To: Alfons Weber

B4 Problem Set 2

Ziyan Li

1. $$\frac{dN}{dt} = -\Gamma N$$

Assumptions: Γ is constant, i.e. the transition rate does not depend on the number of elements ~~and not~~ or explicitly on time. ✓

$$\begin{cases} \frac{dA}{dt} = -\Gamma_A A & (1) \\ \frac{dB}{dt} = \Gamma_A A - \Gamma_B B & (2) \\ \frac{dC}{dt} = \Gamma_B B & (3) \end{cases}$$ ✓

$$(1) \Rightarrow \int_{A(0)}^{A(t)} \frac{dA}{A} = -\int_0^t \Gamma_A \, dt \Rightarrow \ln \frac{A(t)}{A(0)} = -\Gamma_A t$$

$$\Rightarrow A(t) = A(0) e^{-\Gamma_A t}$$ ✓

$$\frac{dB}{dt} = \Gamma_A A(0) e^{-\Gamma_A t} - \Gamma_B B$$

$$\therefore \frac{dB}{dt} + \Gamma_B B = \Gamma_A A(0) e^{-\Gamma_A t}$$

Complementary function B_{CF}

$$\frac{dB_{CF}}{dt} + \Gamma_B B_{CF} = 0 \Rightarrow B_{CF} = C_1 e^{-\Gamma_B t}$$

Particular integral ~~B~~ B_{PI}

try $B_{PI} = C_2 e^{-\Gamma_A t}$ $\quad (\Gamma_A \neq \Gamma_B)$

$\Rightarrow \quad -\Gamma_A C_2 + \Gamma_B C_2 = \Gamma_A A(0)$

$\therefore \quad C_2 = \frac{\Gamma_A A(0)}{\Gamma_B - \Gamma_A} = \frac{A(0)}{\frac{\Gamma_B}{\Gamma_A} - 1}$

$\therefore \quad B(t) = C_1 e^{-\Gamma_B t} + \frac{A(0)}{\Gamma_B/\Gamma_A - 1} e^{-\Gamma_A t}$

$B(0) = C_1 + \frac{A_0}{\frac{\Gamma_B}{\Gamma_A} - 1} \Rightarrow C_1 = B(0) - \frac{A(0)}{\Gamma_B/\Gamma_A - 1}$

$\therefore \quad B(t) = \left(B(0) - \frac{A(0)}{\Gamma_B/\Gamma_A - 1} \right) e^{-\Gamma_B t} + \frac{A(0)}{\Gamma_B/\Gamma_A - 1} e^{-\Gamma_A t}$

$\Rightarrow \quad B(t) = B(0) e^{-\Gamma_B t} + \frac{A(0)}{1 - \Gamma_B/\Gamma_A} \left(e^{-\Gamma_B t} - e^{-\Gamma_A t} \right)$

→ originally contains A only → $B(0) = 0$

~~$\frac{dc}{dt} = \Gamma_B B - B(t)\Gamma_B$~~

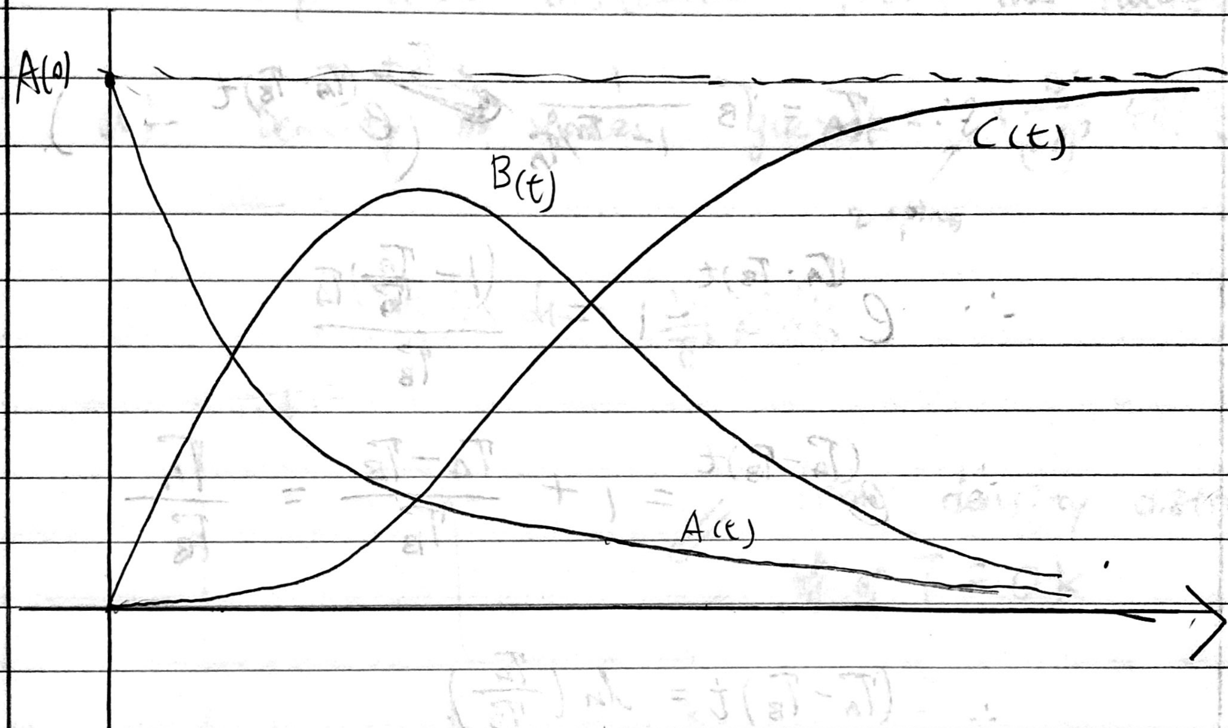
$\therefore \quad B(t) = \frac{A(0)}{1 - \Gamma_B/\Gamma_A} \left(e^{-\Gamma_B t} - e^{-\Gamma_A t} \right)$ ✓

$\frac{dc}{dt} = \Gamma_B B = \frac{A(0)\Gamma_B}{1 - \Gamma_B/\Gamma_A} \left(e^{-\Gamma_B t} - e^{-\Gamma_A t} \right), \quad C(0) = 0$

$c = \frac{A(0)\Gamma_B}{1 - \Gamma_B/\Gamma_A} \int_0^t e^{-\Gamma_B t} - e^{-\Gamma_A t} \, dt.$

$$C = \frac{A(0)\Gamma_B}{1-\Gamma_B/\Gamma_A}\left[\frac{1}{\Gamma_B}\left(1-e^{-\Gamma_B t}\right) - \frac{1}{\Gamma_A}\left(1-e^{-\Gamma_A t}\right)\right]$$

Sketch:



~~decay rate of B: $\frac{dB}{dt} = \Gamma_A A - \Gamma_B B$~~

maximum ~~decrease~~ decay rate requires either $\frac{d^2B}{dt^2} = 0$ ~~(or at t=0)~~

$\therefore\ 0 = \Gamma_A \frac{dA}{dt} - \Gamma_B \frac{dB}{dt}$

$\Rightarrow -\Gamma_A\Gamma_A A - \Gamma_B(\Gamma_A A - \Gamma_B B) = 0$

$\therefore\ \Gamma_A(\Gamma_A + \Gamma_B)A = \Gamma_B^2 B$

Decay rate of B is $\Gamma_B B$

So maximum decay rate occurs at

maximum B $\Rightarrow$ $\frac{dB}{dt} = 0$

$\Rightarrow$ $\Gamma_A A = \Gamma_B B$

$\therefore$ $\Gamma_A A(0) e^{-\Gamma_A t} = \Gamma_B \frac{A(0)}{1 - \Gamma_B/\Gamma_A} (e^{-\Gamma_B t} - e^{-\Gamma_A t})$

$\therefore$ $\Gamma_A = \Gamma_B \frac{1}{1 - \Gamma_B/\Gamma_A} (e^{(\Gamma_A - \Gamma_B)t} - 1)$.

$\therefore$ $e^{(\Gamma_A - \Gamma_B)t} - 1 = \frac{(1 - \frac{\Gamma_B}{\Gamma_A})\Gamma_A}{\Gamma_B}$

$\therefore$ $e^{(\Gamma_A - \Gamma_B)t} = 1 + \frac{\Gamma_A - \Gamma_B}{\Gamma_B} = \frac{\Gamma_A}{\Gamma_B}$

$\therefore$ $(\Gamma_A - \Gamma_B)t = \ln(\frac{\Gamma_A}{\Gamma_B})$.

~~$\Rightarrow$ $t = \ln \Gamma_B - \ln$~~ $\Rightarrow$ $t = \frac{\ln(\Gamma_A/\Gamma_B)}{\Gamma_A - \Gamma_B}$

✓

(+)

2. $\because r = r_0 A^{1/3}$ $\therefore$ Volume $\sim A$

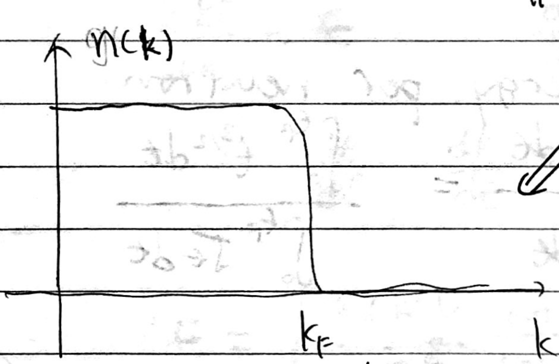
$\Rightarrow$ ~~nucleus~~ nucleons behave like hard spheres — why

$\therefore$ the nuclear force is only present in very close distance from the nucleus

a) Density of states $g(k)\,dk = 2 \times \frac{V}{(2\pi)^3} 4\pi k^2 dk$

(2 spins)

$$\therefore g(k)\,dk = \frac{V}{\pi^2} k^2 dk$$



fermi density distribution at $T = 0K$

$$\therefore N = \int_0^{k_F} g(k)\,dk = \frac{V}{\pi^2}\int_0^{k_F} k^2 dk = \frac{V k_F^3}{3\pi^2}$$

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (r_0 A^{1/3})^3 = \frac{4}{3}\pi r_0^3 A$$

$\therefore N = \frac{4}{9\pi} r_0^3 A k_F^3$ $\quad\because \varepsilon_F = \frac{\hbar^2 k_F^2}{2m} \Rightarrow k_F = \left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{\frac{1}{2}}$

$$\therefore N = \frac{4}{9} r_0^3 A \left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{3/2}$$

$\therefore \left(\frac{9\pi}{4 r_0^3}\frac{N}{A}\right)^{2/3} = \frac{2m\varepsilon_F}{\hbar^2}$ $\quad\therefore \varepsilon_F = \frac{\hbar^2}{2m r_0^2}\left(\frac{9\pi}{4}\frac{N}{A}\right)^{2/3}$

$\because N \approx Z$, $\therefore \frac{N}{A} \approx \frac{Z}{A} \approx \frac{1}{2}$

$\therefore \epsilon_F = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi}{8}\right)^{2/3}$ ✓

(b) for ^{16}O , $N=8$, $Z=8$, $A=16$

For neutrons, ~~$g(k)dk$~~ $\because \epsilon \sim k^2$ $\therefore d\epsilon \sim 2k dk$

$\Rightarrow k \sim \sqrt{\epsilon}$ $\therefore dk \sim \frac{d\epsilon}{\sqrt{\epsilon}}$

$$g(k)dk \sim k^2 dk \sim \epsilon \frac{d\epsilon}{\sqrt{\epsilon}} \sim \sqrt{\epsilon}\, d\epsilon$$ ✓

$\because$ internal energy per neutron

$$\frac{U_N}{N} = \frac{\int_0^{\epsilon_F} g(\epsilon)\epsilon\, d\epsilon}{\int_0^{\epsilon_F} g(\epsilon)\, d\epsilon} = \frac{\int_0^{\epsilon_F} \epsilon^{3/2} d\epsilon}{\int_0^{\epsilon_F} \sqrt{\epsilon}\, d\epsilon}$$

$$= \frac{\frac{2}{5}\epsilon_F^{5/2}}{\frac{2}{3}\epsilon_F^{3/2}} = \frac{3}{5}\epsilon_F$$ ✓

$\therefore U_N = \frac{3}{5}\epsilon_F N$

Same numbers of protons, assume $m_p \approx m_n$

$\because$ total kinetic energy

$U = 2U_N = \frac{6}{5}\epsilon_F N$

$= \frac{6}{5}\left(\frac{\hbar^2}{2m_n r_0^2}\right)\left(\frac{9\pi}{8}\right)^{2/3} \times 8 = 319 \text{ MeV}$ ✓

(c) # of states available to a particle confined to the volume V of a nucleus up to a momentum $\hbar k$ is

$$n = \frac{4\pi k^3}{3} \frac{V}{(2\pi)^3}$$

Near fermi level, the level spacing is

$$\Delta E = \left(\left.\frac{dn}{dE}\right|_{k_F} \right)^{-1} = \left(\frac{4\pi k^2 V}{(2\pi)^3} \left.\frac{dk}{dE}\right|_{k_F} \right)^{-1}$$

$$\frac{\hbar^2 k^2}{2m} = E \qquad \frac{\hbar^2 k\, dk}{m} = dE$$

$$\therefore \frac{dk}{dE} = \frac{m}{\hbar^2 k}$$

$$\therefore \Delta E = \left(\frac{4\pi k^2 V}{(2\pi)^3} \frac{m}{\hbar^2 k_F} \right)^{-1} = \frac{(2\pi)^3 \hbar^2}{4\pi k_F m V}$$

(ΔE = energy between adjacent levels near fermi energy)

If a nucleus Z, $N = A - Z$, $|N - Z| \ll A$

one proton is changed to a neutron, then it has to be moved up the ladder of energy levels to find an unoccupied state
($\because$ they are fermions)

$\because$ Proton/neutron occupancy $= 2$

This must be wrong! you assume ΔE is linear so you lower on and you increase the other net effect should be 0!

$\therefore$ the number of steps $= \frac{N-Z}{2}$ = ~~...~~

$\therefore$ The energy required to change from (Z, N) to $(Z-1, N+1)$ is

$$\frac{N-Z}{2} \Delta E ,$$

in this case $\Delta Z = -1$

$$\therefore \quad -\left(\frac{dE}{dZ}\right)_A = \frac{N-Z}{2} \Delta E = \frac{A-2Z}{2} \Delta E$$

At $Z = N = \frac{A}{2}$ the asymmetry energy $= 0$

$\therefore$ total asymmetry energy

$$E_A = \int_{A/2}^{Z} \left(\frac{dE}{dZ}\right)_A dZ = -\int_{\frac{A}{2}}^{Z} \frac{A-2Z}{2} \Delta E \, dZ$$

$$= \frac{(A-2Z)^2}{8} \Delta E = \underbrace{\left(A \frac{\Delta E}{8}\right)}_{a_A} \frac{(N-Z)^2}{A}$$

$\because$ $r = r_0 A^{1/3}$

$$\therefore \quad a_A = A \frac{\Delta E}{8} = \frac{3\pi}{16} \left(\frac{8}{9\pi}\right)^{\frac{1}{3}} \frac{\hbar^2}{r_0^2 m_n} \simeq 11 \text{ MeV}$$

This is not going to be very accurate $\because$ we need ~~at~~ to add the contribution from the fact that p-n force being more attractive than p-p, n-n force. ✓

(+-)

A, Z assumed to be continuous variables

3. (a)

No longer homogeneously charged spheres

These nuclei are small so SEMF are not very good. Yes, but why?

less continuous / too much discreteness

b) stability depends on the Q-value

$$Q = M(Z,A)c^2 - \sum_i M_i(Z_i, A_i)c^2$$

$$M(Z,A) = ZM_p + (A-Z)M_n - \frac{B}{c^2}$$ ✓

$\Rightarrow {}^9_4Be$ ~~M(4,9) =~~

~~$Q = M(4,9)c^2 - 4m_pc^2 - 5m_nc^2$~~

=

light nucleus consider ~~β~~ β- decay

$${}^9_4Be \rightarrow {}^9_5B + e^- + \bar{\nu}$$

$$Q = 4m_pc^2 + 5m_nc^2 - B(4,9) - 5m_p^2 - 4m_n^2 + B(5,9) \mp m_ec^2$$

$$= -m_pc^2 + m_nc^2 - B(4,9) + B(5,9) \mp m_ec^2$$

$$= (m_n - m_p - m_e)c^2 - B(4,9) + B(5,9)$$

$$= (939.6 - 938.3 - 0.511)\,MeV - 57.5\,MeV + 54.5\,MeV$$

$$= -2.21\,MeV < 0 \Rightarrow \text{stable}$$ ✓

but need to check β^+ and α-decay or p or n emission as well

$^{11}_{4}Be \Rightarrow$ light nucleus, β- decay

$^{11}_{4}Be \rightarrow \, ^{11}_{5}B + e^- + \bar{\nu}$

$Q = (m_n - m_p - m_e)c^2 - B(4,11) + B(5,11)$

$= 14.89$ MeV $> 0 \Rightarrow$ unstable ✓

c) Decay mechanism : $^{10}_{4}Be \rightarrow \, ^{10}_{5}B + e^- + \bar{\nu}$

β-decay

$Q = (m_n - m_p - m_e)c^2 - B(4,10) + B(5,10)$

$= -0.411$ MeV ✓

∴ ~~Need to~~ For decay to occur, $Q > 0$

∴ need to adjust up by 0.411 MeV ✓

d) Electron capture by $^{7}_{4}Be$: $^{7}_{4}Be + e^- \Rightarrow \, ^{7}_{3}Li + \nu$

$Q = 4m_p c^2 + 3m_n c^2 - B(4,7) + m_e c^2 - 3m_p c^2 - 4m_n c^2 + B(3,7)$

$= (m_p c^2 - m_n c^2 + m_e c^2) - B(4,7) + B(3,7)$

$= 1.81$ MeV $> 0 \Rightarrow$ consistent

β^+ decay should not be allowed

$^7_4Be \rightarrow ^7_3Li + e^+ + \bar{\nu}$

$Q = ~~4m_p~~ \; 4m_p + 3M_n - 3m_p - 4m_n - m_e - B(4.7)$
$+ B(3.7)$

$= (m_p - m_n - m_e) - B(4.7) + B(3.7)$

$= 0.79 \text{ MeV} > 0$

inconsistent because this β^+ decay is ~~not~~ allowed in SEMF and should have $Q<0$

e) It is unstable because twice the binding energy of ^4_2He is greater than the binding energy of ^8_4Be ✓

~~system maximum maximising binding energy be by.~~

~~^8_4Be~~

(7)

4. Decay rate per nucleus is — where from?

$$R = \frac{1}{\tau} = a e^{-2G}$$

$$\therefore \log R = \log(a) - (2\log e) G = \log(a) - 2G$$

G is the Gamow factor

$$G = 2\,\text{MeV}^{1/2} \frac{Z}{\sqrt{Q}}$$

$$\therefore \log R = \underbrace{\log a}_{A} - \underbrace{4\,\text{MeV}^{1/2}}_{B} \frac{Z}{\sqrt{Q}}$$ ✓

Rate for ground state R_g

$$\log R_g = 132.8 - 3.97 \times 96 / \sqrt{5.902}$$

$$\therefore R_g = \exp\left(1.328\times 10^{2} - \frac{3.97 \times 96}{\sqrt{5.502}}\right) = 3.49 \times 10^{-11}\,s^{-1}$$

$$R = \text{mean rate} = \frac{R_g}{76.7\%} = 4.55 \times 10^{-11}\,s^{-1}$$

$$\text{lifetime} \quad \tau = \frac{1}{R} = 2.2 \times 10^{10}\,s = 697 \text{ years}$$

✓

$$R_{6^+} = \exp\left(132.8 - \frac{3.97 \times 96}{\sqrt{5.902 - 0.294}}\right)$$

$$= 6 \times 10^{-13}\ s^{-1}$$

$$\text{branching ratio} = \frac{6 \times 10^{-13}}{4.55 \times 10^{-11}} = 1.3\% \gg 0.0036\%$$

→ The reason for this discrepancy is that in the radial schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2 U}{dr^2} + \left\{\frac{zZe^2}{4\pi\varepsilon_0 r} + \frac{l(l+1)\hbar^2}{2mr^2}\right\} U = Q_\alpha U$$

$$\left(U = \frac{R(r)}{r}\right)$$

the term $\frac{l(l+1)\hbar^2}{2mr^2}$ puts an extra angular momentum barrier in addition to the ~~Cot~~ Coulomb term.

larger l → higher barrier → harder to penetrate → less likely to have alpha decay.

(f)

5.

→ The Coulomb and pairing terms (+ asym. term) are responsible for thermal neutron

$$^{238}U + n \rightarrow {}^{239}U^* \rightarrow {}^{238}U + \gamma$$

→ No fission

$$^{235}U + n \rightarrow {}^{236}U^* \rightarrow \begin{cases} \text{fission } 84\% \\ {}^{236}U + \gamma \ 16\% \end{cases}$$

→ fission

The reason is that a transion from an even-odd ~~newde~~ nucleus to an even-even nucleus releases the pairing energy so that the nucleus has now sufficient energy to overcome the Coulomb barrier. ✓

maximising binding energy we have for the most probable Z:

$$\left(\frac{\partial B}{\partial Z}\right)_A = 0 \Rightarrow \frac{N}{Z} = 1 + \frac{a_c}{2a_A} A^{2/3}$$

$(N+Z=A)$

$a_A = 23.285$ MeV
$a_c = 0.697$ MeV

∴ for $A = 140$ → $\frac{N}{Z} \approx 1.4$ ⇒ $Z = 58$

for $A = 93$ → $\frac{N}{Z} = 1.3$ ⇒ $Z = 41$ ✓

daughter nuclei $^{140}_{58}Ce$, $^{93}_{41}Nb$

daughter same neutron proton ratio,
but as A↓, need N close to Z

The fission process

= # of e^-

$$^{235}_{92}U \rightarrow {}^{140}_{58}Ce + {}^{93}_{41}Nb + 2n + 7e^- + 7\bar{\nu}$$

(SEMF $M(Z,A) = Zm_p + (A-Z)m_n - 15.56A + 17.23A^{2/3} + 0.697\frac{Z^2}{A^{1/3}} + 23.285\frac{(A-2Z)^2}{A} \mp 12.0$)

The energy released = Q

~~$Q = 92m_p + 203m_n$~~

~~$Q = 92m_p + 203m_n - 15.56 \times 295$~~

$Q = M(92, 235) - M(58, 140) - M(41, 93) - 2m_n - 7m_e$

amu in $\frac{MeV}{c^2}$

$= 235.04 \times 931.5 - 139.9 \times 931.5 - 92.9 \times 931.5 - 2 \times 939.6 - 7 \times 0.511$

$= \underline{204 \text{ MeV}}$ ✓

Atomic mass of $^{235}_{92}U$ is

$\delta m = 235.04 \times 931.5 \times (1.6 \times 10^{-19}) \times 10^6 / (3 \times 10^8)^2$

$= 3.9 \times 10^{-25}$ kg ✓

rate of consumption of ^{235}U is

$$\frac{dm}{dt} = \frac{dm}{dE}\frac{dE}{dt} = \frac{\delta m}{\delta E}P = \frac{3.9 \times 10^{-25} \text{kg}}{204 \times 1.6 \times 10^{-19} \times 10^6} \times 1 \times 10^9$$

$= \underline{1.19 \times 10^{-5} \text{ kg/s}}$ ✓

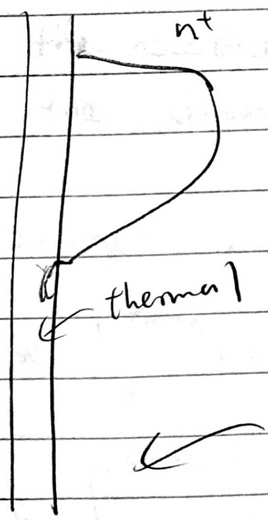
→ water, heavy water and graphite are moderators.

They are used for slowing down neutrons to thermal energy before they can be captured

They are suitable because they ~~cot~~ contain lots of nuclei with similar mass as the neutron (^1H), and have appropriate ~~cross~~ cross-section area

→ fissile material is not completely ~~mi~~ mixed up with the moderator because it mixing is complete then neutrons would be captured before they slow down. ✓

(+)



if mixed
fast neutron would still have chance to pass through U^{238}.

~~[illegible]~~

we need a thin rod.

When fast, should not near ~~[illegible]~~ U