Corrections to Groups PS2 Q5.

5.c) consider the matrix $P = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_n & i\mathbb{1}_n \\ i\mathbb{1}_n & \mathbb{1}_n \end{pmatrix}$.

Define:

$$\tilde{f}(U) = P f(U) P^{-1}.$$

easy to see that $P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_n & -i\mathbb{1}_n \\ -i\mathbb{1}_n & \mathbb{1}_n \end{pmatrix}$.

$$\tilde{f} = P f P^{-1} = \frac{1}{2}\begin{pmatrix} \mathbb{1}_n & i\mathbb{1}_n \\ i\mathbb{1}_n & \mathbb{1}_n \end{pmatrix}\begin{pmatrix} U_R & -U_I \\ U_I & U_R \end{pmatrix}\begin{pmatrix} \mathbb{1}_n & -i\mathbb{1}_n \\ -i\mathbb{1}_n & \mathbb{1}_n \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \mathbb{1}_n & i\mathbb{1}_n \\ i\mathbb{1}_n & \mathbb{1}_n \end{pmatrix}\begin{pmatrix} U_R + iU_I & -iU_R - U_I \\ U_I - iU_R & U_R - iU_I \end{pmatrix}$$

$$= \begin{pmatrix} U_R + iU_I & 0 \\ 0 & U_R - iU_I \end{pmatrix}$$

$$= \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}$$

$\tilde{f} \cong f$ is a ~~2n representation of~~ 2n dimensional representation of $SO(2n)$

and $\tilde{f}$ is an embedding of $SU(n)$ into $SO(2n)$

U is a $\underset{\sim}{n}$ representation of $SU(n)$

U^* is a $\underset{\sim}{\bar{n}}$ representation of $SU(n)$

$\therefore$ ~~&~~ $\underset{\sim}{2n}_{SO(2n)} \to \underset{\sim}{n} \oplus \underset{\sim}{\bar{n}}_{SU(n)}$

d) The dimension of adjoint representation = dimension ~~of~~ of Lie algebra.

For $SO(10)$, this is $\frac{10(10-1)}{2} = 45$.

$\therefore$ adjoint rep of $SO(10)$ is $\underset{\sim}{45}$

$$\underset{SO(10)}{\underset{\sim}{45}} = (\underset{\sim}{10} \otimes \underset{\sim}{10})_A^{SO(10)}$$ (A is antisymmetrized)

This is ~~because~~ explicit in:

$$\underset{\sim}{10} \otimes \underset{\sim}{10} = (\underset{\sim}{10} \otimes \underset{\sim}{10})_A \oplus (\underset{\sim}{10} \otimes \underset{\sim}{10})_S$$
$$= \underset{\sim}{45} \oplus \underset{\sim}{55}$$ for $SO(10)$.

$\therefore$ $SO(10)$ $\Downarrow$ $SU(5)$ $\Downarrow$

$$\underset{\sim}{45} = (\underset{\sim}{10} \times \underset{\sim}{10})_A \longrightarrow ((\underset{\sim}{5} \oplus \underset{\sim}{\bar{5}}) \otimes (\underset{\sim}{5} \oplus \underset{\sim}{\bar{5}}))_A$$
$$= (\underset{\sim}{5} \otimes \underset{\sim}{5} \oplus \underset{\sim}{\bar{5}} \otimes \underset{\sim}{\bar{5}} \oplus \underset{\sim}{5} \otimes \underset{\sim}{\bar{5}} \oplus \underset{\sim}{\bar{5}} \otimes \underset{\sim}{5})_A$$

$$= (\mathbf{J} \otimes \mathbf{J})_A \oplus (\bar{\mathbf{J}} \otimes \bar{\mathbf{J}})_A \oplus (\mathbf{J} \otimes \bar{\mathbf{J}})$$

$$= \mathbf{10} \oplus \overline{\mathbf{10}} \oplus (\mathbf{J} \otimes \bar{\mathbf{J}})$$

$$= \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{1} \oplus \mathbf{24}$$

(This method of calculation follows from Georgi pp. 286–287).