String Theory II

Problem Sheet 4

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[1] a) ∵ $R_{\mu\nu}{}^{\lambda}{}_{\sigma} = \partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\lambda_{\nu\tau}\Gamma^\tau_{\mu\sigma}$

and viel-bein $V^a = e^a_\mu V^\mu$, $V^\mu = e^\mu_a V^a$, $e^a_\mu e^\nu_a = \delta_\mu{}^\nu$

$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$

So $R_{\mu\nu}{}^{ab} = e^{a\rho}e^{b\sigma}R_{\mu\nu\rho\sigma} = e^{a\rho}e^{b\sigma}g_{\rho\lambda}R_{\mu\nu}{}^{\lambda}{}_{\sigma}$

~~$= e^{a\rho}e^{b\sigma}e^c_\rho e^d_\lambda \eta_{cd} R_{\mu\nu}{}^{\lambda}{}_{\sigma}$~~

$= e^{b\sigma}e^a_\lambda R_{\mu\nu}{}^{\lambda}{}_{\sigma}$

$= e^{b\sigma}e^a_\lambda(\partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\lambda_{\nu\tau}\Gamma^\tau_{\mu\sigma})$

$= e^{b\sigma}e^a_\lambda \partial_\mu \Gamma^\lambda_{\nu\sigma} - e^{b\sigma}e^a_\lambda \partial_\nu \Gamma^\lambda_{\mu\sigma} + e^{b\sigma}e^a_\lambda \Gamma^\lambda_{\mu\tau}\Gamma^\tau_{\nu\sigma}$
$- e^{b\sigma}e^a_\lambda \Gamma^\lambda_{\nu\tau}\Gamma^\tau_{\mu\sigma}$

Given that $\partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu}e^a_\rho + \omega_\mu{}^a{}_b e^b_\nu = 0$ (*)

we have:

$R_{\mu\nu}{}^{ab} = e^{b\sigma}\partial_\mu(e^a_\lambda \Gamma^\lambda_{\nu\sigma}) - e^{b\sigma}\partial_\nu(e^a_\lambda \Gamma^\lambda_{\mu\sigma})$
$- e^{b\sigma}\Gamma^\lambda_{\nu\sigma}(\partial_\mu e^a_\lambda) + e^{b\sigma}\Gamma^\lambda_{\mu\sigma}(\partial_\nu e^a_\lambda)$
$+ e^{b\sigma}e^a_\lambda \Gamma^\lambda_{\mu\tau}\Gamma^\tau_{\nu\sigma} - e^{b\sigma}e^a_\lambda \Gamma^\lambda_{\nu\tau}\Gamma^\tau_{\mu\sigma}$

~~$= e^{b\sigma}\partial_\mu(e^a_\lambda \Gamma^\lambda_{\nu\sigma}) - e^{b\sigma}\partial_\nu(e^a_\lambda \Gamma^\lambda_{\mu\sigma})$~~
~~use (*)~~ ~~$- e^{b\sigma}\Gamma^\lambda_{\nu\sigma}(\Gamma^\tau_{\mu\lambda}e^a_\tau - \omega_\mu{}^a{}_b e^b_\lambda)$~~
~~$+ e^{b\sigma}\Gamma^\lambda_{\mu\sigma}(\Gamma^\tau_{\nu\lambda}e^a_\tau - \omega_\nu{}^a{}_b e^b_\lambda)$~~
~~$+ e^{b\sigma}e^a_\lambda \Gamma^\lambda_{\mu\tau}\Gamma^\tau_{\nu\sigma} - e^{b\sigma}e^a_\lambda \Gamma^\lambda_{\nu\tau}\Gamma^\tau_{\mu\sigma}$~~

use (*) $= e^{b\sigma}\partial_\mu(\partial_\nu e^a_\sigma + \omega_\nu{}^a{}_c e^c_\sigma) - e^{b\sigma}\partial_\nu(\partial_\mu e^a_\sigma + \omega_\mu{}^a{}_c e^c_\sigma)$
$- e^{b\sigma}\Gamma^\lambda_{\nu\sigma}\partial_\mu e^a_\lambda + e^{b\sigma}\Gamma^\lambda_{\mu\sigma}\partial_\nu e^a_\lambda$

$$+ e^{b\sigma}\Gamma^{\tau}_{\nu\sigma}\partial_\mu e^a_\tau + e^{b\sigma}\Gamma^{\tau}_{\nu\sigma}\omega^a_{\mu c}e^c_\tau$$

$$- e^{b\sigma}\Gamma^{\tau}_{\mu\sigma}\partial_\nu e^a_\tau - e^{b\sigma}\Gamma^{\tau}_{\mu\sigma}\omega^a_{\nu c}e^c_\tau$$

$$= e^{b\sigma}\partial_\mu(\omega^a_{\nu c}e^c_\sigma) - e^{b\sigma}\partial_\nu(\omega^a_{\mu c}e^c_\sigma)$$

$$+ e^{b\sigma}\Gamma^{\tau}_{\nu\sigma}\omega^a_{\mu c}e^c_\tau - e^{b\sigma}\Gamma^{\tau}_{\mu\sigma}\omega^a_{\nu c}e^c_\tau$$

$$= \underbrace{e^{b\sigma}e^c_\sigma}_{\eta^{bc}}\partial_\mu\omega^a_{\nu c} + e^{b\sigma}\omega^a_{\nu c}(\partial_\mu e^c_\sigma)$$

$$- \underbrace{e^{b\sigma}e^c_\sigma}_{\eta^{bc}}\partial_\nu\omega^a_{\mu c} - e^{b\sigma}\omega^a_{\mu c}(\partial_\nu e^c_\sigma)$$

$$+ e^{b\sigma}\Gamma^{\tau}_{\nu\sigma}\omega^a_{\mu c}e^c_\tau - e^{b\sigma}\Gamma^{\tau}_{\mu\sigma}\omega^a_{\nu c}e^c_\tau.$$

$$= \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + e^{b\sigma}\omega^a_{\nu c}\underbrace{(\partial_\mu e^c_\sigma - \Gamma^\tau_{\mu\sigma}e^c_\tau)}_{= -\omega^c_{\mu d}e^d_\sigma \text{ by } (*)}$$

$$- e^{b\sigma}\omega^a_{\mu c}\underbrace{(\partial_\nu e^c_\sigma - \Gamma^\tau_{\nu\sigma}e^c_\tau)}_{= -\omega^c_{\nu d}e^d_\sigma \text{ by } (*)}$$

$$= \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu - \underbrace{e^{b\sigma}e^d_\sigma}_{\eta^{bd}}\omega^a_{\nu c}\omega^c_{\mu d}$$

$$+ \underbrace{e^{b\sigma}e^d_\sigma}_{\eta^{bd}}\omega^a_{\mu c}\omega^c_{\nu d}$$

$$= \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu - \omega^a_{\nu c}\omega^{cb}_\mu + \omega^a_{\mu c}\omega^{cb}_\nu$$

$$= \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \omega^{ac}_\mu\omega_{\nu c}{}^b - \omega^{ac}_\nu\omega_{\mu c}{}^b$$

D.

This is consistent with $R = d\omega + \omega\wedge\omega$.

b) when acting on ϵ, the covariant derivative acts as $\nabla_\mu = \partial_\mu + \frac{i}{2}\omega_\mu^{ab} T_{ab}$, where

$T_{ab} = -\frac{i}{2}\Gamma_{ab}$ and $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$

T_{ab} is the generator of tangent space of $SO(1,9)$.

Now, $[\nabla_\mu, \nabla_\nu]\epsilon = \nabla_\mu\nabla_\nu\epsilon - \nabla_\nu\nabla_\mu\epsilon$

$$= \nabla_\mu(\partial_\nu\epsilon + \frac{i}{2}\omega_\nu^{ab}T_{ab}\epsilon) - \nabla_\nu(\partial_\mu\epsilon + \frac{i}{2}\omega_\mu^{ab}T_{ab})$$

$$= \partial_\mu(\partial_\nu\epsilon + \frac{i}{2}\omega_\nu^{ab}T_{ab}\epsilon)$$
$$- \Gamma^\rho_{\mu\nu}(\partial_\rho\epsilon + \frac{i}{2}\omega_\rho^{ab}T_{ab}\epsilon)$$
$$+ \frac{i}{2}\omega_\mu^{cd}T_{cd}(\partial_\nu\epsilon + \frac{i}{2}\omega_\nu^{ab}T_{ab}\epsilon)$$
$$- \partial_\nu(\partial_\mu\epsilon + \frac{i}{2}\omega_\mu^{ab}T_{ab}\epsilon)$$
$$+ \Gamma^\rho_{\nu\mu}(\partial_\rho\epsilon + \frac{i}{2}\omega_\rho^{ab}T_{ab}\epsilon)$$
$$- \frac{i}{2}\omega_\nu^{cd}T_{cd}(\partial_\mu\epsilon + \frac{i}{2}\omega_\mu^{ab}T_{ab}\epsilon).$$

$$= \frac{i}{2}(\partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab})T_{ab} + \frac{i}{2}\omega_\nu^{ab}T_{ab}\partial_\mu\epsilon$$
$$+ \frac{i}{2}\omega_\mu^{cd}T_{cd}\partial_\nu\epsilon - \frac{1}{4}\omega_\mu^{cd}\omega_\nu^{ab}T_{cd}T_{ab}\epsilon$$
$$- \frac{i}{2}\omega_\mu^{ab}T_{ab}\partial_\nu\epsilon - \frac{i}{2}\omega_\nu^{cd}T_{cd}\partial_\mu\epsilon + \frac{1}{4}\omega_\nu^{cd}\omega_\mu^{ab}T_{cd}T_{ab}\epsilon$$

$$= \frac{i}{2}(\partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab})T_{ab} - \frac{1}{4}\omega_\mu^{cd}\omega_\nu^{ab}[T_{cd}, T_{ab}]\epsilon$$

II ∵ $[T_{cd}, T_{ab}] = -i(\eta_{ac}T_{bd} - \eta_{ad}T_{bc} - \eta_{bc}T_{ad} + \eta_{bd}T_{ac})$

$\therefore [\nabla_\mu, \nabla_\nu]\epsilon =$

$$= \frac{i}{2}(\partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab})T_{ab}\epsilon$$

$$+ \frac{i}{4}\omega_\mu^{cd}\omega_\nu^{ab}(\eta_{ac}T_{bd} - \eta_{ad}T_{bc} - \eta_{bc}T_{ad} + \eta_{bd}T_{ac})\epsilon$$

$$= \frac{i}{2}(\partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab})T_{ab}\epsilon$$

$$+ \frac{i}{4}(\omega_{\mu a}{}^{d}\omega_\nu^{ab}T_{bd} - \omega_{\mu a}^{c}\omega_\nu^{ab}T_{bc} - \omega_{\mu b}^{d}\omega_\nu^{ab}T_{ad}$$

$$+ \omega_{\mu b}^{c}\omega_\nu^{ab}T_{ac})\epsilon$$

$$= \frac{i}{2}(\partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab})T_{ab}\epsilon$$

$$+ \frac{i}{4}(T_{ab})(\omega_{\mu c}{}^{b}\omega_\nu{}^{ca} - \omega_{\mu c}^{b}\omega_\nu^{ca} - \omega_{\mu c}{}^{b}\omega_\nu^{ac}$$

$$+ \omega_{\mu c}^{b}\omega_\nu^{ac})\epsilon$$

$$= \frac{i}{2}(\partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu{}^{ab} - \frac{1}{2}\omega_\nu^{ac}\omega_{\mu c}{}^{b} - \frac{1}{2}\omega_\nu{}^{ac}\omega_{\mu c}{}^{b}$$

$$-\frac{1}{2}\omega_\nu^{ac}\omega_{\mu c}{}^{b} - \frac{1}{2}\omega_\nu^{ac}\omega_{\mu c}{}^{b})T_{ab}\epsilon$$

$$= \frac{i}{2}(\partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - 2\omega_\nu{}^{ac}\omega_{\mu c}{}^{b})T_{ab}\epsilon$$

we used $\omega_\mu^{ab} = -\omega_\mu^{ba}$

$$\because \quad \Gamma_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] = -\Gamma_{ba}$$

$$\therefore \quad -2\omega_\nu^{\ ac}\,\omega_{\mu c}^{\ \ b}\,\Gamma_{ab} = \omega_\nu^{\ ac}\,\omega_{\mu c}^{\ \ b}\,\Gamma_{ba} - \omega_\nu^{\ ac}\,\omega_{\mu c}^{\ \ b}\,\Gamma_{ab}$$

$$= \left(\omega_\nu^{\ bc}\,\omega_{\mu c}^{\ \ a} - \omega_\nu^{\ ac}\,\omega_{\mu c}^{\ \ b}\right)\Gamma_{ab}$$

$$= \left(-(\omega_{\nu c}^{\ \ b})(-\omega_\mu^{\ ac}) - \omega_\nu^{\ ac}\,\omega_{\mu c}^{\ \ b}\right)\Gamma_{ab}$$

$$= \left(\omega_\mu^{\ ac}\,\omega_{\nu c}^{\ \ b} - \omega_\nu^{\ ac}\,\omega_{\mu c}^{\ \ b}\right)\Gamma_{ab}.$$

$$\Rightarrow \quad [\nabla_\mu, \nabla_\nu]\epsilon = \frac{i}{2}\epsilon\left(\partial_\mu \omega_\nu^{\ ab} - \partial_\nu \omega_\mu^{\ ab} + \omega_\mu^{\ ac}\,\omega_{\nu c}^{\ \ b} - \omega_\nu^{\ ac}\,\omega_{\mu c}^{\ \ b}\right)\Gamma_{ab}$$

$$= \frac{i}{2} R_{\mu\nu}^{\ \ ab}\,\Gamma_{ab}\,\epsilon = \frac{1}{4} R_{\mu\nu}^{\ \ ab}\,\Gamma_{ab}\,\epsilon \qquad \Box.$$

2 a)

A Kähler Manifold satisfies: $g_{ij} = g_{\bar{i}\bar{j}} = 0$ ①,

$\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}$ ② and $\bar{\partial}_i g_{j\bar{k}} = \bar{\partial}_k g_{j\bar{i}}$ ③

$\because$ $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$

$$\therefore \Gamma^k_{ij} = \frac{1}{2} g^{k\bar{l}} (\partial_i g_{j\bar{l}} + \partial_j g_{i\bar{l}} - \bar{\partial}_l \underbrace{g_{ij}}_{=0 \text{ by ①}})$$

$$= \frac{1}{2} g^{k\bar{l}} (\partial_i g_{j\bar{l}} + \underbrace{\partial_i g_{j\bar{l}}}_{\text{by ②}}) = \underline{\underline{g^{k\bar{l}} \partial_i g_{j\bar{l}}}}$$ □

$$\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \frac{1}{2} g^{l\bar{k}} (\bar{\partial}_i g_{l\bar{j}} + \underbrace{\bar{\partial}_j g_{l\bar{i}}}_{= \bar{\partial}_i g_{l\bar{j}} \text{ by ③}} - \partial_l \underbrace{g_{\bar{i}\bar{j}}}_{=0 \text{ by ①}})$$

$$= \underline{\underline{g^{l\bar{k}} \bar{\partial}_i g_{l\bar{j}}}}$$ □

All other connections vanish, explicitly:

$$\Gamma^k_{i\bar{j}} = \Gamma^k_{\bar{j}i} = \frac{1}{2} g^{k\bar{l}} (\partial_i \underbrace{g_{\bar{j}\bar{l}}}_{=0 \text{ by ①}} + \underbrace{\bar{\partial}_j g_{i\bar{l}} - \bar{\partial}_l g_{i\bar{j}}}_{=0 \text{ by ③}}) = \underline{\underline{0.}}$$

$$\Gamma^{\bar{k}}_{i\bar{j}} = \Gamma^{\bar{k}}_{\bar{j}i} = \frac{1}{2} g^{l\bar{k}} (\partial_i g_{l\bar{j}} + \bar{\partial}_j g_{il} - \partial_l g_{i\bar{j}}) = \underline{\underline{0.}}$$

($\bar{\partial}_j g_{il}$: $=0$ by ①; $\partial_i g_{l\bar{j}} - \partial_l g_{i\bar{j}}$: $=0$ by ②)

$$\Gamma^k_{\bar{i}\bar{j}} = \frac{1}{2} g^{k\bar{l}} (\bar{\partial}_i g_{\bar{l}\bar{j}} + \bar{\partial}_j g_{\bar{l}\bar{i}} - \bar{\partial}_l g_{\bar{i}\bar{j}}) = 0 \quad \text{by ①}$$

$\Gamma^{\bar{k}}_{ij} = \frac{1}{2} g^{\ell\bar{k}} (\partial_i g_{\ell j} + \partial_j g_{\ell i} - \partial_\ell g_{ij}) = 0$ by ①.

For Ricci tensor, consider Riemann Tensor

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\tau}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\rho}_{\nu\tau}\Gamma^{\tau}_{\mu\sigma}$$

~~So $R_{ij}{}^{\rho}{}_{\sigma} = \partial_i \Gamma^{\rho}_{j\sigma} - \partial_j \Gamma^{\rho}_{i\sigma} + \Gamma^{\rho}_{i\tau}\Gamma^{\tau}_{j\sigma} - \Gamma^{\rho}_{j\tau}\Gamma^{\tau}_{i\sigma}$~~

~~(i, j holomorphic indices, ρ, σ any indices)~~

$\because \Gamma^{\rho}_{\mu\sigma} \neq 0$ only if it $= \Gamma^{k}_{ij}$ or $\Gamma^{\bar{k}}_{\bar{i}\bar{j}}$ ☆

~~$\therefore R_{\mu\nu}{}^{\rho}{}_{\sigma} \neq 0$ only if it $= R_{ij}{}^{k}{}_{\ell}$ or $R_{\bar{i}\bar{j}}{}^{\bar{k}}{}_{\bar{\ell}}$~~

if $\mu = i$, $\nu = j$ (μ, ν any indices, i, j holomorphic indices)

then

$$R_{ij}{}^{\rho}{}_{\sigma} = \partial_i \Gamma^{\rho}_{j\sigma} - \partial_j \Gamma^{\rho}_{i\sigma} + \Gamma^{\rho}_{i\tau}\Gamma^{\tau}_{j\sigma} - \Gamma^{\rho}_{\nu\tau}\Gamma^{\tau}_{\mu\sigma}$$

if any of $\rho, \sigma = \bar{k}$ or $\bar{\ell}$, then by ☆ $R_{ij}{}^{\rho}{}_{\sigma}$ vanishes, so we only consider $\rho, \sigma = k, \ell$.

$$R_{ij}{}^{k}{}_{\ell} = \partial_i \Gamma^{k}_{j\ell} - \partial_j \Gamma^{k}_{i\ell} + \Gamma^{k}_{im}\Gamma^{m}_{j\ell} - \Gamma^{k}_{jm}\Gamma^{m}_{i\ell}$$

$$= \partial_i (g^{k\bar{m}} \partial_j g_{\ell\bar{m}}) - \partial_j (g^{k\bar{m}} \partial_i g_{\ell\bar{m}})$$

$$+ (g^{k\bar{n}} \partial_i g_{m\bar{n}}) (g^{m\bar{p}} \partial_j g_{\ell\bar{p}})$$

$$- (g^{k\bar{n}} \partial_j g_{m\bar{n}}) (g^{m\bar{p}} \partial_i g_{\ell\bar{p}})$$

$$= g^{k\bar{m}} (\partial_i \partial_j - \partial_j \partial_i) g_{\ell\bar{m}} + (\partial_i g^{k\bar{m}})(\partial_j g_{\ell\bar{m}})$$

$$- (\partial_j g^{k\bar{m}})(\partial_i g_{\ell\bar{m}}) - \delta^{\bar{p}}_{\bar{n}} (\partial_i g^{k\bar{n}})(\partial_j g_{\ell\bar{p}})$$

$$+ \delta^{\bar{p}}_{\bar{n}} (\partial_j g^{k\bar{n}})(\partial_i g_{\ell\bar{p}}) = \underline{\underline{0}}$$

So in the above line we used.

$$0 = \partial_i (\delta^k_m) = \partial_i (g^{k\bar{n}} g_{m\bar{n}}) = g^{k\bar{n}} \partial_i g_{m\bar{n}} + g_{m\bar{n}} \partial_i g^{k\bar{n}}.$$

So, we have $R_{ij**} = 0$ and all components related to R_{ij**} by symmetries vanish as well. (Similarly $R_{\bar{i}\bar{j}**} = 0$)

So we can only have $R_{i\bar{j}**}$, but $\because R_{ij**} = 0$ so, by symmetry, the last two indices of $R_{i\bar{j}**}$ also need to be one holomorphic and one antiholomorphic.

So only $R_{i\bar{j}k\bar{l}}$ terms are non-zero.

$$R_{i\bar{j}k\bar{l}} = g_{k\bar{m}} R_{i\bar{j}}{}^{\bar{m}}{}_{\bar{l}} = g_{k\bar{m}} \left(\partial_i \Gamma^{\bar{m}}_{\bar{j}\bar{l}} - \bar{\partial}_j \Gamma^{\bar{m}}_{i\bar{l}} + \Gamma^{\bar{m}}_{i\bar{t}} \Gamma^{\bar{t}}_{\bar{j}\bar{l}} - \Gamma^{\bar{m}}_{\bar{j}\bar{t}} \Gamma^{\bar{t}}_{i\bar{l}} \right).$$

$$= g_{k\bar{m}} \partial_i \Gamma^{\bar{m}}_{\bar{j}\bar{l}}$$

$$= g_{k\bar{m}} \partial_i (g^{n\bar{m}} \bar{\partial}_j g_{n\bar{l}})$$

$$= \delta^n_k \partial_i \bar{\partial}_j g_{n\bar{l}} + g_{k\bar{m}} (\partial_i g^{n\bar{m}}) (\bar{\partial}_j g_{n\bar{l}})$$

$$= \partial_i \bar{\partial}_j g_{k\bar{l}} - g^{n\bar{m}} (\partial_i g_{k\bar{m}}) (\bar{\partial}_j g_{n\bar{l}})$$

$$= \partial_i \bar{\partial}_j g_{k\bar{l}} - g^{m\bar{n}} (\partial_i g_{k\bar{n}}) (\bar{\partial}_j g_{m\bar{l}})$$

Riemann to Ricci :

$$R_{i\bar{j}} = -R_{i\bar{j}k}{}^{k} = -g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$$

$$= -g^{k\bar{l}} \left(\partial_i \bar{\partial}_{\bar{j}} g_{k\bar{l}} - g^{m\bar{n}} (\partial_i g_{k\bar{n}})(\bar{\partial}_{\bar{j}} g_{m\bar{l}})\right) \quad ①$$

use the identity that.

$$\bar{\partial}_{\bar{j}} (\log \det(g)) = \frac{1}{\det(g)} \bar{\partial}_{\bar{j}} \det(g)$$

$$= \frac{1}{\det(g)} \det(g)\, g^{k\bar{l}} \bar{\partial}_{\bar{j}} g_{k\bar{l}} = g^{k\bar{l}} \bar{\partial}_{\bar{j}} g_{k\bar{l}}$$

Jacobi formula.

And

$$\partial_i \bar{\partial}_{\bar{j}} \log(\det(g)) = \partial_i (g^{k\bar{l}} \bar{\partial}_{\bar{j}} g_{k\bar{l}})$$

$$= g^{k\bar{l}} \partial_i \bar{\partial}_{\bar{j}} g_{k\bar{l}} + (\partial_i g^{k\bar{l}})(\bar{\partial}_{\bar{j}} g_{k\bar{l}}) \quad ②$$

And

$$g^{k\bar{l}} g^{m\bar{n}} (\partial_i g_{k\bar{n}})(\bar{\partial}_{\bar{j}} g_{m\bar{l}})$$

$$= -g^{m\bar{n}} g_{k\bar{n}} (\partial_i g^{k\bar{l}})(\bar{\partial}_{\bar{j}} g_{m\bar{l}})$$

$$= -\delta^m_k (\partial_i g^{k\bar{l}})(\bar{\partial}_{\bar{j}} g_{m\bar{l}}) = -(\partial_i g^{k\bar{l}})(\bar{\partial}_{\bar{j}} g_{k\bar{l}}) \quad ③$$

use above ~~two~~ 3 equations ~~we have~~ ①, ②, ③ we have

$$R_{i\bar{j}} = -\partial_i \bar{\partial}_{\bar{j}} (\log \det g)$$

b) i)

Define charts

$\phi_r : U_r \to \mathbb{C}^n$ ~~such that~~ : $(z^0, \cdots, z^n) \to \left(\frac{z^0}{z^r}, \cdots, \widehat{\frac{z^r}{z^r}}, \cdots \frac{z^n}{z^r}\right)$

and transition functions

$\phi_r \circ \phi_s^{-1} : \phi_s(U_r \cap U_s) \to \phi_r(U_r \cap U_s)$

$: (w^0, \cdots, \widehat{w^s}, \cdots, w^n) \xrightarrow{\phi_s^{-1}} (w^0, \cdots, 1, \cdots, w^n)$

$\xrightarrow{\phi_r} \left(\frac{w_0}{w^r}, \cdots, \frac{\widehat{w^r}}{w^r}, \cdots, \frac{1}{w^r}, \cdots, \frac{w^n}{w^r}\right)$

$\because$ $w^r \neq 0$ on $\phi_s(U_r \cap U_s)$,

$\therefore$ coordinate functions of $\phi_r \circ \phi_s^{-1}$ ~~are~~ take the form $\frac{w^t}{w^r}$ or $\frac{1}{w^r}$, so they are holomorphic on the domain of $\phi_r \circ \phi_s^{-1}$.

So $\mathbb{P}^n$ is a complex manifold.

ii) Kähler potential

$$K_{(r)} = \log\left(1 + \sum_{i=1}^{n} |z^i_{(r)}|^2\right)$$

$$\therefore g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log\left(1 + \sum_{i=1}^{n} |z^i_{(r)}|^2\right)$$

$$= \frac{1}{\left(1 + \sum_{i=1}^{n} |z^i_{(r)}|^2\right)^2} \left(\left(1 + \sum_{i=1}^{n} |z^i_{(r)}|^2\right)\delta_{ij} - \bar{z}^i_{(r)} z^j_{(r)}\right).$$

D.

(a iii) $\det(g_{i\bar{j}}) = \frac{1}{(1+\sum_{i=1}^{n}|z^i_{(r)}|^2)^{n+1}}$

$\Rightarrow R_{i\bar{j}} = -\partial_i\bar{\partial}_{\bar{j}} \log(g_{i\bar{j}})$

$= -\partial_i\bar{\partial}_{\bar{j}} \log\left(\frac{1}{(1+\sum_{i=1}^{n}|z^i_{(r)}|^2)^{n+1}}\right).$

$= (n+1)\frac{1}{(1+\sum_{i=1}^{n}|z^i_{(r)}|^2)^2}\left((1+\sum_{i=1}^{n}|z^i_{(r)}|^2)\delta_{i\bar{j}} - \bar{z}^i_{(r)} z^j_{(r)}\right)$

$= (n+1)\, g_{i\bar{j}}$

iv) ~~for n=1, the transition~~ function.

~~$\phi_r \circ \phi_s^{-1} = \phi_0 \circ \phi_0^{-1}, \phi_0 \circ \phi_1^{-1}, \phi_1 \circ \phi_0^{-1}$~~, $\phi_1 \circ \phi_1^{-1}$

~~$\phi_0 \circ \phi_1^{-1}: (w^0, \hat{w}^1) \to (\frac{\hat{w}^0}{w^0}, \frac{1}{w^0})$.~~

The map (for n=1)

~~$f: \mathbb{P}^1 \to S^2 = (z^0, z^1) \to (2\mathrm{Re}(z^1\bar{z}^0), 2\mathrm{Im}(z^1\bar{z}^0))$~~

$f: \mathbb{P}^1 \to S^2 : (z^0, z^1) \to \frac{(2\mathrm{Re}(z^1\bar{z}^0),\ 2\mathrm{Im}(z^1\bar{z}^0),\ |z^1|^2 - |z^0|^2)}{(|z^1|^2+|z^0|^2)}$

gives a homeomorphism so $\mathbb{P}^1 \cong S^2$.

□.

3 a) Bosonic Field contents:

Type IIB : (W_6)

$g_{\mu\nu}$ $(C_4^+)_{\mu\nu ijk}$ Φ a, $B_{\mu\nu}$ $(C_2)_{\mu\nu}$ $\underbrace{g_{i\bar{j}} \quad B_{i\bar{j}}}_{h^{1,1}}$

$\underbrace{(C_2)_{i\bar{j}} \quad (C_4^+)_{\mu\nu i\bar{j}}}_{h^{1,1}}$ $\underbrace{(C_4^+)_{\mu ij\bar{k}} \quad g_{ij}}_{h^{2,1}}$ $g_{\bar{i}\bar{j}}$

Type IIA : (M_6)

$g_{\mu\nu}$ $(C_1)_\mu$ Φ $B_{\mu\nu}$ $(C_3)_{ijk}$ $(C_3)_{\bar{i}\bar{j}\bar{k}}$

g_{ij} $\underbrace{g_{\bar{i}\bar{j}} \quad (C_3)_{ij\bar{k}} \quad (C_3)_{\bar{i}jk}}_{h^{2,1}}$ $(C_3)_{\mu i\bar{j}}$ $\underbrace{g_{i\bar{j}} \quad B_{i\bar{j}}}_{h^{1,1}}$

they agree via $h^{1,1}(W_6) = h^{1,2}(M_6)$
$h^{1,2}(W_6) = h^{1,1}(M_6)$

b)