Problem sheet 2a

1. To provide a model for a collapsing star, consider the $k = 1$ dust FRW universe
$$ds^2 = dT^2 - a(T)^2(d\psi^2 + \sin^2\psi d\Omega^2)\,,$$
where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the round sphere metric, and the Friedmann equations give for dust in terms of cosmological time T
$$\left(\frac{da}{dT}\right)^2 + 1 = \frac{8\pi\rho_0}{3a}\,.$$
Show that it is possible to glue the hypersurface $\psi = \psi_0$ to Schwarzschild
$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2 d\Omega^2\,,$$
along a timelike 3-surface $(t, r) = (t(T), R(T))$ such that the curves $(\theta, \phi) =$ constant are radial time-like geodesics and the metric is continuous.

2. Let $\mathcal{H}$ be a bifurcate Killing horizon associated to Killing vector k_a and let the surface gravity κ be defined by $k^a\nabla_a k_b = \kappa k_b$. Show that $\nabla_a k^b k_b = -2\kappa k_a$. Use the fact that on $\mathcal{H}$, k_a is hypersurface-orthogonal, i.e., $k_{[a}\nabla_b k_{c]} = 0$, and Killing to show that
$$\kappa^2 = -\tfrac{1}{2}(\nabla_a k_b)(\nabla^a k^b) \qquad (146)$$
Show that if the Killing horizon is bifurcate, then κ is constant. [*Hint: κ is constant up the generators, so work on the bifurcation surface where $k_a = 0$. First prove that for any Killing vector $\nabla_a\nabla_b k_c = -R_{bcad}k^d$.*]

3. Compute the surface gravity for the Reissner-Nordstrom metric
$$ds^2 = \frac{\Delta}{r^2}dt^2 - \frac{r^2}{\Delta}dr^2 - r^2 d\Omega^2\,, \qquad \Delta = (r - r_+)(r - r_-) = r^2 - 2mr + Q^2,$$
at the horizon $r = r_+ > r_-$.

[*Note that the original coordinates are singular at the horizon, whereas Eddington-Finkelstein coordinates make the horizon smooth with $dv = dt + \frac{r^2 dr}{\Delta}$ and $\partial/\partial t$ is $\partial/\partial v$ in these (v, r, θ, ϕ) coordinates.*]

4. Show that if a Killing vector k^a is thought of as a 1-form, $k_a dx^a$, then the 2-form ${}^*dk$ satisfies the identity

$$d^*dk = R_{ab}k^{a*}dx^b \,. \tag{147}$$

Assuming Einstein's equations, deduce that

$$\nabla^b J_b = 0 \,, \qquad J_a = T_{ab}k^b - \tfrac{1}{2}Tk_a \,, \quad T = T^a_a \,.$$

Evaluate the integral of ${}^*dk$ on a sphere of constant r in Reissner-Nordstrom. Interpret the answer.