QFT in Curved Spacetime

Problems 3

1. By evaluating the integral

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, \mathrm{e}^{-x^2}$$

in two different ways, show that the area of the n-sphere, A_n, is given by the expression

$$A_n \;=\; \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \; .$$

Let $G(x, x')$ denote the Feynman propagator for a massive scalar field. Thus

$$\langle 0 | T(\varphi(x)\varphi(x')) | 0 \rangle \;=\; -\mathrm{i} G(x, x') \quad \text{and} \quad (\Box - m^2) G(x, x') \;=\; -g^{-1/2}\, \delta(x, x') \; .$$

By making use of the representation

$$\frac{1}{k^2 + m^2} \;=\; \mathrm{i} \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-\mathrm{i}s(k^2 + m^2)} \; ,$$

(notice that this builds in the iϵ prescription for m^2) show that, in Minkowski space, for the standard vacuum

$$G(x, x') \;=\; \frac{1}{2\pi} \left(\frac{m^2}{8\pi^2 \sigma} \right)^{\frac{n-2}{4}} K_{\frac{n-2}{2}} \left(\sqrt{2 m^2 \sigma} \right)$$

where K_ν denotes the modified Bessel function and

$$\sigma \;=\; \frac{1}{2}(x - x')^2 + \mathrm{i}\epsilon \; .$$

Take the limit $m^2 \to 0$ to show that, in the massless case,

$$G(x, x') \;=\; \frac{1}{8\pi^2 \sigma} \; .$$

2. The Kontorovich-Lebedev transform (see Erdélyi et al. *Higher Transcendental Functions* vol II) is classically given by the pair of relations

$$
\begin{aligned}
f(\nu) &= \int_0^\infty \mathrm{d}\xi\, \tilde{f}(\xi) K_{\mathrm{i}\nu}(\xi) \ , \\
\tilde{f}(\xi) &= \frac{2}{\pi^2 \xi} \int_0^\infty \mathrm{d}\nu\, \nu \sinh(\pi\nu) K_{\mathrm{i}\nu}(\xi) f(\nu) \ .
\end{aligned}
$$

Show that these relations are equivalent to the relations

$$
\begin{aligned}
\int_0^\infty \frac{\mathrm{d}\xi}{\xi} K_{\mathrm{i}\nu}(\xi) K_{\mathrm{i}\nu'}(\xi) &= \frac{\pi^2}{2} \frac{\delta(\nu - \nu')}{\nu \sinh \nu\pi} \\
\int_0^\infty \mathrm{d}\nu\, \nu \sinh \nu\pi\, K_{\mathrm{i}\nu}(\xi) K_{\mathrm{i}\nu}(\xi') &= \frac{\pi^2}{2} \xi \delta(\xi - \xi') \ .
\end{aligned}
$$

Show that the modes

$$
v_{\mathbf{k}\,\nu}(x) = \frac{1}{2\pi^2} \sqrt{\sinh \nu\pi}\, \mathrm{e}^{-\mathrm{i}\nu\tau} K_{\mathrm{i}\nu}(\mu\xi) \mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{y}}
$$

are orthonormal with respect to the inner product

$$
(v_{\mathbf{k}\,\nu},\, v_{\mathbf{k}'\,\nu'}) = \mathrm{i} \int \mathrm{d}^2\mathbf{y} \int_0^\infty \frac{\mathrm{d}\xi}{\xi}\, v^*_{\mathbf{k}\,\nu} \frac{\overleftrightarrow{\partial}}{\partial\tau} v_{\mathbf{k}'\,\nu'} \ .
$$

Expand a scalar field in modes and define a Feynman propagator appropriate to the modes we have just defined. Show that this Feynman propagator is given by the expression

$$
G(x, x') = \int_0^\infty \frac{\mathrm{d}\nu}{\pi^2} \sinh \nu\pi \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} K_{\mathrm{i}\nu}(\mu\xi) K_{\mathrm{i}\nu}(\mu'\xi')\, \mathrm{e}^{-\mathrm{i}|\tau - \tau'| + \mathrm{i}\mathbf{k}.(\mathbf{y} - \mathbf{y}')} \ .
$$