Groups and Representations

Problem Sheet 1

Date: TBA, Deadline: TBA

1) (Normal sub-groups and group homomorphisms)

a) Consider a group G, a normal sub-group H of G and the quotient G/H together with the multiplication defined by $(gH)(\tilde{g}H) := g\tilde{g}H$. Show that this multiplication is well-defined (that is, the definition is independent on the choice of representative) and that G/H together with this multiplication forms a group. [5]

b) Let G and $\tilde{G}$ be two groups and $F : G \to \tilde{G}$ a group homomorphism. Show that $\mathrm{Im}(F)$ is a sub-group of $\tilde{G}$ and that $H := \mathrm{Ker}(F)$ is a normal sub-group of G. [7]

c) For the situation described in b) define a map $f : G/H \to \mathrm{Im}(F)$ by $f(gH) := F(g)$. Show that this map is well-defined and that it is a group isomorphism. [8]

2) (Characters of $\mathbb{Z}_n$) Consider the group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with multiplication defined by $kl := (k + l) \bmod n$. What are the conjugacy classes of $\mathbb{Z}_n$?

a) Write down the irreducible complex representations R_q (where $q = 0, 1, \ldots, n-1$) of Z_n and compute their characters χ_q. [5]

b) Show explicitly that the characters obtained in a) are ortho-normal, that is, show that $(\chi_q, \chi_p) = \delta_{qp}$. [7]

c) Focus on $\mathbb{Z}_3$. For the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

define a map $R : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{C}^3)$ by $R(k) := M^k$. Show that R is a representation of $\mathbb{Z}_3$ and determine which irreducible representations it contains. [8]

3) (Permutation groups) Denote by S_n the group of permutations of n objects, that is $S_n = \{\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \,|\, \sigma \text{ bijective}\}$. It is often useful to denote a particular permutation σ by the symbol

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix} .$$

a) Verify that S_n forms a group for all n which is non-Abelian for $n > 2$. [3]

b) Focus on S_3. Determine its conjugacy classes and show that the complete set of its complex irreducible representations consists of one two-dimensional and two one-dimensional representations. [3]

c) Find the character table of S_3. [6]

d) Consider the regular representation of S_3 and write down the projectors which correspond to the various irreducible representations. [4]

e) Find the irreducible representation content of $R_2 \otimes R_2$ where R_2 is the irreducible two-dimensional representation. [4]

4) (Generalisation of Schur's Lemma) Consider a group G and a complex representation R which can be written as $R = n_1 R_1 \oplus \cdots \oplus n_r R_r$ where R_i, $i = 1, \cdots, r$ are irreducible representations of dimensions d_i and the integers n_i indicate how often R_i appears in R.

a) Convince yourself that the representation matrices $R(g)$ can then be written as $R(g) = \mathbf{1}_{n_1} \times R_1(g) \oplus \cdots \oplus \mathbf{1}_{n_r} \times R_r(g)$. (Here, the tensor product $A \times B$ of two matrices A and B denotes the matrix obtained when every entry of A is replaced by this entry times the matrix B). [4]

b) Show that a matrix P with $[P, R(g)] = 0$ for all $g \in G$ has the general form $P = P_1 \times \mathbf{1}_{d_1} \oplus \cdots \oplus P_r \times \mathbf{1}_{d_r}$ where P_i are $n_i \times n_i$ matrices. [10]

c) Consider the representation of $U(1)$ defined by $R(e^{i\alpha}) := \mathrm{diag}(e^{i\alpha}, e^{i\alpha}, e^{-2i\alpha})$. What is the most general form of complex 3×3 matrices which commute with all representation matrices $R(e^{i\alpha})$? [6]

5) (Dihedral group D_4) The dihedral group D_4 is generated by the two matrices

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

a) Show that D_4 is of order eight, that it is non-Abelian and that it has five conjugacy classes. [6]

b) Find the complex irreducible representations and the character table of D_4. [10]

c) Work out which irreducible representations are contained in the tensor product of the two-dimensional irreducible representation with itself. [4]