Quantum Field Theory

Homework #3

Hand-in time and place (week 7):

Class	Hand-in time	Hand-in place	Teaching Assistant
Tuesday 15.30-17.00	Sunday 6pm	Mathematics†	Johan Henriksson
Friday 14.30-16.00	Tuesday noon	Mathematics†	Johan Henriksson
Thursday 8.30-10.00	Monday 6pm	Mathematics‡	Matteo Parisi
Friday 8.30-10.00	Monday 6pm	Mathematics‡	Matteo Parisi

† Mezzanine level in the Mathematical Institute (Andrew Wiles Building, Woodstock Road)

`DO NOT FORGET TO PUT THE NAME OF YOUR TEACHING ASSISTANT ON THE SHEET.`

Useful formulae:

- Schwinger parametrisation:

$$\frac{1}{A^x} = \frac{1}{\Gamma(x)} \int_0^\infty dt\, t^{x-1} e^{-At} \,. \tag{1}$$

- Momentum integrals in d dimensions

$$\int \frac{d^d k}{(2\pi)^d} e^{-\alpha k^2} = (4\pi\alpha)^{-d/2} \,. \tag{2}$$

- Miscellaneous useful integrals:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1) \frac{xyz}{(xy+xz+yz)^3} = \frac{1}{2} \,, \tag{3}$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1) \frac{xyz}{(xy+xz+yz)^3} \times \times \log \frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz)^2} = -\frac{3}{4} \,. \tag{4}$$

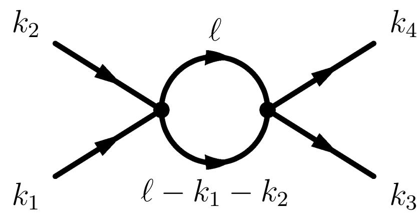
8. **Feynman parametrisation**

Using induction in n, prove the following formula

$$\frac{1}{a_1 \cdots a_n} = (n-1)! \int_0^1 \prod_{i=1}^n dx_j \, \frac{\delta(\sum_{j=1}^n x_j - 1)}{(a_1 x_1 + \ldots + a_n x_n)^n} \,. \tag{5}$$

9. **Evaluating Feynman integrals**

In this homework we evaluate a contribution to the four-point correlator in the ϕ^4 theory in Minkowski space at one-loop, given by the diagram



(a) Write the dimensionless coupling constant g_0 in $4-2\epsilon$ dimensions as the bare coupling λ_0 multiplied by a suitable power of the mass scale μ.

(b) Show that using the Feynman rules in momentum space and after Wick rotation the diagram yield

$$A(\epsilon) = \frac{g_0^2}{2}\mu^{4\epsilon} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2 + m^2} \frac{1}{(\ell - k_1 - k_2)^2 + m^2} \,. \tag{6}$$

(c) Use Feynman's identity to rewrite the loop integration as

$$A(\epsilon) = \frac{g_0^2}{2}\mu^{4\epsilon} \int_0^1 dx \int \frac{d^{4-2\epsilon}\tilde{\ell}}{(2\pi)^{4-2\epsilon}} \frac{1}{(\tilde{\ell}^2 + x(1-x)(k_1+k_2)^2 + m^2)^2} \,. \tag{7}$$

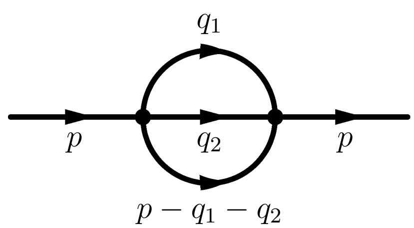
(d) Show that the momentum integral evaluates to

$$\begin{aligned} A(\epsilon) =& \mu^{2\epsilon} \frac{g_0^2}{32\pi^2} \left[\frac{1}{\epsilon} - \gamma_E - \int_0^1 dx \ln \left(\frac{m^2 + (k_1+k_2)^2 x(1-x)}{4\pi\mu^2} \right) \right. \\ & \left. + \mathcal{O}(\epsilon) \right] , \end{aligned} \tag{8}$$

where γ_E is the Euler constant.

10. **Wave-function renormalisation**

In order to find the leading correction to wave-function renormalization constant in $\lambda_0\phi^4$ theory in Minkowski space we will evaluate the so-called sunrise diagram:

$iA(p^2) =$ 

(a) Write down the Feynman integral for sunrise diagram in four dimensions. Remember to include the proper symmetry factor.

(b) Use Feynman parametrisation and rewrite the Feynman integral as

$$\begin{aligned} A(p^2) =& C\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\times \\ &\times \int \frac{d^4q_1}{(2\pi)^4}\int \frac{d^4q_2}{(2\pi)^4}\frac{1}{\left(\mathcal{D}(x,y,z,q_1^2,q_2^2,p^2)\right)^3}\,, \end{aligned} \tag{9}$$

where C is a constant and the function $\mathcal{D}$ depends only on the squares of momenta q_1 and q_2.

(c) In order to find the contribution to wave-function renormalization we need to evaluate the derivative of $A(p^2)$ with respect to p^2. Working in Euclidean space and using dimensional regularization in $d = 4 - 2\epsilon$ dimensions show that

$$\begin{aligned} \frac{dA(p^2)}{dp^2} =& \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)F(x,y,z)\times \\ &\times \left\{\frac{2}{\epsilon} + const + \log G(x,y,z,p^2/m^2)\right\}\,, \end{aligned} \tag{10}$$

for some functions F and G.

(d) Using the explicit form of this functions and working in the leading order perturbative expansion in dimensionless coupling evaluate

$$\begin{aligned} \left.\frac{dA(p^2)}{dp^2}\right|_{p^2=m^2} =& -\frac{g_0^2}{6144\pi^4}\left(\frac{1}{\epsilon} + 2\log\frac{\mu^2}{m^2} + 2\log(4\pi) - 2\gamma_E - \frac{3}{2}\right. \\ &\left. + \mathcal{O}(\epsilon)\right)\,. \end{aligned} \tag{11}$$

Some of the integrals given in the introduction can be useful here.

(e) Evaluate the anomalous dimension for the field ϕ from the formula

$$\gamma(g_0) = \mu \frac{\partial}{\partial \mu} \log Z_\phi \,, \tag{12}$$

where up the order we are working at

$$Z_\phi^{-1} = \frac{1}{1 + \frac{dA(p^2)}{dp^2}\Big|_{p^2=m^2}} \,. \tag{13}$$

Compare the result with the one found in lecture.