To: Alfons Weber

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B4 Problem Set 1

1. $E = \gamma m c^2 \qquad p = \gamma m v = \cancel{\gamma m c}\ \gamma m c \beta$

$$\therefore E^2 - c^2 p^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 c^4 \beta^2 = m^2 c^4 (\underbrace{\gamma^2(1-\beta^2)}_{1})$$

$$= m^2 c^4$$

Total momentum 4-vector $P_t = \begin{pmatrix} E_t/c \\ p_t \end{pmatrix}$

$$\therefore P_t \cdot P_t = \text{invariant} = -\frac{E_t^2}{c^2} + p_t^2$$

$$\Rightarrow E_t^2 - c^2 p_t^2 = \text{invariant}$$

$P_1' \cdot P_2' = P_1 \cdot P_2$

In cm frame $P_t' \cdot P_t' = \frac{1}{c}\begin{pmatrix} E_{t,cm} \\ 0 \end{pmatrix} \cdot \frac{1}{c}\begin{pmatrix} E_{t,cm} \\ 0 \end{pmatrix}$

$P_T = P_1 + P_2 + \dots + P_N$

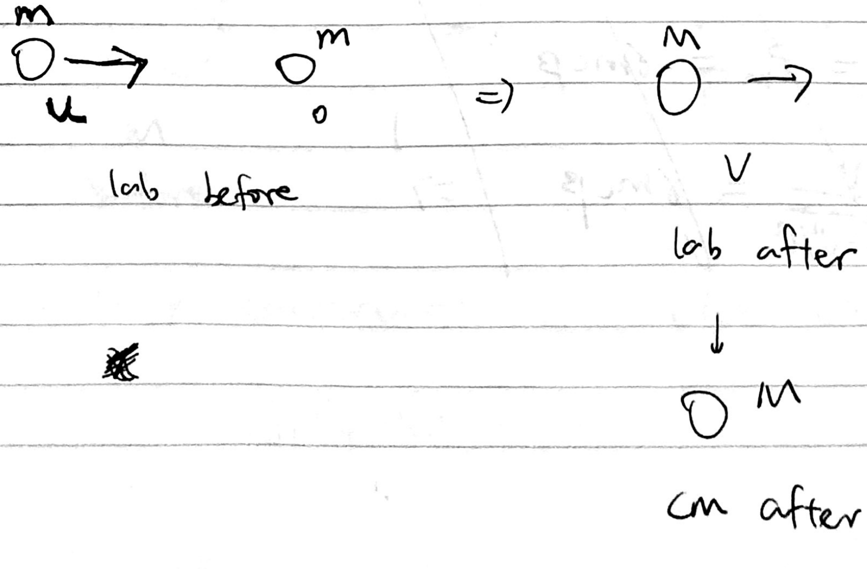
$P_T^2 = \text{const.}$

$$= -\frac{E_{t,cm}^2}{c^2}$$

$\because$ In CM frame total momentum is 0

why

$\therefore E_t^2 - c^2 p_t^2 = E_{t,cm}^2$ = Available Energy in CM frame



Use the invariant

$$E_t^2 - P_t^2 c^2 = M^2 c^4$$

$$\Rightarrow \quad \cancel{(\gamma mc)} \quad (\gamma mc^2 + mc^2)^2 - (\gamma m\beta c)^2 c^2 = M$$

$$\left(\gamma = (1 - u^2/c^2)^{-\frac{1}{2}} \quad \beta = \frac{u}{c} \right)$$

$$\therefore \; (\gamma+1)^2 m^2 \cancel{c^4} - \gamma^2 \beta^2 m^2 \cancel{c^4} = M^2 \cancel{c^4}$$

$$\rightarrow \quad M^2 = ((\gamma+1)^2 - \gamma^2\beta^2) m^2$$

$$= (\gamma^2 + 2\gamma + 1 - \gamma^2\beta^2) m^2$$

$$= (\underbrace{\gamma^2(1-\beta^2)}_{1} + 2\gamma + 1) m^2$$

$$= 2(\gamma+1) m^2$$

$$\therefore \quad M = m\sqrt{2(\gamma+1)}$$

~~Conservation of Energy ~~momentum~~ :~~

~~momentum of M is~~

$$\cancel{P_M = P_m = \gamma m c \beta}$$

$$\cancel{\therefore \frac{Mv}{\sqrt{1-v^2/c^2}} = \gamma m c \beta \Rightarrow}$$

Use conservation of energy.

$$\gamma m \cancel{c^2} + m \cancel{c^2} = \frac{1}{\sqrt{1-v^2/c^2}} M \cancel{c^2}$$

$$\therefore \quad \frac{(\gamma+1)m}{M} = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-v^2/c^2}} = \frac{(\gamma+1)\cancel{m}}{\sqrt{2(\gamma+1)}\,\cancel{m}} = \sqrt{\tfrac{1}{2}(\gamma+1)}$$

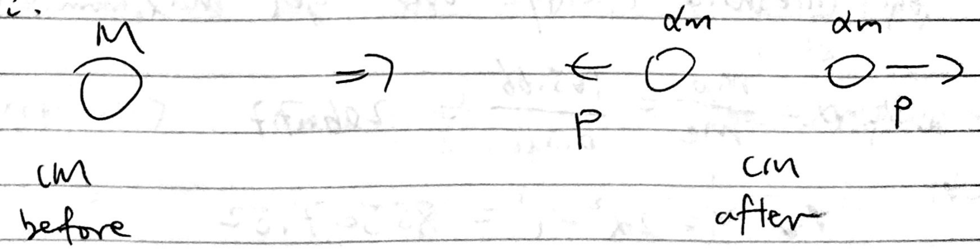
$$\therefore \quad \sqrt{\frac{2}{\gamma+1}} = \sqrt{1-\frac{v^2}{c^2}} \quad \Rightarrow \quad 1-\frac{v^2}{c^2} = \frac{2}{\gamma+1}$$

$$\therefore \quad \frac{v^2}{c^2} = 1-\frac{2}{\gamma+1} = \frac{\gamma+1-2}{\gamma+1} = \frac{\gamma-1}{\gamma+1}$$

$$\therefore \quad \boxed{v = c\sqrt{\frac{\gamma-1}{\gamma+1}}}$$

In the CM frame two particles after breakdown have equal ~~vel~~ speed.

$\therefore$



Energy conservation: (let $c=1$)

$$2\sqrt{p^2+\alpha^2 m^2} = M$$

$\therefore$ ~~$4p^2 + 4\alpha^2 m^2 = M^2 = 2(\gamma+1)$~~ m^2

$\Rightarrow$ ~~$\therefore p^2 =$~~

$\Rightarrow \quad 4p^2 + 4\alpha^2 m^2 = M^2 = 2(\gamma+1) m^2$

$\therefore \quad p^2 = \frac{1}{2}(\gamma + 1 - 2\alpha) m^2$

take square root and put c back :

$$P = mc\sqrt{\frac{1+\gamma-2\alpha^2}{2}}$$

P is the magnitude of momentum, so $P \geq 0$

maximum α occurs for $P = 0$, in which case

$1 + \gamma - 2\alpha^2 = 0 \quad \Rightarrow \quad \alpha = \sqrt{\frac{1+\gamma}{2}}$

In threshold energy we get maximum α

$\therefore \alpha = \frac{m_\mu}{m_e} = \frac{105.66}{0.511} = 206.77$

$\gamma = 2\alpha^2 - 1 = 85507.52$

Threshold energy $E_{th} = \gamma m_e c^2 = 4.37 \times 10^4$ MeV

+(-)

2. Consider scattering potential V and H_0 is the Hamiltonian if no scattering centre is present

Then TISE $(H_0 + V)|\psi\rangle = E|\psi\rangle$

$$\therefore \quad (H_0 - E)|\psi\rangle = -V|\psi\rangle \quad ①$$

We try to rewrite this $|\psi\rangle = \frac{V}{E-H_0}|\psi\rangle$, but since operator $\frac{V}{E-H_0}$ can be singular, and we are always to add a solution to $(H_0 - E)|\phi\rangle = 0$ to the Solution for $|\psi\rangle$

→ we write ① as follows:

$$|\psi\rangle = |\phi\rangle + \frac{V}{E-H_0 \pm i\epsilon}|\psi\rangle \quad ②$$

where ϵ is small and real

→ This is the Lippmann-Schwinger equation

Note that since $E-H_0$ acting on $\frac{1}{E-H_0}|\psi\rangle$ should return $|\psi\rangle$, and $\frac{1}{E-H_0 \pm i\epsilon}$ is the replacement of $\frac{1}{E-H_0}$, we can say that $\frac{1}{E-H_0 \pm i\epsilon}$ is the Green's function of $E-H_0$

We consider the 1-Dimensional case and use the position representation

$$\psi(x) = \langle x|\psi\rangle = \langle x|\phi\rangle + \langle x|\frac{V}{E-H_0 \pm i\epsilon}|\psi\rangle$$

$|\phi\rangle$ is the incoming plane wave, so

$$\langle x|\phi\rangle = e^{ikx}$$

$|A\rangle = \frac{1}{E-H_0 \pm i\epsilon}|\psi\rangle$

$(E-H_0 \pm i\epsilon)|A\rangle = |\psi\rangle$

For the second term

integral operator

$$\langle x|\frac{V}{E-H_0 \pm i\epsilon}|\psi\rangle = \int \langle x|\frac{1}{E-H_0 \pm i\epsilon}|x'\rangle\langle x'|V|\psi\rangle dx'$$

$$= \int G_{\pm}(x,x',E)\, V(x')\, \psi(x')\, dx'$$

where $G_{\pm}(x,x',E) = \langle x|\frac{1}{E-H_0 \pm i\epsilon}|x'\rangle$ is the Greens function of $E-H_0$, with

$$(E-H_0)\, G_{\pm}(x,x',E) = \delta(x-x')$$

To evaluate $G_{\pm}$, we insert the momentum identity operator

$$G_{\pm} = \iint \langle x|p\rangle\langle p|\frac{1}{E-H_0 \pm i\epsilon}|p'\rangle\langle p'|x'\rangle dp\,dp'$$

$\because$ $H_0|p\rangle = \frac{p^2}{2m}|p\rangle$ $\therefore$ ~~$\langle p|\frac{1}{E-H_0 \pm i\epsilon}|p'\rangle$~~

$$\therefore \langle p | \frac{1}{E - H_0 \pm i\epsilon} | p' \rangle = \frac{1}{E - \frac{p^2}{2m} \pm i\epsilon} \langle p | p' \rangle$$

$$= \frac{1}{E - \frac{p^2}{2m} \pm i\epsilon} \delta(p - p')$$

$$\therefore G_\pm = \iint \langle x | p \rangle \frac{1}{E - \frac{p^2}{2m} \pm i\epsilon} \delta(p - p') \langle p' | x' \rangle \, d^3p \, d^3p'$$

$$= \int \langle x | p \rangle \frac{1}{E - \frac{p^2}{2m} \pm i\epsilon} \langle p | x' \rangle \, dp$$

$|A\rangle = \frac{1}{E - H_0 \pm i\epsilon} |P\rangle$

use $\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

$$\langle p | x' \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx'/\hbar}$$

$(E - H_0 \pm i\epsilon)|A\rangle = |P\rangle$

$(E - H_0 + i\epsilon)|P\rangle = (E - \frac{p^2}{2m} + i\epsilon)|P\rangle$

$$\therefore G_\pm = \frac{1}{2\pi\hbar} \int \frac{\exp\left(\frac{ip}{\hbar}(x - x')\right)}{E - \frac{p^2}{2m} \pm i\epsilon} \, dp$$

Now let $k^2 = \frac{2mE}{\hbar^2}$, $q = \frac{p}{\hbar}$, $x_0 = x - x'$

$\rightarrow dq = \frac{1}{\hbar} dp \rightarrow dp = \hbar \, dq$

$$G_\pm = \frac{1}{2\pi\hbar} \left(\int \frac{e^{iqx_0} \, dq \, (\hbar)}{\frac{2mE}{\hbar^2} - \frac{p^2}{\hbar^2} \pm \frac{i\epsilon 2m}{\hbar^2}} \right) \left(\frac{2m}{\hbar^2} \right)$$

$\because$ ϵ is an arbitrary small quantity

$\therefore$ ignore the ~~sca~~ scaling $\frac{i\epsilon 2m}{\hbar^2} = i\epsilon$

$$\therefore \quad G_\pm = \frac{m}{\pi\hbar^2}\int dq \frac{e^{iqx_0}}{k^2 - q^2 \pm i\epsilon}$$

Note that $k^2 - q^2 \pm i\epsilon \approx k^2 - q^2 \pm i\epsilon + \left(\frac{i\epsilon}{2k}\right)^2$

$$= \left[k^2 \pm i\epsilon + \left(\frac{i\epsilon}{2k}\right)^2\right] - q^2$$

$$= \left(k \pm \frac{i\epsilon}{2k}\right)^2 - q^2 = \left(k + q \pm \frac{i\epsilon}{2k}\right)\left(k - q \pm \frac{i\epsilon}{2k}\right)$$

We can do this because ϵ is small and adding a quantity of order $O(\epsilon^2)$ is effectively adding 0

~~Agg~~ Again we ignore the rescaling $\frac{i\epsilon}{2k} \approx i\epsilon$

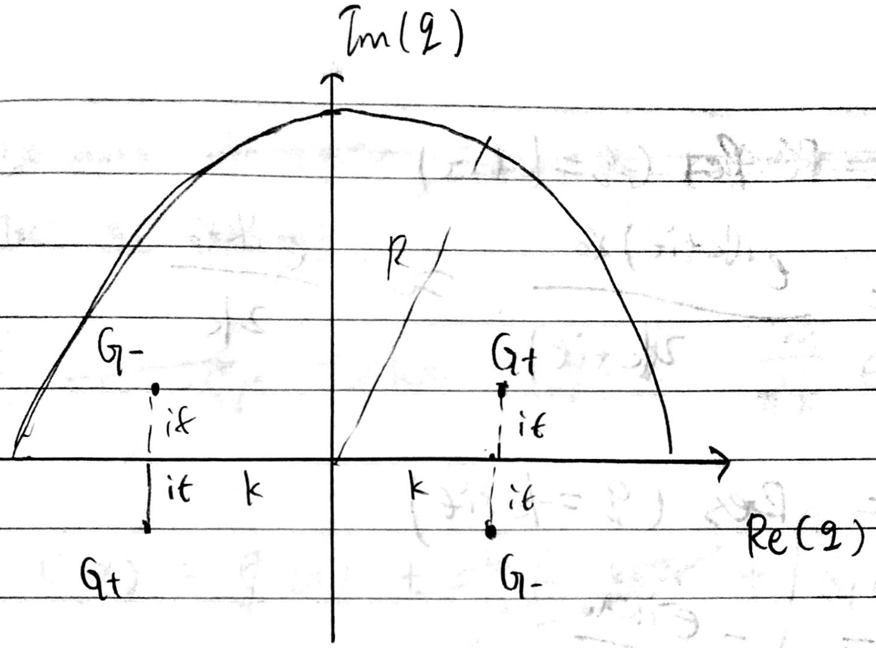
$$\therefore \quad k^2 - q^2 \pm i\epsilon \approx (k + q \pm i\epsilon)(k - q \pm i\epsilon)$$

$$= \cancel{-(k+q \mp i\epsilon)(}$$

$$= -(q + k \pm i\epsilon)(q - k \mp i\epsilon)$$

$$\therefore \quad G_\pm = -\frac{m}{\pi\hbar^2}\int dq \frac{e^{iqx_0}}{(q + k \pm i\epsilon)(q - k \mp i\epsilon)}$$

To evaluate the integral we use the Cauchy's Residue Theorem



The poles are at :

$G_+ \Rightarrow \quad q = -k - i\epsilon \; , \; q = k + i\epsilon$

$G_- \Rightarrow \quad q = -k + i\epsilon \; , \; q = k - i\epsilon$

We choose the contour of a semicircle shown with its radius $R \to \infty$

$\because$ The integrand is proportional to e^{iqx_0} and e^{iqx_0} will tend to 0 if q has a large positive imaginary part

$\therefore$ The contour of upper semicircle is 0

$$\therefore \int_{-\infty}^{\infty} dq \frac{e^{iqx_0}}{(q+k\pm i\epsilon)(q-k\mp i\epsilon)} = 2\pi i \times Res_{\pm}$$ (residue of poles enclosed by semicircle)

~~$= 2\pi i ($~~

$$\text{Res}_+ = \cancel{R_0} \text{ Res } (q = k + i\epsilon)$$

$$= \frac{e^{i(k+i\epsilon)x_0}}{2(k+i\epsilon)} \approx \frac{e^{ikx_0}}{2k}$$

$$\text{Res}_- = \text{Res } (q = -k + i\epsilon)$$

$$= -\frac{e^{-ikx_0}}{2k}$$

but This is for $x_0 > 0$

If $x_0 < 0$, we need to use the lower ~~semicircle~~ semicircle

$$\rightarrow \text{ overall } \quad \text{Res}_\pm = \pm \frac{e^{\pm ik|x_0|}}{2k}$$

$$G_\pm = \mp \frac{m}{\pi \hbar^2} (2\pi i) \left(\frac{e^{\pm ik|x_0|}}{2k} \right)$$

$$= \mp \frac{im}{\hbar^2} \frac{e^{\pm ik|x_0|}}{\cancel{2}k}$$

$$= \mp \frac{im}{\hbar^2 k} e^{\pm ik|x_0|}$$

$e^{ik|x_0|}$ is wave travel ~~to~~ the ~~right~~ out from centre and

$e^{-ik|x_0|}$ is wave travel ~~from~~ into the centre ~~to the left~~

We only consider wave travel ~~to~~ the ~~right~~ out from centre

~~because incoming wave $\phi(x) = Ae^{ikx}$ is travelling to the right~~

$\Rightarrow$ $\therefore$ ~~G~~ take $G_+ = -\frac{im}{\hbar^2 k} e^{ik|x|} = -\frac{im}{\hbar^2 k} e^{ik|x-x'|}$

$$\psi(x) = \Phi(x) + \cancel{\int G_+} + \int G_+(x,x') V(x') \psi(x') \, dx'$$

$$\therefore \psi(x) = \Phi(x) - \frac{im}{\hbar^2 k} \int_{-\infty}^{\infty} e^{ik|x-x'|} V(x') \psi(x') \, dx'$$

QED.

b) use the Born Approximation $\Phi(x) = Ae^{ikx}$

and $\psi(x') \approx \Phi(x') = Ae^{ikx'}$

and $x \ll 0$ so that $|x-x'| = x'-x$

$$\therefore \psi(x) = Ae^{ikx} - \frac{im}{\hbar^2 k} \int_{-\infty}^{\infty} e^{ik(x'-x)} V(x') Ae^{ikx'} \, dx'$$

$$= Ae^{ikx} - \frac{im}{\hbar^2 k} Ae^{-ikx} \int_{-\infty}^{\infty} e^{2ikx'} V(x') \, dx'$$

$$\therefore \psi(x) \propto e^{ikx} + \left(\frac{-im}{\hbar^2 k} \int_{-\infty}^{\infty} e^{2ikx'} V(x') \, dx' \right) e^{-ikx}$$

coefficient in front of the reflected wave, which is e^{-ikx}, is $-\frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} e^{2ikx'} V(x')\, dx'$

$\therefore$ Reflection coefficient (for $\psi = e^{ikx} + r e^{-ikx}$)

$$R = \left|\frac{J_{reflected}}{J_{incident}}\right| = |r|^2 = \left(\frac{m}{\hbar^2 k}\right)^2 \left|\int_{-\infty}^{\infty} e^{2ikx} V(x)\, dx\right|^2$$

c) transmission coefficient is

$$T = 1 - R = 1 - \left(\frac{m}{\hbar^2 k}\right)^2 \left|\int_{-\infty}^{\infty} e^{2ikx} V(x)\, dx\right|^2$$

For delta potential $V(x) = -\alpha\delta(x)$:

$$T_\delta = 1 - \left(\frac{m}{\hbar^2 k}\right)^2 \left|\int_{-\infty}^{\infty} e^{2ikx} \alpha\,\delta(x)\, dx\right|^2$$

$$= 1 - \left(\frac{m}{\hbar^2 k}\right)^2 \alpha^2 = 1 - \frac{m^2\alpha^2}{\hbar^4 k^2}$$

$$= 1 - \frac{m^{\cancel{2}}\alpha^2 \cancel{\hbar^2}}{\hbar^{\cancel{4}}\, 2\cancel{m} E} = 1 - \frac{m\alpha^2}{2E\hbar^2}$$

Exact $T_\delta = \left(1 + \frac{m\alpha^2}{2\hbar^2 E}\right)^{-1} \simeq 1 - \frac{m\alpha^2}{2E\hbar^2}$

Agrees to first order.

For T_{sq} $\quad V(x) = \begin{cases} -V_0 & -a<x<a \\ 0 & \text{otherwise} \end{cases}$

$$\therefore \quad T_{sq} = 1 - R = 1 - \left(\frac{m}{\hbar^2 k}\right)^2 \left| \int_{-\infty}^{\infty} e^{2ikx} \, V(x) \, dx \right|^2$$

$$= 1 - \left(\frac{mV_0}{\hbar^2 k}\right)^2 \left| \int_{-a}^{a} e^{2ikx} \, dx \right|^2$$

$$\int_{-a}^{a} e^{2ikx} dx = \frac{1}{2ik}\left[e^{2ika} - e^{-2ika}\right]$$

$$= \frac{1}{2\cancel{i}k}\left(2\cancel{i} \sin(2ka)\right) = \frac{\sin(2ka)}{k}$$

$$\therefore \quad T_{sq} = 1 - \left(\frac{mV_0}{\hbar^2 k}\right)^2 \frac{\sin^2(2ka)}{k^2}$$

$$= 1 - \left(\frac{mV_0}{\hbar^2 k^2}\right)^2 \sin^2(2ka)$$

$$\because \quad k^2 = \frac{2mE}{\hbar^2} \quad \therefore \hbar^2 k^2 = 2mE$$

$$\therefore T_{sq} = 1 - \left(\frac{\cancel{m} V_0}{2\cancel{m}E}\right)^2 \sin^2\left(\frac{2a}{\hbar}\sqrt{2mE}\right)$$

$$\rightarrow \quad T_{sq} = 1 - \frac{V_0^2}{4E^2} \sin^2\left(\frac{2a}{\hbar}\sqrt{2mE}\right)$$

The exact solution:

$$T_{sq} = \left(1 + \frac{V_0^2}{4E(E+V_0)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)\right)^{-1}$$

$$\approx \left(1 + \frac{V_0^2}{4E^2} \sin\left(\frac{2a}{\hbar}\sqrt{2mE}\right)\right)^{-1}$$

$E >> V_0$

$$\rightarrow \approx 1 - \frac{V_0^2}{4E^2} \sin^2\left(\frac{2a}{\hbar}\sqrt{2mE}\right)$$

Agrees with the approximation. ✓

(+)

3.

The Klein-Gordon equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\Phi(\underline{x},t) = 0 \qquad ①$$

a) For $\Phi(\underline{x},t) = \exp(-iEt/\hbar + i\underline{p}\cdot\underline{x}/\hbar)$

$$= \exp\left(\frac{i}{\hbar}(\underline{p}\cdot\underline{x} - Et)\right)$$

Substitute this into the LHS of ①

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi + \frac{m^2c^2}{\hbar^2}\Phi$$

$$= \left(\frac{iE}{\hbar}\right)^2\frac{1}{c^2}\Phi - \left(\frac{i\underline{p}}{\hbar}\right)\cdot\left(\frac{i\underline{p}}{\hbar}\right)\Phi + \frac{m^2c^2}{\hbar^2}\Phi$$

$$= \left(-\frac{E^2}{\hbar^2c^2} + \frac{p^2}{\hbar^2} + \frac{m^2c^2}{\hbar^2}\right)\Phi$$

$$= -\left(E^2 - (p^2c^2 + m^2c^4)\right)\frac{\Phi}{\hbar^2c^2}$$

$\because$ For a relativistic particle $E^2 = p^2c^2 + m^2c^4$

$\therefore \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi + \frac{m^2c^2}{\hbar^2}\Phi = 0$ satisfied

b) $k \cdot x = k^{\mu} x_{\mu} = \cancel{k} g_{\mu\nu} k^{\mu} x^{\nu}$

$= \cancel{\cdot} (E/c\hbar)(ct) - \frac{\underline{p} \cdot \underline{x}}{\hbar} = \frac{1}{\hbar}(Et - \underline{p} \cdot \underline{x})$

$[\hbar] = J \cdot s \qquad [E] = J \quad [t] = s$

$\cancel{E} \therefore \frac{[E][t]}{[\hbar]} = \frac{J \cdot s}{J \cdot s} = 1 \rightarrow$ dimensionless

$[P] = kg \cdot m/_s = \cancel{E} \; J \cdot s \cdot m^{-1} \qquad [x] = m$

$\therefore \frac{[P][x]}{[\hbar]} = \frac{J \cdot s \cdot m^{-1} \cdot m}{J \cdot s} = 1 \rightarrow$ dimensionless ✓

c)

Substitute the trial solution into ~~equat~~ Klein Gordon operator

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x'-x)} G(k)$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ikx'} \left(\cancel{\partial} \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(e^{ikx}) - \nabla^2(e^{ikx}) + \frac{m^2c^2}{\hbar^2}(e^{ikx})\right)$$

$$\times G(k)$$

$\cancel{= \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{c^2} \ldots \right.}$

$\because kx = k_0 ct - \underline{k} \cdot \underline{x}$

∴ Above equals to

$$= \int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{c^2}(-k_0^2c^2) - (-\underline{k}\cdot\underline{k}) + \frac{m^2c^2}{\hbar^2}\right)$$

$$e^{-ik\cdot(x'-x)} \; G(k)$$

$$\because \quad G(k) = -\frac{1}{k_0^2 - \underline{k}\cdot\underline{k} - \frac{m^2c^2}{\hbar^2}}$$

∴ Above

$$= \int \frac{d^4k}{(2\pi)^4}\left(-\frac{1}{k_0^2 - \underline{k}\cdot\underline{k} - \frac{m^2c^2}{\hbar^2}}\right)\left(-k_0^2 + \underline{k}\cdot\underline{k} + \frac{m^2c^2}{\hbar^2}\right)$$

$$\times\, e^{-ik\cdot(x'-x)}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x'-x)} = \underline{\underline{\delta^4(x'-x)}}$$

∴ $G(x'-x)$ is the Green's function of Klein-Gordon equation.

4.

The radial part $R(r)$ of the wavefunction satisfies the radial Schrodinger equation

$$\left(-\frac{d^2}{dr^2} + \frac{2m}{\hbar^2}V\right)(u) = k^2(u)$$

where $u = rR(r)$ $\quad k^2 = \frac{2mE}{\hbar^2}$

~~For r<a~~ $\because$ $V(r) = \alpha\delta(r-a)$

$\therefore$ For $r<a$, $V=0$

$$\frac{d^2u}{dr^2} + k^2u = 0$$

$\therefore$ $u = A\cos(kr) + B\sin(kr)$

$\therefore$ $R = \frac{A\cos(kr)}{r} + \frac{B\sin(kr)}{r}$

to prevent R from blowing up at $r=0$

need $A=0$

$\therefore$ $R(r) = B\frac{\sin(kr)}{r}$ for $r<a$

For $r>a$, same equation holds

$$R = \frac{1}{r}\left(C'\cos(kr) + D\sin(kr)\right) = C\frac{\sin(kr+\delta_0)}{r}$$

$\cancel{= \frac{1}{r} C\frac{\sin(kr+\delta_0)}{r}}$

~~Bo~~ At $r=a$, $u(r)$ is continuous

$\therefore$ $B\sin(ka) = C\sin(ka+\delta_0)$ ①

Also at $r=a$

$$-\frac{d^2u}{dr^2} + \frac{2m\alpha}{\hbar^2}\delta(r-a)u = k^2u.$$

Integrate from $a-\epsilon$ to $a+\epsilon$ for $\epsilon \ll a$

$$-\int_{a-\epsilon}^{a+\epsilon}\frac{d^2u}{dr^2}\,dr + \frac{2m\alpha}{\hbar^2}\int_{-\infty}^{\infty}u(r)\,\delta(r-a) = k^2\int_{a-\epsilon}^{a+\epsilon}u(r)\,dr$$

~~take the li~~

take the limit as $\epsilon \to 0$ ($\because u$ is continuous)

$$\left.\frac{du}{dr}\right|_{a^-} - \left.\frac{du}{dr}\right|_{a^+} + \frac{2m\alpha}{\hbar^2}u(\alpha) = 0$$

$$\therefore \left.\frac{du}{dr}\right|_{a^+} - \left.\frac{du}{dr}\right|_{a^-} = \frac{2m\alpha}{\hbar^2}u(\alpha)$$

$$\therefore kC\cos(ka+\delta_0) - kB\cos(ka) = \frac{2m\alpha}{\hbar^2}B\sin(ka)$$

✓ ②

① $\rightarrow$ $$C = \frac{B\sin(ka)}{\sin(ka+\delta_0)}$$

$$\therefore \frac{k\cancel{B}\sin(ka)}{\sin(ka+\delta_0)}\cos(ka+\delta_0) - k\cancel{B}\cos(ka)$$

$$= \frac{2m\alpha}{\hbar^2}\cancel{B}\sin(ka)$$

$$\therefore k\sin(ka)\cot(ka+\delta_0) - k\cos(ka) = \frac{2m\alpha}{\hbar^2}\sin(ka)$$

$$\therefore \cot(ka+\delta_0) = \cot(ka) + \frac{2m\alpha}{k\hbar^2}$$

$\therefore$ For small ka and δ_0 ($ka \ll 1$, $\delta_0 \ll 1$)

We have

$$\frac{1}{ka+\delta_0} = \frac{1}{ka} + \frac{2m\alpha}{\hbar^2 k}$$

$$\therefore \cancel{1} = \cancel{1} + \frac{\delta_0}{ka} + \frac{2m\alpha}{\hbar^2 k}(ka+\delta_0)$$

$$\cancel{\Rightarrow \frac{\delta_0}{ka} = -}$$

$$\therefore 1 = 1 + \frac{\delta_0}{ka} + \frac{2m\alpha}{\hbar^2 k}(ka+\delta_0)$$

$$\therefore 0 = \frac{\delta_0}{ka} + \frac{2ma\alpha}{\hbar^2} + \frac{2ma\alpha}{\hbar^2}\left(\frac{\delta_0}{ka}\right)$$

$$\therefore \frac{\delta_0}{ka} + \Phi + \Phi\left(\frac{\delta_0}{ka}\right) = 0 \Rightarrow \frac{\delta_0}{ka} = -\frac{\Phi}{1+\Phi}$$

$$\therefore \delta_0 = \frac{-\Phi}{1+\Phi} ka.$$

Scattering amplitude for S-wave

$$f(\theta) = \frac{1}{k} e^{i\delta_0} \sin\delta_0 \underset{\uparrow}{=} \cancel{\frac{1}{k}\sin\delta_0} \frac{1}{k} \cdot 1 \cdot \delta_0 = \frac{\delta_0}{k}$$

for small δ_0

$$\Rightarrow f(\theta) = \frac{-a\Phi}{1+\Phi}$$

differential cross-section

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{a^2\Phi^2}{(1+\Phi)^2}$$

Cross-section

$$\sigma = \frac{4\pi}{k^2}\sin^2(\delta_0) = \frac{4\pi}{k^2}\delta_0^2 = 4\pi\left(\frac{\delta_0}{k}\right)^2$$

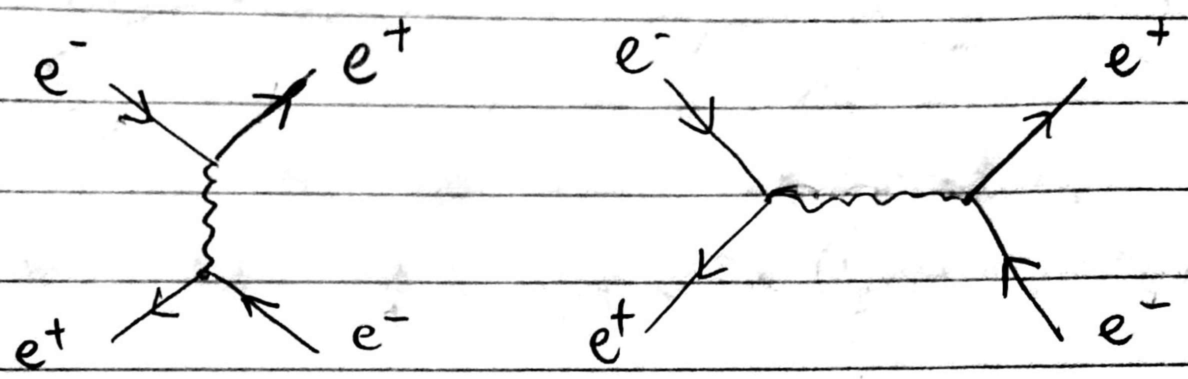
$$= \frac{4\pi\Phi^2 a^2}{(1+\Phi)^2}$$

7

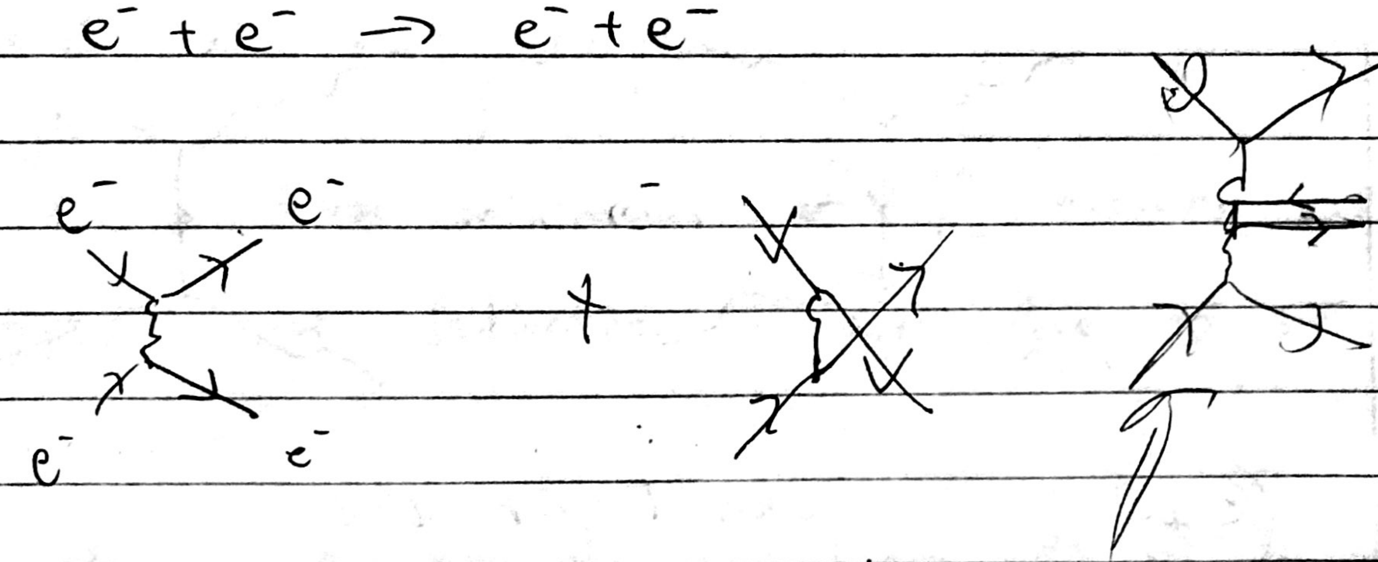
5. Feynman diagrams

a)

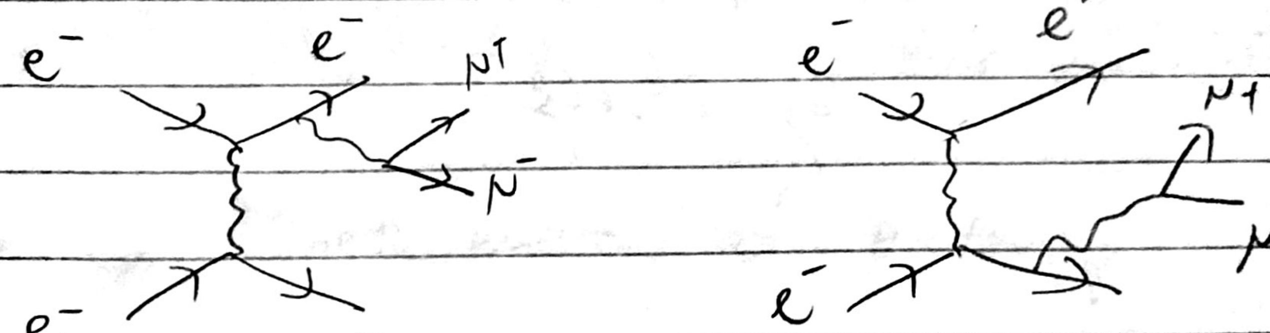
$e^- + e^+ \to e^- + e^+$

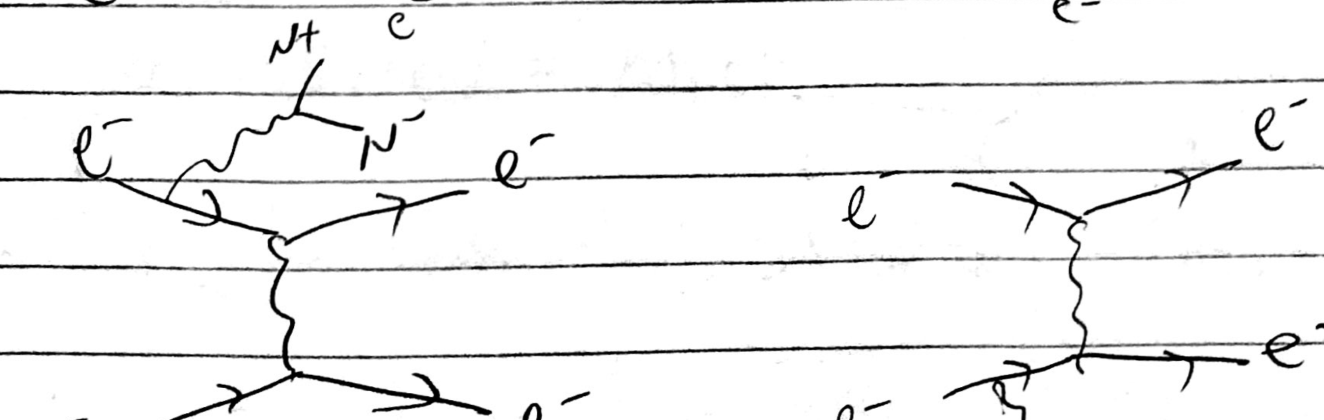


b) $e^- + e^- \to e^- + e^-$



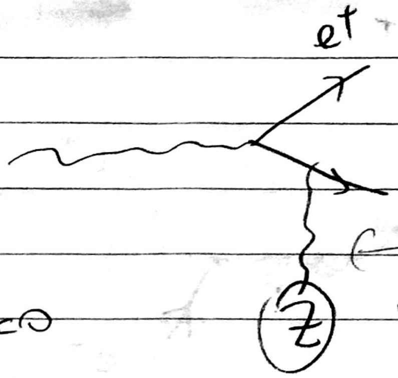
c) $e^- + e^- \to e^- + e^- + \mu^+ + \mu^-$

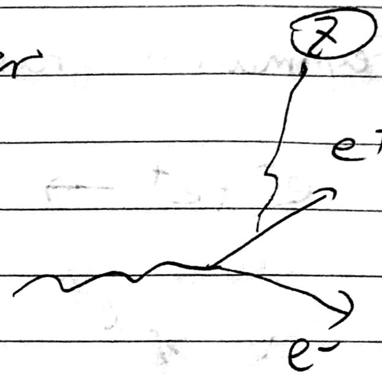




1 more

d) $\gamma \to e^+ + e^-$ in matter

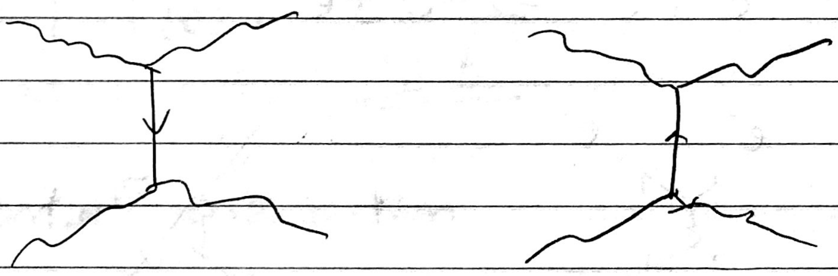




$p \cdot p = 0$ $\quad$ $+\, p \cdot p \neq 0$ $\quad$ ✓

unless another photon

e)

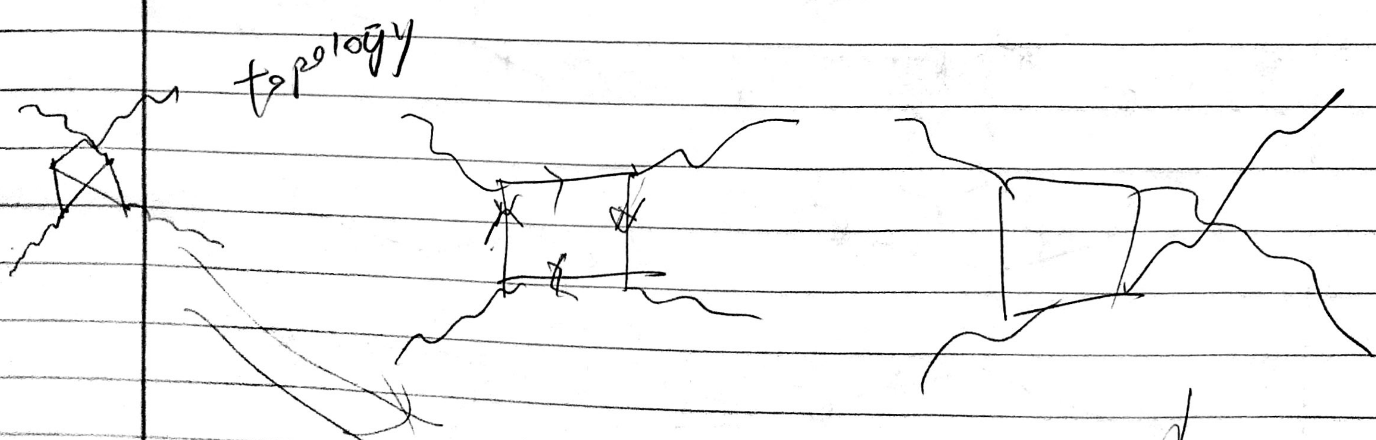


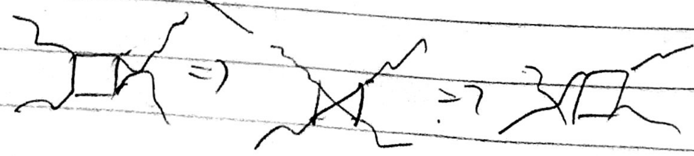
A+

+ -

can't create e^- from nothing by itself $\quad$ X

topology





identical particle

6.

$\because$ The Form Factor is the Fourier Transform of charge density ~~p(r)~~ $\rho(r)$ into k space

$\therefore$ If we double the radius, the scale in the Δk axis will be multiplied by $\frac{1}{2}$

Consider the first minimum of $\frac{\sigma}{\sigma_R}$ vs θ graph which corresponds to the first zero of the $|F(\Delta k)|^2$ vs. $r\Delta k$ graph

For Ag, first minimum of $\frac{\sigma}{\sigma_R}$ occurs at $\theta = 40°$ (the deflection angle due to scattering)

The magnitude of momentum ~~is~~ of incident proton is $T = P^2/2m$

$P = (E^2 - m_p^2 c^4)^{1/2} \left(\frac{1}{c}\right)$, $\because$ ~~Energy of proton~~ is

$\because$ Kinetic energy of proton is $T = 17\,MeV$

$\therefore T \ll 938\,MeV = M_p c^2$

$\therefore$ Take the classical limit, $T = \frac{p^2}{2m_p}$

$\therefore P = \sqrt{2Tm_p} = \sqrt{2(17)(938)} = 178.6\ MeV/c$

momentum transfer $\Delta P = 2P\sin\frac{\theta}{2}$

$= 2(178.6)\sin(20°) = 122.2\ MeV/c$

For a sphere of radius 1

At the first zero of the $|F(\Delta k)|^2$ vs. $r\Delta k$ graph,
~~$r\Delta k = 4$~~

we find $r\Delta k = 4$

$\therefore$ For a silver atom, the radius

$$r = \frac{4}{\Delta k} = \frac{4}{\Delta p/\hbar} = \frac{4\hbar}{\Delta p} = \cancel{\frac{4 \times 6.63 \times}{}}$$

$$= \frac{4 \times 1.05 \times 10^{-34}}{(122.2)(1.6 \times 10^{-19})} \frac{\times 3 \times 10^{8}}{\times 10^{6}} = 6.4 \times 10^{-15} m$$

$= 6.4\ fm$

expectation of ~~these~~ incompressible nucleus

$r = r_0 A^{1/3}$

A = atomic mass number of silver = 108

$\therefore$ $r = \cancel{(1.25)}\ (1.25 fm)(108)^{1/3} = 6.0\ fm$

close to ~~the sa~~ our estimate

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