Problem Sheet 1

1. Show that Maxwell's equations are conformally invariant under $g_{ab} \to \Omega^2 g_{ab}$. [*Hint: show that Hodge duality on 2-forms is conformally invariant.*]

   Show that the stress-energy tensor is trace-free for a conformally invariant theory.

   In d-dimensions, find the power of the conformal factor required to rescale a scalar field so that the action for the conformally invariant wave equation is scale invariant.

   Harder (optional): show that the wave equation with scalar curvature term $R/6$ is conformally invariant.

   [*Hint: prove invariance of the action up to boundary terms. For the wave equation, use the conformal variation of R from the trace of that for the Schouten tensor,* (110) *in the notes.*]

2. Show that in 2-component spinors, the stress-energy tensors for Maxwell theory is

$$T_{ab} = \phi_{AB}\bar{\phi}_{A'B'} ,$$

   and for the scalar massless wave equation ($a = 0$)

$$T_{ab} = \nabla_{AB'}\phi\nabla_{BA'}\phi .$$

   Hence prove the weak version of the dominant energy condition, that $T_{ab}l^a l^b \geq 0$ when l^a is a future-pointing null vector.

3. (This question is harder and optional, i.e. not required for homework completion:) Use the spinor Ricci identities to prove the relation

$$d(idx^{AA'}d\chi_A) = -\frac{1}{2}{}^*dx^b G_b^{AA'} \wedge \chi_A ,$$

   where χ_A is an indexed form of degree 0 or 1.

   Explain briefly how this relation justifies the consistency of the Rarita-Schwinger equations. [The point is that there are more Rarita-Schwinger equations than components of the field minus the number of gauge degrees of freedom. Show that the identity provides as many Bianchi identities for the field equation as this excess number of equations.]

Deduce the Sen-Witten identity

$$d(idx^{AA'}\bar{\xi}_{A'}d\xi_A) = -\frac{1}{2}G_b^{AA'}\bar{\xi}_{A'}\xi_A{}^* dx^b - idx^{AA'} \wedge d\bar{\xi}_{A'} \wedge d\xi_A\,. \quad (144)$$

This lies at the heart of the Witten positive energy proof. Show that the first term on the RHS is positive by the dominant energy condition when the Einstein equations are satisfied. Harder: show that if ξ_A satisfies the Witten equation on a space-like surface Σ, i.e., it satisfies the 4d massless Dirac equation together with the condition that $n^b\nabla_b\xi_A = 0$ where n^a is the normal to Σ, then the second term on the right is positive definite. [*This material is covered p430 on of Vol 2 of Penrose & Rindler's Spinors and Space-time; to complete the argument, the integral of the left hand side gives a boundary term by Stoke's theorem that gives the 4-momentum component along $\xi^A\bar{\xi}^{A'}$ at space-like infinity when ξ_A is asymptotically constant.*]

4. Prove the Sachs equation and geodesic deviation equation in terms of both $(\zeta, \bar{\zeta})$ and in terms of (ρ, σ) starting from the definitions in section §2.2. Show that in flat space with vanishing twist and shear, the expansion is given in terms of an affine parameter s by

$$\rho = \frac{\rho_0}{1 - \rho_0 s}\,,$$

where $\rho_0 = \rho(0)$. What is the interpretation of the blow up ρ at $s = 1/\rho_0$? More generally, show that if the dominant energy condition is satisfied in curved space, and $\rho_0 > 0$ then ρ blows up in finite time.

5. What is the Domain of dependence of the space-like hypersurface $t = 0$, $x > 0$ in Minkowski space.

Consider the metric

$$ds^2 = a^2\xi^2 d\tau^2 - d\xi^2 - dy^2 - dz^2\,, \qquad \xi > 0\,. \quad (145)$$

Find a coordinate transformation to the flat metric and explain its relation to the domain of dependence above. What are the trajectories $(\xi, y, z) =$ constant and the interpretation of the parameter a? [These are Rindler coordinates on which there is a good wikipedia entry.]

The Milne universe is somewhat analogous with metric

$$ds^2 = d\tau^2 - \tau^2 d\chi^2 - dy^2 - dz^2\,.$$

Give its relation to flat space. Is it globally hyperbolic? Why?