varepsilon^{ijk}\underbrace{\varepsilon_{ijr}}_{2\delta^k{}_r} t^r \begin{pmatrix} i\sigma_k & 0 \\ 0 & i\sigma_k \end{pmatrix}\right)$$

$$= \exp\left(\begin{pmatrix} -\frac{1}{2}s^{\rho}\sigma^{\rho} & 0 \\ 0 & \frac{1}{2}s^{\rho}\sigma^{\rho} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} it^{k}\sigma^{k} & 0 \\ 0 & it^{k}\sigma^{k} \end{pmatrix} \right).$$

$$= \cancel{\exp\left(\frac{1}{4}\left(-\frac{1}{2}t^{i}\right.\right.} = \exp\left(\begin{pmatrix} \frac{1}{2}(-s^{i}\sigma^{i} - it^{i}\sigma^{i}) & 0 \\ 0 & \frac{1}{2}(s^{i}\sigma^{i} - it^{i}\sigma^{i}) \end{pmatrix} \right)$$

$$= \cancel{\exp} \begin{pmatrix} e^{\frac{1}{2}(-s^{i} - it^{i})\sigma_{i}} & 0 \\ 0 & e^{\frac{1}{2}(s^{i} - it^{i})\sigma_{i}} \end{pmatrix}.$$

$$= \begin{pmatrix} R_L(M) & 0 \\ 0 & R_R(M) \end{pmatrix} = R_D(M)$$

$\Rightarrow$ $R_D(M)$ can be expressed in terms of $R_D(M) = \Omega = \exp(\frac{1}{4} i T^{\mu\nu}\sigma_{\mu\nu})$.

let $\varepsilon^{\mu\nu} = \frac{1}{4}T^{\mu\nu}$ $\therefore$ $R_D(M) = \exp(i\varepsilon^{\mu\nu}\sigma_{\mu\nu})$.

Hence ~~transfor~~ infinitesimal transformation

$$R_D(M)\psi = \exp(i\varepsilon^{\mu\nu}\sigma_{\mu\nu})\psi \simeq (1 + i\varepsilon^{\mu\nu}\sigma_{\mu\nu})\psi$$

$$\stackrel{!}{=} \psi + \delta\psi$$

$\Rightarrow$ $\delta\psi = i\varepsilon^{\mu\nu}\sigma_{\mu\nu}\psi$ with

$\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ $\square$

b) The ~~represen~~ Lie algebra of the Lorentz group is

$$\mathcal{L}(L) = \{ T \in \mathrm{End}(\mathbb{R}^4) \mid T = -\eta T^T \eta \}$$

where $\eta = \mathrm{diag}(-1,1,1,1)$

$\cancel{\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]}$

Lie algebra can also be written as

$\mathcal{L}(L) = \mathrm{span}\{X_{\mu\nu}\}$ for generators $X_{\mu\nu}$

and the commutation relation of $\mathcal{L}(L)$ is

$$(*) \quad [X_{\alpha\beta}, X_{\gamma\delta}] = \eta_{\alpha\delta} X_{\beta\gamma} + \eta_{\alpha\gamma} X_{\delta\beta} + \eta_{\beta\delta} X_{\gamma\alpha} + \eta_{\beta\gamma} X_{\alpha\delta}$$

$\because T = -\eta T^T \eta$. So we have freedom to choose the upper-right corner of T

$$\Rightarrow \quad \dim(\mathcal{L}(L)) = 1+2+3 = \underline{\underline{6.}}$$

Now consider $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] = i\gamma_\mu\gamma_\nu$

$$[\sigma_{\mu\nu}, \gamma_\rho] = i[\gamma_\mu\gamma_\nu, \gamma^\rho] = i\gamma_\mu\gamma_\nu\gamma_\rho - i\gamma_\rho\gamma_\mu\gamma_\nu$$

$$= i\gamma_\mu \underbrace{\{\gamma_\nu, \gamma_\rho\}}_{2\eta^{\nu\rho}} - \cancel{i\gamma_\mu\gamma_\rho\gamma_\nu} - i\underbrace{\{\gamma_\rho, \gamma_\mu\}}_{2\eta^{\rho\mu}}\gamma_\nu + \cancel{i\gamma_\mu\gamma_\rho\gamma_\nu}$$

$= \cancel{2i\gamma_\mu\eta^{\nu\rho}} = \cancel{2i\gamma_\nu\gamma}$

$$= 2i\gamma_\mu\eta_{\nu\rho} - 2i\gamma_\nu\eta_{\rho\mu}$$

so

$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = i[\sigma_{\mu\nu}, \gamma_\rho\gamma_\sigma]$$

$$= i[\sigma_{\mu\nu}, \gamma_\rho]\gamma_\sigma + i\gamma_\rho[\sigma_{\mu\nu}, \gamma_\sigma]$$

$$= 2i\left(i\gamma_\mu\gamma_\sigma\eta_{\nu\rho} - i\gamma_\nu\gamma_\sigma\eta_{\rho\mu} + i\gamma_\rho\gamma_\mu\eta_{\nu\sigma} - i\gamma_\rho\gamma_\nu\eta_{\sigma\mu}\right) \quad (1)$$

$$\cancel{= \because i\sigma_{\mu\sigma} + \eta_{\mu\sigma} = 2[\gamma_\mu, \gamma_\sigma] + \eta_{\mu\sigma}}$$

$$\cancel{= i\gamma_\mu\gamma_\sigma = 2\gamma_\sigma\gamma_\mu + \eta_{\mu\sigma} =}$$

$$\because \; -i\sigma_{\mu\sigma} + \eta_{\mu\sigma} = \frac{1}{2}[\gamma_\mu, \gamma_\sigma] + \eta_{\mu\sigma}$$

$$= \frac{1}{2}\gamma_\mu\gamma_\sigma - \frac{1}{2}\gamma_\sigma\gamma_\mu + \eta_{\mu\sigma} = \gamma_\mu\gamma_\sigma - \cancel{\frac{1}{2}\underbrace{\{\gamma_\mu, \gamma_\sigma\}}_{2\eta_{\mu\sigma}}} + \cancel{\eta_{\mu\sigma}}$$

$$= \gamma_\mu\gamma_\sigma$$

$$\therefore \; \gamma_\mu\gamma_\sigma = -i\sigma_{\mu\sigma} + \eta_{\mu\sigma}$$

$\because$ (1) $\Rightarrow \quad [\sigma_{\mu\nu}, \sigma_{\rho\sigma}]$

$$= 2i\left(-\sigma_{\mu\sigma}\eta_{\nu\rho} + i\cancel{\eta_{\mu\sigma}\eta_{\nu\rho}} + \sigma_{\nu\sigma}\eta_{\rho\mu} - i\cancel{\eta_{\nu\sigma}\eta_{\rho\mu}} - \sigma_{\rho\mu}\eta_{\nu\sigma} + i\cancel{\eta_{\rho\mu}\eta_{\nu\sigma}} + \sigma_{\rho\nu}\eta_{\sigma\mu} - i\cancel{\eta_{\rho\nu}\eta_{\sigma\mu}}\right)$$

$$= 2i\left(\sigma_{\sigma\mu}\eta_{\nu\rho} + \sigma_{\nu\sigma}\eta_{\rho\mu} + \sigma_{\mu\rho}\eta_{\nu\sigma} + \sigma_{\rho\nu}\eta_{\sigma\mu}\right).$$

$\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$

$\because$ commutator is antisymmetric

$$= -2i\left(\eta_{\mu\sigma}\sigma_{\nu\rho} + \eta_{\mu\rho}\sigma_{\cancel{\alpha}\sigma\nu} + \eta_{\nu\sigma}\sigma_{\rho\mu} + \eta_{\nu\rho}\sigma_{\mu\sigma}\right)$$

~~pe~~ relabel $\left.\begin{array}{l} \mu \to \alpha \\ \nu \to \beta \\ \rho \to \gamma \\ \sigma \to \delta \end{array}\right\}$

$$\Rightarrow \; [\cancel{\sigma_{x}}\, [\sigma_{\alpha\beta}, \sigma_{\gamma\delta}] = -2i\left(\cancel{\eta_{\alpha\beta}}\; \eta_{\alpha\delta}\sigma_{\beta\gamma} + \eta_{\alpha\gamma}\sigma_{\delta\beta} + \eta_{\beta\delta}\sigma_{\gamma\alpha} + \eta_{\beta\gamma}\sigma_{\alpha\sigma}\right).$$

which is identical to the ~~com~~ commutation relation of $(*)$ up to a rescale-able constant.

$\Rightarrow$ $\sigma_{\mu\nu}$ form a representation of Lie algebra $\mathcal{L}(L)$ $\square$

c) $\sigma_\mu = R_V(M)_\mu{}^\nu R_L(M)\,\sigma_\nu\, R_L(M)^\dagger$

$= \Lambda_\mu{}^\nu R_L(M)\,\sigma_\nu\, R_L(M)^\dagger.$ (From notes)

$$R_L(M)^{-1}\sigma_\mu \underbrace{\left(R_L(M)^{-1}\right)^\dagger}_{R_R(M)} = R_V(M)_\mu{}^\nu R_L(M)^{-1}R_L(M)\,\sigma_\nu\, R_L(M)^\dagger \times \left(R_L(M)^{-1}\right)^\dagger$$

$$= R_V(M)_\mu{}^\nu \sigma_\nu$$

$$R_V(M)_\mu{}^\nu\sigma_\nu = R_L(M)^{-1}\sigma_\mu R_R(M) \qquad (*)$$

For $\mu = i = \{1,2,3\}$

$$R_D^{-1}(M)\,\gamma_\mu R_D(M) = \begin{pmatrix} R_L^{-1}(M) & 0 \\ 0 & R_R^{-1}(M)\end{pmatrix}\begin{pmatrix} 0 & \sigma_\mu \\ -\sigma_\mu & 0\end{pmatrix}\begin{pmatrix} R_L(M) & 0 \\ 0 & R_R(M)\end{pmatrix}$$

$$= \begin{pmatrix} R_L^{-1}(M) & 0 \\ 0 & R_R^{-1}(M)\end{pmatrix}\begin{pmatrix} 0 & \sigma_\mu R_R(M) \\ -\sigma_\mu R_L(M) & 0\end{pmatrix}$$

$$= \begin{pmatrix} 0 & R_L^{-1}(M)\,\sigma_\mu R_R(M) \\ -R_R^{-1}(M)\,\sigma_\mu R_L(M) & 0\end{pmatrix}. \quad (1)$$

$$R_V(M)_\mu{}^\nu\gamma_\nu = \cancel{\begin{pmatrix} 0 & R_V(M)_\mu{}^\nu\sigma_\nu \\ -R_V(M)_\mu{}^\nu\sigma_\nu & 0\end{pmatrix}}.$$

$$= \begin{pmatrix} 0 & R_V(M)_\mu{}^\nu\sigma_\nu \\ (R_V(M)_\mu{}^0)\mathbb{1} - R_V(M)_\mu{}^i\sigma_i & 0\end{pmatrix}$$

For $\sigma_0 = \mathbb{1}$.

the top-right corner :

$$R_V(M)_\mu{}^\nu \sigma_\nu = R_L^{-1}(M)\sigma_\mu R_R(M) \quad \text{by } (*)$$

$\therefore$ they are equal

the bottom left ~~corne~~ corner :

$$\left(R_V(M)_\mu{}^0 \mathbb{1} - R_V(M)_\mu{}^i \sigma_i\right)\left(R_V(M)_\mu{}^\nu \sigma_\nu\right)$$

$$= \left(R_V(M)_\mu{}^0 \mathbb{1} - R_V(M)_\mu{}^i \sigma_i\right)\left(R_V(M)_\mu{}^0 \mathbb{1} + R_V(M)_\mu{}^j \sigma_j\right)$$

$$= \left(R_V(M)_\mu{}^0\right)^2 \mathbb{1} - \left(R_V(M)_\mu{}^i R_V(M)^j \sigma_i \sigma_j\right).$$

$$= \left[\left(R_V(M)_\mu{}^0\right)^2 - \left(R_V(M)_\mu{}^i R_V(M)_\mu{}^i\right).\right] \mathbb{1}$$

~~$\sigma_i\sigma_j = \delta_{ij}$~~

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})I + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

$$= -\Big(\underbrace{\sum_\alpha \eta_{\alpha\alpha} R_V(M)_\mu{}^\alpha R_V(M)_\mu{}^\alpha}_{=\eta_{\mu\mu} \text{ (no sum)}}\Big)\mathbb{1}$$

$$= -\eta_{\mu\mu}\mathbb{1} = \begin{cases} -\mathbb{1} & \text{for } \mu = i = \{1,2,3\} \\ \mathbb{1} & \text{for } \mu = 0 \end{cases}$$

$\therefore$ For $\mu = i = \{1,2,3\}$

$$\left(R_V(M)_\mu{}^0 \mathbb{1} - R_V(M)_\mu{}^i \sigma_i\right) = -\left(R_V(M)_\mu{}^\nu \sigma_\nu\right)^{-1}$$

$$= -\left(R_L^{-1} \sigma_\mu R_R(M)\right)^{-1} = -R_R^{-1}(M)\, \sigma_\mu R_L(M)$$

gives the equality for the left-bottom corner.

Hence for $\mu = i = \{1,2,3\}$

$$R_D(M)^{-1} \gamma_\mu R_D(M) = R_V(M)_\mu{}^\nu \gamma_\nu$$

For $\mu = 0$

$$R_D^{-1}(M) \gamma_\mu R_D(M) = \begin{pmatrix} 0 & R_L^{-1}(M)\, \sigma_0 R_R(M) \\ R_R^{-1}(M)\, \sigma_0 R_L(M) & 0 \end{pmatrix}$$

$$R_V(M)_\mu{}^\nu \gamma_\mu = \begin{pmatrix} 0 & R_V(M)_\mu{}^\nu \sigma_\nu \\ \underbrace{(R_V(M)_\mu{}^0)\mathbb{1} - R_V(M)_\mu{}^i \sigma_i}_{=(R_V(M)_\mu{}^\nu \sigma_\nu)^{-1} \text{ for } \mu=0.} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & R_V(M)_\mu{}^\nu \sigma_\nu \\ (R_V(M)_\mu{}^\nu \sigma_\nu)^{-1} & 0 \end{pmatrix}$$

We know $R_L^{-1}(M)\, \sigma_0 R_R(M) = R_V(M)_0{}^\nu \sigma_\nu$ for $\mu = 0$ from $(*)$

$$\therefore \left(R_V(M)_0{}^\nu \sigma_\nu\right)^{-1} = \left(R_L^{-1}(M)\, \sigma_0 R_R(M)\right)^{-1}$$

$$= R_R^{-1}(M)\, \sigma_0 R_L(M)$$

→ Both top-right and bottom left corners are satisfied.

→ We conclude for $\mu = 0, 1, 2, 3$

$$R_D(M)^{-1} \gamma_\mu R_D(M) = R_V(M)_\mu{}^\nu \gamma_\nu$$

~~d). Dirac equation~~

$$\cancel{i\gamma^\mu \partial_\mu \psi - m\psi} = 0.$$

$$\therefore \cancel{i\gamma^\mu R_V(M)_\mu{}^\nu \partial_\nu \psi - m}$$

$$\cancel{i\gamma^\mu \partial_\mu \psi - m}$$

$$\cancel{\left(i\gamma^\mu R_V(M)_\mu{}^\nu \partial_\nu \psi - m\right) R_D(M)\psi = 0}$$

$$\cancel{= \left(i R_D(R) \gamma^\mu R_D(R)^{-1} \partial_\mu \psi - m\right) R_D(M)\psi = 0}$$

$$\cancel{\therefore \quad R_D(R) \left[i\gamma^\mu \partial_\mu \psi - m\psi \right] = 0}$$

$$\cancel{\Rightarrow \quad i\gamma^\mu \partial_\mu \psi - m\psi = 0}$$

~~⇒ Dirac equation is invariant.~~

d)

The Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \Rightarrow (i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\Rightarrow (i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \qquad x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

transformation

$$\cancel{(i\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu' - m)\psi'(x')}$$

$$\cancel{= (i R_V(M)_\mu{}^\nu \gamma^\mu \partial_\nu' - m)\psi(x')}$$

$$\cancel{= (i R_D^{-1}(M)\gamma^\mu R_D(M))}$$

$$i(\gamma^\mu \partial_\mu' - m)\psi'(x') = 0 \quad \text{gives}$$

$$\cancel{= i \Lambda \gamma^\mu R_\mu{}^\nu \partial_\nu}$$

$$0 = i(\gamma^\mu R_V(M)_\mu{}^\nu \partial_\nu - m) R_D^{-1}(M)\psi.$$

$$= i R_D^{-1}(M)\gamma^\mu \underbrace{R_D(M) R_D^{-1}(M)}_{1}\partial_\nu\psi - i m R_D^{-1}(M)\psi.$$

$$= R_D^{-1}(M)(i\gamma^\mu \partial_\nu - m)\psi = 0$$

$$\Rightarrow (i\gamma^\mu \partial_\nu - m)\psi = 0 \qquad \square$$

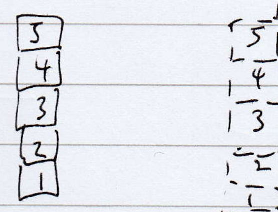
Dirac equation Lorentz invariant.

4

a)

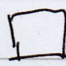
~~1 =~~

1



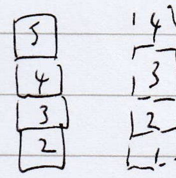
5/5 = 1

5



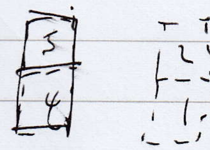
5/1 = 5

5



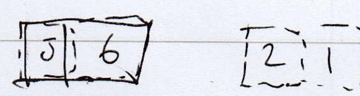
~~24~~ 5!/4! = 5

10



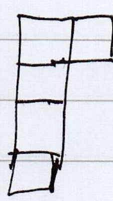
20/2 = 10

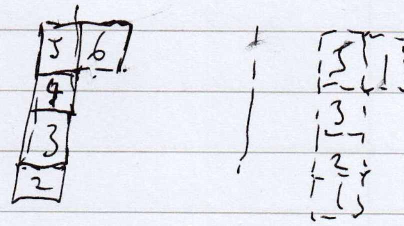
15



30/2 = 15

24





~~~~ The tensors. are

$\underset{\sim}{1}$ $\begin{array}{|c|}\hline i_1\\\hline i_2\\\hline i_3\\\hline i_4\\\hline i_5\\\hline\end{array}$ Tensor $= T^{[i_1 i_2 i_3 i_4 i_5]}$

where [ ] indicates full antisymmetrisation.

$\underset{\sim}{5}$ Tensor $= T^{i}$

$\underset{\sim}{\bar{5}}$ Tensor $= T^{[i_1 i_2 i_3 i_4]}$

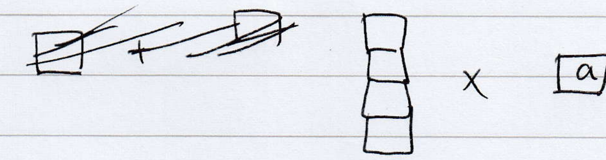
$\underset{\sim}{10}$ Tensor $= T^{[i_1 i_2]} = \frac{1}{2!}(T^{i_1 i_2} - T^{i_2 i_1})$

$\underset{\sim}{15}$ $\begin{array}{|c|c|}\hline i_1 & i_2\\\hline\end{array}$ Tensor $= T$

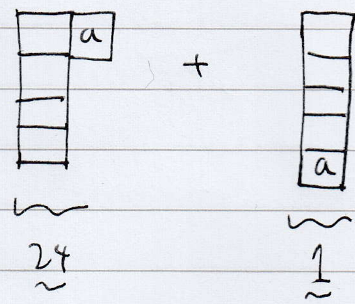
$\underset{\sim}{24}$ $\begin{array}{|c|c|}\hline i_1 & j_1\\\hline i_2 \\\cline{1-1} i_3 \\\cline{1-1} i_4 \\\cline{1-1}\end{array}$ Tensor $= T^{[i_1 j_1]} + T^{[i_1 i_2 i_3 i_4]}$

( ) indicates symmetrisation
[ ]indicates anti-symmetrisation
~~~~

b) $\underset{\sim}{5} \times \underset{\sim}{\bar{5}} =$

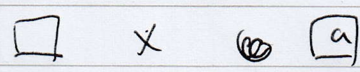


$=$

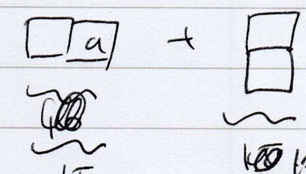


$= \underset{\sim}{1} + \underset{\sim}{24} .$ □

$\underset{\sim}{5} \times \underset{\sim}{5} =$

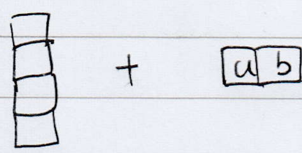


$=$

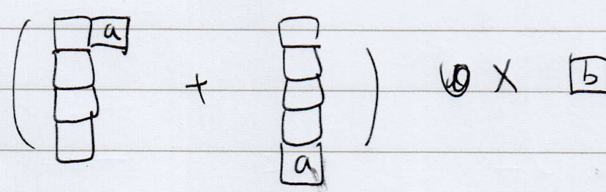


$= \underset{\sim}{10} + \underset{\sim}{15}$ □

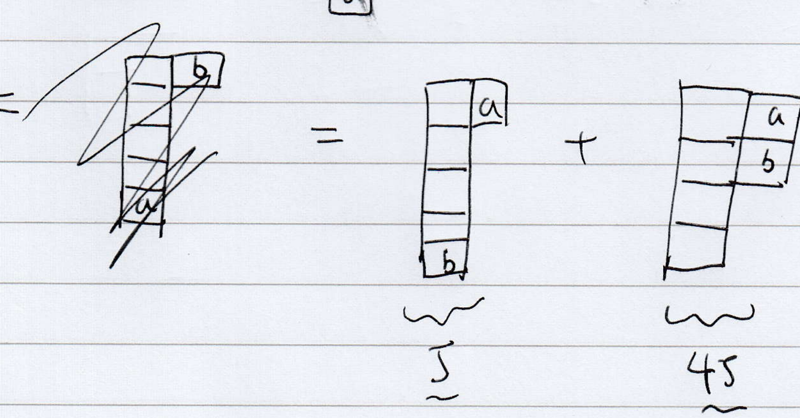
$\underset{\sim}{\bar{5}} \times \underset{\sim}{10} =$



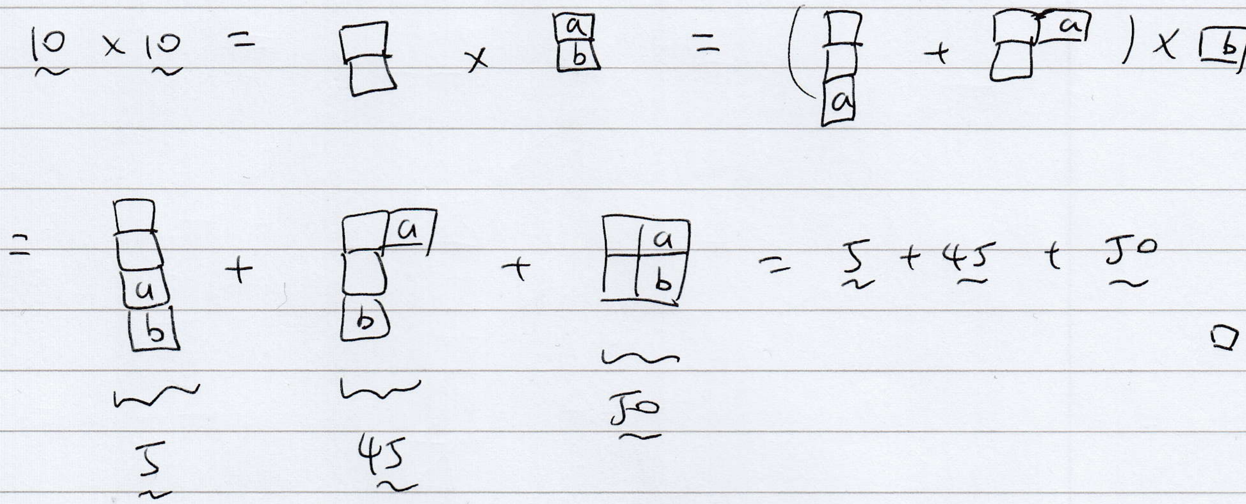
$=$



$=$



$= \underset{\sim}{5} + \underset{\sim}{45}$ □



$\bar{5}$ or 5 ?

~~$\underset{\sim}{10}$~~

c) $U = \begin{pmatrix} U_3 & 0 \\ 0 & U_2 \end{pmatrix} \in SU(5)$

$SU(3) \times SU(2) \subset SU(5)$

Branching ($\cdot$ is trivial tableaux).

$\underset{\sim}{5} = \square = (\square \;\cdot)_2 + (\cdot\; \square)_{-3}$

$= (\underset{\sim}{3}, \underset{\sim}{1})_2 + (\underset{\sim}{1}, \underset{\sim}{2})_{-3}$

$\underset{\sim}{\bar{5}} = \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} = \left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \;\square\right)_3 + \left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \;\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)_{-2}$

~~$+ \left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \;\square\square\right)_{-2}$~~

$= (\underset{\sim}{1}, \underset{\sim}{2})_3 + (\underset{\sim}{\bar{3}}, \underset{\sim}{1})_{-2}$

$\underset{\sim}{10} = \begin{smallmatrix}\square\\\square\end{smallmatrix} = \left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \;\cdot\right)_4 + (\square \;\square)_{-1}$

$\bar{3}$

$+ \left(\cdot \;\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)_{-6}$

~~$= (\bar{3}, 1) \times 3$~~

$= (\underset{\sim}{\bar{3}}, \underset{\sim}{1})_4 + (\underset{\sim}{3}, \underset{\sim}{2})_{-1} + (\underset{\sim}{1}, \underset{\sim}{1})_{-6}$

d) let $P \underbrace{}_{\text{a representation of}} U(1)$ such that $[P, U] = 0$.

and $P = \begin{pmatrix} P_{33} & P_{32} \\ P_{23} & P_{22} \end{pmatrix}$ with P_{ij} means

a ~~block~~ matrix with i rows and j ~~columns~~ columns.

ie $[P, U] = 0 \Rightarrow PU = UP$

$$\Rightarrow \begin{pmatrix} P_{33} & P_{32} \\ P_{23} & P_{22} \end{pmatrix}\begin{pmatrix} U_3 & 0 \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} U_3 & 0 \\ 0 & U_2 \end{pmatrix}\begin{pmatrix} P_{33} & P_{32} \\ P_{23} & P_{22} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} P_{33}U_3 & P_{32}U_2 \\ P_{23}U_3 & P_{22}U_2 \end{pmatrix} = \begin{pmatrix} U_3P_{33} & U_3P_{32} \\ U_2P_{23} & \cancel{P_{22}}\, U_2P_{22} \end{pmatrix}.$$

$\therefore \quad P_{32}U_2 = U_3P_{32} \quad , \quad P_{23}U_3 = U_2P_{23}$

$\because U_3, U_2$ irreducible representations

$\therefore$ By schur's lemma, $P_{23} = P_{32} = 0$

$P_{33}U_3 = U_3P_{33} \quad , \quad P_{22}U_2 = U_2P_{22}$

By schur's lemma $\quad P_{33} = \lambda_3 \mathbb{1}_3$
$\quad P_{22} = \lambda_2 \mathbb{1}_2$

$\lambda_3, \lambda_2 \in \mathbb{C}$

$\therefore$ ~~ρ $\otimes$ $U(1)$~~

$\therefore$ ρ a representation of $U(1)$ and

$\rho \in SU(5)$

$\therefore$ $\det(\rho) = 1$

$\rightarrow$ $\lambda_3^3 \lambda_2^2 = 1$ $\rightarrow$ can write

$\lambda_3 = e^{-2i\alpha}$ $\lambda_2 = e^{3i\alpha}$

$\therefore$ $\rho = \mathrm{diag}(e^{-2i\alpha}, e^{-2i\alpha}, e^{-2i\alpha}, e^{3i\alpha}, e^{3i\alpha})$

The $U(1)$ charges ~~are given~~ for

$SU(N) \times SU(M) \subset SU(M+N)$ ~~is~~ are.

$Q = nM - mN$

In our case $N=3$, $M=2$

$\therefore$ $\boxed{Q = 2n - 3m}$ where

n is number of squares in $SU(N)$ and m is number of squares in $SU(M)$

All values calculated and given as subscript in question part c)

(5) $U = U_R + iU_I \in SU(n)$

a)

$$f : SU(n) \to Gl(\mathbb{R}^{2n})$$

$$\forall \; f(U) = \begin{pmatrix} U_R & -U_I \\ U_I & U_R \end{pmatrix}. \in Im(f).$$

$$f^T(U) f(U) = \begin{pmatrix} U_R^T & U_I^T \\ -U_I^T & U_R^T \end{pmatrix} \begin{pmatrix} U_R & -U_I \\ U_I & U_R \end{pmatrix}.$$

$$= \begin{pmatrix} U_R^T U_R + U_I^T U_I & -U_R^T U_I + U_I^T U_R \\ -U_I^T U_R + U_R^T U_I & U_R^T U_R + U_I^T U_I \end{pmatrix}.$$

$\because \; U \in SU(n)$

$\therefore \; U^\dagger U = \mathbb{1}_n$ ~~$\det(U) = 1$~~ $\det(U) = 1$

$\therefore \; (U_R^T - iU_I^T)(U_R + iU_I) = \mathbb{1}_n$

$\therefore \; (U_R^T U_R + U_I^T U_I) + i(U_R^T U_I - U_I^T U_R) = \mathbb{1}_n$

Equating real & ~~comp~~ imaginary parts gives

$$U_R^T U_R + U_I^T U_I = \mathbb{1}_n \,, \quad U_R^T U_I - U_I^T U_R = 0$$

Hence $f^T(U) f(U) = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n \end{pmatrix} = \mathbb{1}_{2n}$ (1)

$1 = \det(\mathbb{I}_{2n}) = \det(f^T f) = \det(f(U))^2$

$\because f(U)$ is a real matrix

$\therefore \det(f(U)) = \pm 1$

If $U_R = \mathbb{I}_n$, $U_I = 0$ $f(U) = \mathbb{I}_{2n}$ clearly has $\det(f(U)) = 1$

if we vary U_R and U_I smoothly $\det(f(U))$ should not change abruptly from 1 to -1

$\therefore \det(f(U)) = \underline{1}$ ②

①, ② $\Longrightarrow f(U) \in SO(2n) \;\forall\, U \in SU(n)$

$\rightarrow \operatorname{Im}(f) \subset SO(2n)$ $\square$

So $\cancel{f(U) \in SO(2n) \ \forall \ f(U)}$

$\cancel{\rightarrow \mathrm{Im}(f) \in SO(2n)}$

b) $f(U_1) = f(U_2) \rightarrow \begin{pmatrix} U_{1R} & -U_{1I} \\ U_{1I} & U_{1R} \end{pmatrix} = \begin{pmatrix} U_{2R} & -U_{2I} \\ U_{2I} & U_{2R} \end{pmatrix}.$

$\rightarrow U_{1R} = U_{2R}$ and $U_{1I} = U_{2I}$

$\rightarrow U_{1R} + iU_{1I} = U_{2R} + iU_{2I}$

$\forall \ U_1 = U_{1R} + iU_{1I}, \ U_2 = U_{2R} + iU_{2I} \in SU(n).$

$\rightarrow U_1 = U_2$

$\rightarrow f$ is injective. $\square$

$$f(U_1)f(U_2) = \begin{pmatrix} U_{1R} & -U_{1I} \\ U_{1I} & U_{1R} \end{pmatrix} \begin{pmatrix} U_{2R} & -U_{2I} \\ U_{2I} & U_{2R} \end{pmatrix}.$$

$$= \begin{pmatrix} U_{1R}U_{2R} - U_{1I}U_{2I} & -(U_{1R}U_{2I} + U_{1I}U_{2R}) \\ U_{1I}U_{2R} + U_{1R}U_{2I} & U_{1R}U_{2R} - U_{1I}U_{2I} \end{pmatrix}.$$

$\cancel{f(U_1U_2) = (\mathrm{Re}(U_1U_2)} \quad \cancel{U_1U_2 = (U_{1R} + iU_{1I}}$

$$U_1U_2 = (U_{1R} + iU_{1I})(U_{2R} + iU_{2I})$$

$$= (U_{1R}U_{2R} - U_{1I}U_{2I}) + i(U_{1R}U_{2I} + U_{1I}U_{2R})$$

So

$$f(U_1U_2) = \begin{pmatrix} U_{1R}U_{2R} - U_{1I}U_{2I} & -(U_{1R}U_{2I} + U_{1I}U_{2R}) \\ (U_{1R}U_{2I} + U_{1I}U_{2R}) & U_{1R}U_{2R} - U_{1I}U_{2I} \end{pmatrix}.$$

$$= f(U_1)f(U_2)$$ group homomorphism $\square$