String Theory II

Problem sheet 3

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$\boxed{1}$ 1)

$$\mathcal{L} = -\frac{1}{4g_{YM}^2}\left(\mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + 2i\,\mathrm{Tr}(\bar\lambda \Gamma^\mu D_\mu \lambda)\right)$$

we specify some notations.

- $F_{\mu\nu} = F^a_{\mu\nu} T^a$, T^a are the generators of the Gauge group.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$$

- ~~$\bar\lambda \Gamma^\mu D_\mu \lambda = \lambda^a_\alpha \Gamma^\mu_{\alpha\beta} (D_\mu)^{ab} \lambda^b_\beta$~~

$(1,\cdots,N^2-1) \leftarrow a, b, \cdots$ adjoint indices ~~$(1,\cdots,N^2-1)$~~ ~~$(1,\cdots,N^2-1)$~~

$(0,\cdots,9) \leftarrow \mu, \nu \cdots$ spacetime indices ~~$(0,\cdots,N-1)$~~

~~$\alpha, \beta, \cdots \rightarrow (1,\cdots,N)$~~

$(1,\cdots,16) \leftarrow \alpha, \beta, \cdots \nearrow$ spinor indices.

D_μ acts on the adjoint representation, that is,

$$D_\mu \Phi = \partial_\mu \Phi - i[A_\mu, \Phi] \quad , A_\mu = A_\mu^a T^a .$$

$$(D_\mu \Phi)^a = \partial_\mu \Phi^a + g f^{abc} A^b \Phi^c .$$

$$\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] = F^a_{\mu\nu} F^{b,\mu\nu}\, \mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab} F^a_{\mu\nu} F^{b\mu\nu}$$

$$= \frac{1}{2} F^a_{\mu\nu} F^{a,\mu\nu}$$

(Trace over adjoint indices.)

$$\mathrm{Tr}(\bar\lambda \Gamma^\mu D_\mu \lambda) = \bar\lambda^a_\alpha \Gamma^\mu_{\alpha\beta} (D_\mu\lambda)^b_\beta\, \mathrm{Tr}(T^a T^b)$$

$$= \frac{1}{2} \bar\lambda^a_\alpha \Gamma^\mu_{\alpha\beta} (D_\mu \lambda)^a_\beta$$

$$= \underline{\frac{1}{2} \bar\lambda^a_\alpha \Gamma^\mu_{\alpha\beta} (D_\mu)^{ab} \lambda^b_\beta} ,$$

where $\underline{(D_\mu)^{ab} = \partial_\mu \delta^{ab} + g f^{acb} A^c_\mu}$

Now prove some useful properties.

~~D_μ~~ Lebnitz rule: △1

$$D_\mu(\Phi_1\Phi_2) = \partial_\mu(\Phi_1\Phi_2) - i[A_\mu, \Phi_1\Phi_2]$$

$$= \Phi_1\partial_\mu\Phi_2 + (\partial_\mu\Phi_1)\Phi_2 - i[A,\Phi_1]\Phi_2 - i\Phi_1[A,\Phi_2]$$

$$= (\partial_\mu\Phi_1 - i[A_\mu,\Phi_1])\Phi_2 + \Phi_1(\partial_\mu\Phi_2 - i[A_\mu,\Phi_2])$$

$$= (D_\mu\Phi_1)\Phi_2 + \Phi_1(D_\mu\Phi_2) \qquad \square$$

Bianchi identity: △2

$$\cancel{D_{[\lambda\mu\nu}}\ \underbrace{3D_{[\lambda}F_{\mu\nu]} =}_{\because F_{\mu\nu} = \cancel{0} -F_{\nu\mu}} (D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu})$$

$= \cancel{\partial_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda}}$

$$= \partial_\lambda F_{\mu\nu} - i[A_\lambda, F_{\mu\nu}] + \partial_\mu F_{\nu\lambda} - i[A_\mu, F_{\nu\lambda}]$$
$$+ \partial_\nu F_{\lambda\mu} - i[A_\nu, F_{\lambda\mu}]$$

$$= \partial_\lambda(\cancel{\partial_\mu A_\nu} - \cancel{\partial_\nu A_\mu}) - i\partial_\lambda[A_\mu, A_\nu] - i[A_\lambda, F_{\mu\nu}]$$
$$+ \partial_\mu(\cancel{\partial_\nu A_\lambda} - \cancel{\partial_\lambda A_\nu}) - i\partial_\mu[A_\nu, A_\lambda] - i[A_\mu, F_{\nu\lambda}]$$
$$+ \partial_\nu(\cancel{\partial_\lambda A_\mu} - \cancel{\partial_\mu A_\lambda}) - i\partial_\nu[A_\lambda, A_\mu] - i[A_\nu, F_{\lambda\mu}]$$

$$= -i\{\partial_\lambda[A_\mu, A_\nu] + [A_\lambda, \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]]$$
$$+ \partial_\mu[A_\nu, A_\lambda] + [A_\mu, \partial_\nu A_\lambda - \partial_\lambda A_\nu - i[A_\nu, A_\lambda]]$$
$$+ \partial_\nu[A_\lambda, A_\mu] + [A_\nu, \partial_\lambda A_\mu - \partial_\mu A_\lambda - i[A_\lambda, A_\mu]]\}$$

$$= -i[[\partial_\lambda[A_\mu, A_\nu] - [A_\mu, \partial_\lambda A_\nu] - [\partial_\lambda A_\mu, A_\nu]$$

$$+\ \partial_\mu [A_\nu, A_\lambda] - [A_\nu, \partial_\mu A_\lambda] - [\partial_\mu A_\nu, A_\lambda]$$
$$+\ \partial_\nu [A_\lambda, A_\mu] - [A_\lambda, \partial_\nu A_\mu] - [\partial_\nu A_\lambda, A_\mu]\,]\,)$$

$$-\ (\,[A_\lambda, [A_\mu, A_\nu]] + [A_\mu, [A_\nu, A_\lambda]] + [A_\nu, [A_\lambda, A_\mu]]\,)\,.$$

$= 0$ (first term $= 0$ by Lebnitz
~~sec~~ second term $= 0$ by Jacobi identity).

$$\Rightarrow \quad \underline{D_{[\lambda} F_{\mu\nu]} = 0}$$

Now, vary the Lagrangian, (with integral over $\int d^{10}x$ implicit)

$$\delta \mathcal{L} = + \frac{1}{4 g_{YM}^2}\, \delta \Big(\underbrace{-\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})}_{①} - 2i\, \mathrm{Tr}(\underbrace{\bar\lambda \Gamma^\mu D_\mu \lambda}_{②}) \Big).$$

$$\Rightarrow \quad -\delta\, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\mathrm{Tr}(\delta(F_{\mu\nu}F^{\mu\nu}))$$
$$= -\mathrm{Tr}(\delta F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu}\delta F^{\mu\nu})$$
$$= -2\,\mathrm{Tr}(F_{\mu\nu}\delta F^{\mu\nu})$$

Now, another identity ③

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu - i\,\delta[A_\mu, A_\nu]$$
$$= \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu - i[\delta A_\mu, A_\nu] - i[A_\mu, \delta A_\nu]$$
$$= (\partial_\mu \delta A_\nu - i[A_\mu, \delta A_\nu]) - (\partial_\nu \delta A_\mu - i[A_\nu, \delta A_\mu])$$
$$= D_\mu \delta A_\nu - D_\nu \delta A_\mu$$

use ③

$$\Rightarrow \; -\delta \,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -2\,\mathrm{Tr}(F_{\mu\nu}\,\delta F^{\mu\nu})$$

$$= -2\,\mathrm{Tr}(F_{\mu\nu} D^{\mu}\delta A^{\nu} - F_{\mu\nu} D^{\nu}\delta A^{\mu})$$

$F^{\mu\nu} = -F^{\nu\mu}$

$\mu \leftrightarrow \nu$ for ~~second~~ second term

$$\rightarrow \; = -4\,\mathrm{Tr}[F_{\mu\nu} D^{\mu}\delta A^{\nu}]$$

$$= -4\left(\mathrm{Tr}(D^{\mu}(F_{\mu\nu}\delta A^{\nu})) - \mathrm{Tr}((D^{\mu}F_{\mu\nu})\delta A^{\nu})\right)$$

lebnitz

$$= -4\Big(\mathrm{Tr}(\underbrace{\partial^{\mu}(F_{\mu\nu}\delta A^{\nu})}_{\text{total derivative} \to 0}) - i\,\mathrm{Tr}(\underbrace{[A^{\mu}, F_{\mu\nu}\delta A^{\nu}]}_{\text{Trace of commutator } = 0})\Big) \quad ⊛$$

$$+4\,\mathrm{Tr}((D^{\mu}F_{\mu\nu})\delta A^{\nu})$$

$$= 4\,\mathrm{Tr}((D^{\mu}F_{\mu\nu})\delta A^{\nu})$$

$$\therefore \text{ Now } \quad \delta\mathcal{L} = \frac{1}{g_{YM}^2}\Big[\mathrm{Tr}(\underbrace{(D^{\mu}F_{\mu\nu})\delta A^{\nu}}_{①}) - \frac{i}{2}\underbrace{\delta(\mathrm{Tr}(\bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda))}_{②}\Big]$$

$$\therefore \; g_{YM}^2\,\delta\mathcal{L} = \mathrm{Tr}(\overbrace{(D^{\mu}F_{\mu\nu})\delta A^{\mu}}^{①}) - \frac{i}{2}\mathrm{Tr}(\overbrace{\delta\bar{\lambda}\Gamma^{\mu}}^{③}D_{\mu}\lambda)$$

$$-\frac{i}{2}\mathrm{Tr}(\underbrace{\bar{\lambda}\Gamma^{\mu}D_{\mu}}_{④}\delta\lambda) - \frac{i}{2}\mathrm{Tr}(\underbrace{\bar{\lambda}\Gamma^{\mu}[\delta A_{\mu}, \lambda]}_{⑤})$$

and we have the SUSY transformation

$$\delta A_{\mu} = -i\bar{\epsilon}\Gamma_{\mu}\lambda \quad \text{and} \quad \delta\lambda = \frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon$$

(with $\Gamma^{\mu\nu} = \frac{1}{2}\Gamma^{[\mu}\Gamma^{\nu]}$)

– We also need $\delta\bar{\lambda}$, note that λ is a majorana spinor so $\bar{\lambda} = \lambda^{\dagger}\Gamma^{0} = \lambda^{T}C$ where C is the charge conjugation operator.

So, $\delta\bar{\lambda} = (\delta\lambda)^T C = \left(\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon\right)^T C$

$$= \frac{1}{2}F_{\mu\nu}\,\epsilon^T(\Gamma^{\mu\nu})^T C$$

$$(\Gamma^{\mu\nu})^T = \frac{1}{2}\left(\Gamma^{[\mu}\Gamma^{\nu]}\right)^T = \frac{1}{2}\Gamma^{T[\nu}\Gamma^{T\mu]}$$

(Also note that △4 $\underline{-C\Gamma^{\mu} = (\Gamma^{\mu})^T C}$, $\underline{C^T = -C}$ △5

and △6 $\Gamma^{\mu\nu}\Gamma^{\rho} = \Gamma^{\mu\nu\rho} + \eta^{\nu\rho}\Gamma^{\mu} - \eta^{\mu\rho}\Gamma^{\nu}$)

So $\delta\bar{\lambda} = \frac{1}{2}F_{\mu\nu}\,\epsilon^T\,\frac{1}{2}\Gamma^{T[\nu}\Gamma^{T\mu]}C$

$$= -\frac{1}{2}F_{\mu\nu}\,\epsilon^T\,\frac{1}{2}\Gamma^{T[\nu}C\Gamma^{\mu]}$$

$$= +\frac{1}{2}F_{\mu\nu}\,\epsilon^T C\,\frac{1}{2}\underbrace{\Gamma^{[\nu}\Gamma^{\mu]}}_{-\Gamma^{\mu\nu}}$$

$$= -\frac{1}{2}F_{\mu\nu}\underbrace{\epsilon^T C}_{\bar{\epsilon}}\Gamma^{\mu\nu} = -\frac{1}{2}\bar{\epsilon}F_{\mu\nu}\Gamma^{\mu\nu}$$

we equivalently derived
$(\Gamma^{\mu\nu})^T C = -C\Gamma^{\mu\nu}$
△7

- Note we trace over adjoint indices previously, but here we ~~&~~ transpose over spinor indices.
- (Note ϵ is also a Majorana-Weyl spinor, like λ^a, $\bar{\lambda}^a$. ~~Plug in δA^ν, $\delta\bar{\lambda}$, $\delta\lambda$ for~~ , but no adjoint ~~indic~~ index,)

Also, observe that terms ①, ③, ④ depends linearly on λ, but term ⑤ $\sim \lambda^3$

So ~~we~~ in order to expect a cancellation in the end, we need

① + ③ + ④ = 0 , ⑤ = 0 <u>separately</u>.

Focus on ①, ③, ④, plug in $\delta A^\mu, \delta\lambda, \delta\bar\lambda$ gives

①+③+④

$$= \cancel{\mathrm{Tr}}\ -2i\,\mathrm{Tr}\big((D^\mu \overbrace{F_{\mu\nu})\,\bar\epsilon\,\Gamma^\nu}^{①}\lambda\big) + \frac{i}{2}\mathrm{Tr}(\overbrace{\bar\epsilon F_{\mu\nu}\Gamma^{\mu\nu}\Gamma^\rho}^{③} D_\rho\lambda)$$

$$- \frac{i}{2}\mathrm{Tr}(\underbrace{\bar\lambda\,\Gamma^\rho(D_\rho F_{\mu\nu})}_{④}\Gamma^{\mu\nu}\epsilon).$$

- for ③, we use an ~~fact~~ observation from previous calculation (*), we saw that inside the trace, we can integrate by parts the covariant derivative just ~~as~~ like we usually do to ordinary partial derivatives, because the fact that $\mathrm{Tr}([A,B]) = 0$.

So $③ = +\frac{i}{2}\mathrm{Tr}(\bar\epsilon F_{\mu\nu}\Gamma^{\mu\nu}\Gamma^\rho D_\rho\lambda)$

by parts $\rightarrow$ $= -\frac{i}{2}\mathrm{Tr}(\bar\epsilon(D_\rho F_{\mu\nu})\Gamma^{\mu\nu}\Gamma^\rho\lambda)$

$\bar\epsilon$ and $\Gamma^{\mu\nu}\Gamma^\rho$ carres no adjoint indices, so $[A^\mu, \Gamma] = [A^\mu, \epsilon] = 0$ they can be taken out of D_μ

- For ④,

$$④ = -\frac{i}{2}\mathrm{Tr}(\bar\lambda\Gamma^\rho(D_\rho F_{\mu\nu})\Gamma^{\mu\nu}\epsilon)$$

$$= -\frac{i}{4}\bar\lambda^a_\alpha\Gamma^\rho_{\alpha\beta}\Gamma^{\mu\nu}_{\beta\gamma}\epsilon_\gamma(D_\rho F_{\mu\nu})^a$$

$$= -\frac{i}{2}\mathrm{Tr}(\bar\lambda_\alpha\Gamma^\rho_{\alpha\beta}\Gamma^{\mu\nu}_{\beta\sigma}\epsilon_\sigma(D_\rho F_{\mu\nu}))$$

I didn't do anything here, just to make ~~explicit~~ explicit that there is a sum over spinor indices $\alpha, \beta, \dots$ independent of the trace, which is over adjoint indices.

look inside the trace, ($D_\rho F_{\mu\nu}$ is a constant as far as a spinor space is concerned)

$$\bar\lambda_\alpha \Gamma^\rho_{\alpha\beta}\Gamma^{\mu\nu}_{\beta\gamma}\epsilon_\gamma$$

$$= \underbrace{\bar\lambda_\alpha\epsilon_\gamma}_{}\Gamma^\rho_{\alpha\beta}\Gamma^{\mu\nu}_{\beta\gamma} = \cancel{-\epsilon_\gamma\Gamma^\rho_{\beta\alpha}} - \epsilon_\gamma\bar\lambda_a\Gamma^\rho_{\alpha\beta}\Gamma^{\mu\nu}_{\beta\gamma}$$

Grassmann -valued

$$= -\epsilon_\gamma(\Gamma^{\mu\nu})^T_{\gamma\beta}(\Gamma^\rho)^T_{\beta\alpha}\bar\lambda_\alpha$$

so $\bar\lambda\Gamma^\rho\Gamma^{\mu\nu}\epsilon = -(\bar\lambda\Gamma^\rho\Gamma^{\mu\nu}\epsilon)^T$

$$= -\epsilon^T(\Gamma^{\mu\nu})^T(\Gamma^\rho)^T(\bar\lambda)^T$$

$$\because\ \bar\lambda = \lambda^T C \quad \therefore (\bar\lambda)^T = C^T\lambda \underbrace{=}_{\triangle 5} -C\lambda$$

$$\Rightarrow \bar\lambda\Gamma^\rho\Gamma^{\mu\nu}\epsilon = +\epsilon^T(\Gamma^{\mu\nu})^T(\Gamma^\rho)^T C\lambda.$$

$$\underbrace{=}_{\triangle 4} -\epsilon^T(\Gamma^{\mu\nu})^T C\Gamma^\rho\lambda \underbrace{=}_{\triangle 7} +\underbrace{\epsilon^T C}_{\bar\epsilon}\Gamma^{\mu\nu}\Gamma^\rho\lambda$$

$$= \bar\epsilon\Gamma^{\mu\nu}\Gamma^\rho\lambda$$

so ④ $= -\frac{i}{2}\mathrm{Tr}(\bar\epsilon D_\rho F_{\mu\nu}\Gamma^{\mu\nu}\Gamma^\rho\lambda) =$ ③

and ① + $\underbrace{③ + ④}_{=⑥}$ $= -2i\,\mathrm{Tr}((D^\mu F_{\mu\nu})\bar\epsilon\Gamma^\nu\lambda)$ ← ①

$- i\,\mathrm{Tr}(\bar\epsilon(D_\rho F_{\mu\nu})\Gamma^{\mu\nu}\Gamma^\rho\lambda)$ ← ⑥ = ③ + ④

$\underbrace{⑥}_{} = -i\,\mathrm{Tr}(\bar\epsilon(D_\rho F_{\mu\nu})(\Gamma^{\mu\nu\rho} + \eta^{\nu\rho}\Gamma^\mu - \eta^{\mu\rho}\Gamma^\nu)\lambda)$

use $\triangle 6$

~~term $(D_\rho F_{\mu\nu})$~~ $\Gamma^{\mu\nu\rho} = \frac{1}{3!}\Gamma^{[\mu}\Gamma^{\nu}\Gamma^{\rho]}$ totally antisymmetric

$$\therefore \quad (D_\rho F_{\mu\nu})\Gamma^{\mu\nu\rho} = \underbrace{(D_{[\rho}F_{\mu\nu]})}_{\text{this by } \triangle 2 \text{ is } 0}\Gamma^{\mu\nu\rho} = 0$$

$(D_\rho F_{\mu\nu})(\eta^{\nu\rho}\Gamma^{\mu} - \eta^{\mu\rho}\Gamma^{\nu})$ ~~$= D\rho F \; D^\rho F_{\mu\rho} \; D^\rho F$~~

$$= D^\rho F_{\mu\rho}\Gamma^{\mu} - D^\rho F_{\rho\nu}\Gamma^{\nu} \underbrace{=}_{F_{\mu\nu}=-F_{\nu\mu}} -2D^{\mu}F_{\mu\nu}\Gamma^{\nu}$$

$$\therefore ⑥ = (-i)(-2)\,\mathrm{Tr}(\bar{\epsilon} D^{\mu}F_{\mu\nu}\Gamma^{\nu}\lambda)$$

$$= 2i\,\mathrm{Tr}(\bar{\epsilon} D^{\mu}F_{\mu\nu}\Gamma^{\nu}\lambda) = -①$$

$$\therefore ① + ③ + ④ = ① + ⑥ = ① - ① = \underline{\underline{0}} \qquad \square$$

we have term ⑤ left.

$$⑤ = -\tfrac{1}{2}\,\mathrm{Tr}(\bar{\lambda}\Gamma^\nu[\delta A_\nu,\lambda])$$ (its a single term,

so the $-\frac{1}{2}$ prefactor doesn't matter)

$$= \mathrm{Tr}(\bar{\lambda}^a T^a \Gamma^\nu \delta A^b_\mu \lambda^c \underbrace{[T^b, T^c]}_{if^{bcd}T^d})$$

$$= i\,\bar{\lambda}^a \Gamma^\nu \underbrace{\delta A^b_\mu}_{-i\bar{\epsilon}\Gamma_\mu\lambda^b} \lambda^c f^{bcd}\,\mathrm{Tr}\,(\underbrace{T^a T^d}_{\frac{1}{2}\delta^{ad}}).$$

$$= \frac{i}{2}\,\bar{\lambda}^a\Gamma^\nu((\underbrace{-i\bar{\epsilon}}_{\text{a number in spinor sense.}})\Gamma_\nu\lambda^b)\lambda^c f^{bca}$$

f^{abc} totally antisymmetric

$$= -\frac{1}{2} f^{bac}(\bar{\epsilon}\Gamma_\nu\lambda^b)\,\bar{\lambda}^a\Gamma^\nu\lambda^c$$

$$\rightsquigarrow \quad \underline{f^{abc}(\bar{\epsilon}\Gamma_\nu\lambda^a)(\bar{\lambda}^b\Gamma^\nu\lambda^c)}$$

rename a↔b
ignore $-\frac{1}{2}$ factor

we want to prove this term vanishes.

~~write $\bar{\epsilon} = \epsilon^\dagger\Gamma^0$, $\bar{\lambda}^b = \lambda^{\dagger b}\Gamma^0$.~~

~~$⑤ = f^{abc}\epsilon^\dagger\Gamma^0\Gamma_\mu\lambda^a\lambda^{\dagger b}\Gamma^0\Gamma^\nu\lambda^c$~~

~~$= f^{abc}\epsilon^\dagger_\alpha\Gamma^0_{\alpha\beta}\Gamma_{\mu,\beta\gamma}\lambda^a_\gamma\underbrace{\lambda^{\dagger b}_\delta}_{=\lambda^{*b}_\delta}\Gamma^0_{\delta\epsilon}\Gamma^\mu_{\epsilon\chi}\lambda^c_\chi$~~

~~$= f^{abc}\epsilon^*_\alpha\Gamma^0_{\alpha\beta}\Gamma_{\mu,\beta\gamma}\lambda^a_\gamma B_{\delta\eta}\lambda^b_\eta\Gamma^0_{\delta\epsilon}\Gamma^\mu_{\epsilon\chi}\lambda^c_\chi$~~

~~$\lambda^* = B\lambda$~~

we first note that $(C\Gamma^\nu)$ is symmetric since

$$(C\Gamma^\nu)^T = (\Gamma^\nu)^T C^T = -(\Gamma^\nu)^T C = C\Gamma^\nu \;\square$$

With $\bar{\epsilon} = \epsilon^T C$, $\bar{\lambda}^b = \lambda^{Tb} C$ we have

~~Ⓙ $= 2f^{abc} \epsilon^T_\alpha (C\Gamma^\mu)_{\alpha\beta} \lambda^a_\beta$~~

Ⓙ $= f^{abc} \epsilon^T C \Gamma_\mu \lambda^a \lambda^{Tb} C \Gamma^\mu \lambda^c$

$$= f^{abc} \epsilon_\alpha (C\Gamma^\mu)_{\alpha\beta} \lambda^a_\beta \lambda^b_\gamma (C\Gamma_\mu)_{\gamma\delta} \lambda^c_\delta$$

Now, $\underline{\underline{\text{if}}}$ $(C\Gamma^\mu)_{\alpha\beta} (C\Gamma_\mu)_{\gamma\delta}$

$$+ (C\Gamma^\mu)_{\alpha\gamma} (C\Gamma_\mu)_{\delta\beta} + (C\Gamma^\mu)_{\alpha\delta} (C\Gamma_\mu)_{\beta\gamma}$$

$$= 0 \quad (**)$$

then Ⓙ $= -f^{abc} \epsilon_\alpha (C\Gamma^\mu)_{\alpha\gamma} \lambda^a_\beta \lambda^b_\gamma (C\Gamma_\mu)_{\delta\beta} \lambda^c_\delta$

$$-f^{abc} \epsilon_\alpha (C\Gamma^\mu)_{\alpha\delta} \lambda^a_\beta \lambda^b_\gamma (C\Gamma_\mu)_{\beta\gamma} \lambda^c_\delta$$

$\lambda^a, \lambda^b, \lambda^c$ anticommutes $\Rightarrow$

$$= f^{abc} \epsilon_\alpha (C\Gamma^\mu)_{\alpha\gamma} \lambda^b_\gamma \lambda^a_\beta (C\Gamma_\mu)_{\beta\delta} \lambda^c_\delta$$

$(C\Gamma_\mu)$ symmetric $\Rightarrow$

$$- f^{abc} \epsilon_\alpha (C\Gamma^\mu)_{\alpha\delta} \lambda^c_\delta \lambda^a_\beta (C\Gamma_\mu)_{\beta\gamma} \lambda^b_\gamma$$

$$= (f^{bac} - f^{bca}) \epsilon^T C\Gamma_\mu \lambda^a \lambda^{Tb} C\Gamma^\mu \lambda^c$$

$$= -2f^{abc} \epsilon^T C\Gamma_\mu \lambda^a \lambda^{Tb} C\Gamma^\mu \lambda^c$$

$= -2$ Ⓙ $\Rightarrow$ $\underline{\text{Ⓙ} = 0}$

So if (**) is true, then Ⓙ $=0$

So to show Ⓙ $=0$, it is sufficient to show that ~~Ⓙ~~ (**) is true.

we sandwhich (**) with two <u>anticommuting spinors</u> $\phi_1{}^\gamma$ and $\phi_2{}^\delta$, use the facts that $C\Gamma_\mu$ is symmetryic and $\phi_1{}^\gamma C_{\gamma\epsilon} = (\phi_1^T C)_\epsilon = \bar{\phi}_{2\epsilon}$, we have:

~~have (**) ⇒ $\Gamma^\mu_{\alpha\beta}\, \bar{\psi}_1 \Gamma_\mu \psi_2 + (\Gamma^\mu \psi_1)_\alpha (\bar{\psi}_2 \Gamma_\mu)_\beta$~~

~~$- (\Gamma^\mu \psi_2)_\alpha (\bar{\psi}_1 \Gamma_\mu)_\beta$~~

~~the minus sign in last term~~

~~have~~

$$\phi_1^\gamma (**) \phi_2^\delta = (C\Gamma^\mu)_{\alpha\beta}\, \underbrace{\phi_1{}^\gamma (C}_{\bar{\psi}_1} \Gamma_\mu)_{\gamma\delta}\, \phi_2{}^\delta$$

$$+ (C\Gamma^\mu)_{\alpha\gamma}\, \phi_1{}^\gamma \cancel{(C\Gamma_\mu)_{\delta\beta}}\, \underbrace{\phi_2{}^\delta (C}_{\bar{\psi}_2} \Gamma_\mu)_{\delta\beta}$$

$$- (C\Gamma^\mu)_{\alpha\delta}\, \phi_2{}^\delta\, \underbrace{\phi_1{}^\gamma (C}_{\bar{\psi}_1} \Gamma_\mu)_{\gamma\beta}.$$

in the last line "−" coms from swapping ϕ_1^γ, ϕ_2^δ. Multiplying C^{-1} on the left gives.

$$(***)\quad \Gamma^\mu{}_{\alpha\beta}\, \bar{\phi}_1 \Gamma_\mu \phi_2 + (\Gamma^\mu \phi_1)_\alpha (\bar{\phi}_2 \Gamma_\mu)_\beta + (\Gamma^\mu \psi_2)_\alpha (\bar{\phi}_1 \Gamma_\mu)_\beta$$

∵ ϕ_1, ϕ_2 arbitrary, ∴ (***) = 0 implies (**) = 0 implies Ⓢ = 0.

- An arbitrary matrix $N_{\alpha\beta}$ can be expanded in the complete basis of Gamma matrices

Note that (***) is a matsix with two indices α and β.

So it can be expanded in terms of

$\Gamma_{\rho_1\rho_2\cdots\rho_k}$ for $k=0,1,2,\cdots,10$, where $k=0$ is the identity matrix and k can't be >10 since in 10d we can only antisymmetrise 10 ~~indices~~ spacetime indices.

use the identity $\Gamma_{\rho_1\cdots\rho_k} = \cancel{0}-\frac{(i)^{k(k-1)-4}}{(10-k)!}\varepsilon_{\rho_1\cdots\rho_{10}}\Gamma^{\rho_{k+1}\cdots\rho_{10}}\Gamma_{11}$

The spinors in the Lagrangian are Majorana-Weyl, so ~~it is~~ they are Weyl makes them have definite Chirality

without loss of generality we say Chirality is positive.

so $\Gamma_{11}\lambda=\lambda$ and $\Gamma_{11}=-i\Gamma^0\Gamma^1\cdots\Gamma^9$.

so we can write $\Gamma_{\rho_1\cdots\rho_k}=\frac{-(i)^{k(k-1)}}{(10-k)!}\varepsilon_{\rho_1\cdots\rho_{10}}\Gamma^{\rho_{k+1}\cdots\rho_{10}}$

Hence we observe that $\Gamma_{\rho_1\cdots\rho_k}$ and $\Gamma^{\rho_{k+1}\cdots\rho_{10}}$ are not independent, so we only need to consider either Γ with k indices or with $10-k$ indices. It is sufficient, therefore to consider only

~~Also by Weyl projections terms with even k~~

— So if we represent (***) by a matrix $N_{\alpha\beta}$. first note that $\because$ (**) is symmetric in α,β so $(CN)_{\alpha\beta}$ is symmetric ($\because$ (***) $= C^{-1}\psi_1(**)\psi_2$)

we expand $N_{\alpha\beta}$ as

$$N_{\alpha\beta}= a_0\mathbb{1}_{16} + a_1\Gamma_{\rho_1} + a_2\Gamma_{\rho_1\rho_2} + \cdots + a_5\Gamma_{\rho_1\cdots\rho_5}$$

~~By Weyl projection, k even terms vanish → the~~

~~$\rightarrow a_0 = a_2 = a_4 = 0$~~

so ~~first~~ we consider k = odd. $(1,3,5)$

$\because$ $(CN)_{\alpha\beta}$ is symmetric $\therefore$ we looks for basis $\Gamma_{\rho_1\cdots\rho_k}$ such that $C\Gamma_{\rho_1\cdots\rho_k}$ symmetric.

$k=1$ $\rightarrow$ $C\Gamma^{\mu}$ is symmetric (proven before)

$k=3$ $\rightarrow$ $(C\Gamma^{\mu\nu\rho})^T = (\Gamma^{\mu\nu\rho})^T C^T = -\Gamma^{T[\rho}\Gamma^{T\nu}\Gamma^{T\mu]}C$

$$\underbrace{=}_{3\text{ swaps}} +C\Gamma^{[\rho}\Gamma^{\nu}\Gamma^{\mu]} \underbrace{=}_{1+2\text{ swaps}} -C\Gamma^{\mu\nu\rho} \quad \underline{\text{antisymmetric.}}$$

$k=5$ $\rightarrow$ $(C\Gamma^{abcde})^T = -(\Gamma^{abcde})^T C = -\Gamma^{T[e}\Gamma^{Td}\Gamma^{Tc}\Gamma^{Tb}\Gamma^{Ta]}C$

$$\underbrace{=}_{5\text{ swaps}} +C\Gamma^{[e}\Gamma^{d}\Gamma^{c}\Gamma^{b}\Gamma^{a]} \underbrace{=}_{\substack{4+3+2+1\\ \text{swaps}}} +C\Gamma^{abcde} \quad \underline{\text{symmetric}}$$

so we ~~set~~ have $a_3 = 0$, use $k=1$, $k=5$. only.

and $N_{\alpha\beta} = (a_1\Gamma_{\rho_1} + a_5\Gamma_{\rho_1\cdots\rho_5})_{\alpha\beta}$. for odd k

~~use (***), we find by contracting $(\Gamma_\rho)_{\beta\alpha}$~~

~~gives $\alpha_1 \propto \mathrm{Tr}(\Gamma^{\mu}\Gamma_{\rho})\bar{\Phi}_1\Gamma_\nu\Phi_2 - \bar{\Phi}_2\Gamma_\nu\Gamma_\rho\Gamma^{\mu}$~~

Now since there are 10 Γ_{ρ_1} matrices, and

$\dfrac{10\times 9\times\cdots\times 6}{5!}$ $\Gamma_{\rho_1\cdots\rho_5}$ matrices. But due to

$$\Gamma_{\rho\cdots\rho_5} = \frac{-(i)^{20}}{5!}\varepsilon_{\rho_1\cdots\rho_{10}}\Gamma^{\rho_6\cdots\rho_{10}},$$ we have

only half of $\Gamma_{\rho_1\cdots\rho_5}$ that contributes

So we have $10 + \frac{1}{2}\frac{10\times\cdots\times 6}{5!} = 136$ ~~de~~ ~~indep~~ independent basis, just from Γ_{ρ_1} and $\Gamma_{\rho_1\cdots\rho_5}$.

But this is precisely $136 = \frac{16\times 17}{2}$ = number of entries of a symmetric 16×16 matrix. $(CN)_{\alpha\beta}$.

So we do not need a_0, a_2, a_4 ~~a_1~~ any more. In fact, they are all 0 ~~4~~ because of Weyl projections.

So $N_{\alpha\beta} = a_1 (\Gamma_{\rho_1})_{\alpha\beta} + a_5 (\Gamma_{\rho_1\cdots\rho_5})_{\alpha\beta}$

use $(***)$ we find by contracting $(\Gamma_{\rho_1})_{\beta\alpha}$ gives

$$\rightarrow \quad a_1 \propto \mathrm{Tr}(\Gamma^\mu\Gamma_{\rho_1})\,\bar\phi_1\Gamma_\mu\phi_2 - \bar\phi_2\Gamma_\mu\Gamma_{\rho_1}\Gamma^\mu\phi_1 + \bar\phi_1\Gamma_\mu\Gamma_\rho\Gamma^\mu\phi_2$$

use
$$\begin{cases} \mathrm{Tr}(\Gamma^\mu\Gamma^\nu) = d\,\eta^{\mu\nu} \\ \Gamma_\mu\Gamma_{\rho_1\cdots\rho_k}\Gamma^\mu = (-1)^k(D-2k)\,\Gamma_{\rho_1\cdots\rho_k} \end{cases}$$

where ~~D~~ d = spinor space dimension $= 16$

D = ~~spact~~ spacetime dimension $= 10$

we have for $k=1$

$$a_1 \propto 16\,\bar\phi_1\Gamma_{\rho_1}\phi_2 + 8\,\bar\phi_2\Gamma_{\rho_1}\phi_1 - 8\,\bar\phi_1\Gamma_{\rho_1}\phi_2$$

$$\because \bar\phi_2\Gamma_{\rho_1}\phi_1 = -(\bar\phi_2\Gamma_{\rho_1}\phi_1)^T = +\phi_1^T\Gamma_{\rho_1}^T C\phi_2 = -\bar\phi_1\Gamma_{\rho_1}\phi_2$$

$$\therefore \underline{\underline{a_1 = 0}}$$

For the $k=5$ case, $Tr[\Gamma^{\mu}\Gamma_{\rho_1\cdots\rho_5}]=0$

so $a_5\alpha + \bar{\phi}_2\Gamma_{\mu}\Gamma_{\rho_1\cdots\rho_5}\Gamma^{\mu} \mp \bar{\phi}_1\Gamma_{\mu}\Gamma_{\rho_1\cdots\rho_5}\Gamma^{\mu}\phi_2$

$=2\bar{\phi}_1\Gamma_{\mu}\Gamma_{\rho_1\cdots\rho_5}\Gamma^{\mu}\phi_2$

but $\Gamma_{\mu}\Gamma_{\rho_1\cdots\rho_5}\Gamma^{\mu} = (-1)^5(10-2\times 5)\Gamma_{\rho_1\cdots\rho_5}$

$=0$ in $\underline{D=10}$

so $a_5=0$ ~~W~~ Hence $N_{\alpha\beta}=0$ indentically in 10 dimensions. (spacetime)

so $(**)=0 \Rightarrow ⑤=0 \Rightarrow \underrightarrow{①+③+④+⑤=0}$

$\Rightarrow$ $\mathcal{L}$ invariant and SUSY transformations given. $\square$

Now $\mathcal{L}_{10} = -\frac{1}{4g_{ym}^2}\left(\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + 2i\,\mathrm{Tr}(\bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda)\right)$

where $\mu = 0,1,\cdots,9$.

We decompose μ into ~~as i,j,k~~, $i = 0,1,\cdots,9-d$ and $p = 10-d, 11-d, \cdots, 9$

(e.g, $d=6$, $\mu,\nu,\cdots = 0,\cdots,9$
$i,j,k\cdots = 0,1,2,3$
$p,q,r\cdots = 4,5,\cdots,9.$)

We ~~first~~ do Kaluza-Klein reduction to bosons.

Assume that ~~all quantities~~ in the lower dimensional theory, physical quantities (fields) do not depend on the compactified dimensions ~~p,q~~ (coordinates)

So $\underline{\partial_p = 0}$

Hence, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ becomes

$$F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]$$ ~~$i,j = 0,1,2,3$~~ $i,j = 0,1,\cdots,9-d$

$$F_{ip} = \partial_i A_p - i[A_i, A_p]$$ $i = 0,1,\cdots,9-d$
$p = 10-d,\cdots,9.$

$$F_{pq} = -i[A_p, A_q]$$ $p,q = 10-d,\cdots,9$

So $F_{\mu\nu}F^{\mu\nu} = F_{ij}F^{ij} + F_{ip}F^{ip} + F_{pi}F^{pi} + F_{pq}F^{pq}$.

$\because$ $F_{ip} = -F_{pi}$ so

$$F_{\mu\nu}F^{\mu\nu} = F_{ij}F^{ij} + 2F_{ip}F^{ip} + F_{pq}F^{pq}$$

$$= F_{ij}F^{ij} + 2(\partial_i A_p - i[A_i, A_p])(\partial^i A^p - i[A^i, A^p])$$

$$+ (-i[A_p, A_q])(-i[A^p, A^q])$$

Note that $\partial_i A_p - i[A_i, A_p] = D_i A_p$ by definition

$$\therefore F_{\mu\nu}F^{\mu\nu} = F_{ij}F^{ij} + 2(D_i A_p)(D^i A^p) - [A_p, A_q][A^p, A^q]$$

Note that A_i is a 10-d dimensional gauge ~~fiel~~ field, and ~~the~~ A_p ~~are~~ on T^d are d number of ~~sca~~ adjoint scalars.

Now consider the fermions:

Note that $D_\nu \lambda$ becomes

$D_i \lambda = \partial_i \lambda - i[A_i, \lambda]$ $\quad i = 0, \dots, 9-d$

$D_p \lambda = -i[A_p, \lambda]$ $\quad p = 10-d, \dots, 9$ $(\because \partial_p = 0)$

So ~~2i Tr~~ $\bar\lambda \Gamma^\nu D_\nu \lambda = \bar\lambda \Gamma^i D_i \lambda + \bar\lambda \Gamma^p D_p \lambda$

$$= \cancel{\bar\lambda \Gamma^\nu D_\nu \lambda}\ \bar\lambda \Gamma^i D_i \lambda - i \bar\lambda \Gamma^p [A_p, \lambda]$$

So overall, the ~~action~~ Lagrangian becomes.

$$\boxed{\mathcal{L}_{10-d} = -\frac{1}{4g_{YM}^2}\Big(\mathrm{Tr}(F_{ij}F^{ij}) + 2\mathrm{Tr}((D_iA_p)(D^iA^p)) - \mathrm{Tr}([A_p,A_q][A^p,A^q]) + 2i\,\mathrm{Tr}(\bar\lambda\Gamma^i D_i\lambda) + 2\,\mathrm{Tr}(\bar\lambda\Gamma^p[A_p,\lambda])\Big)}$$

~~we are not done yet~~

and SUSY transformations $\delta A_\mu = -i\bar{\epsilon}\Gamma_\mu\lambda$

becomes
$$\boxed{\begin{aligned}\delta A_i &= -i\bar{\epsilon}\Gamma_i\lambda \\ \delta A_p &= -i\bar{\epsilon}\Gamma_p\lambda\end{aligned}}$$
$i = 0, \dots, 9-d$

$p = 10-d, \dots, 9$

and $\delta\lambda = \frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon = \frac{1}{2}F_{ij}\Gamma^{ij}\epsilon + F_{ip}\Gamma^{ip}\epsilon + \frac{1}{2}F_{pq}\Gamma^{pq}\epsilon$

where in the second term we used that F_{ip}, Γ^{ip} both antisymmetric.

use $F_{ip} = D_iA_p$, $F_{pq} = -i[A_p, A_q]$

$$\boxed{\delta\lambda = \left(\frac{1}{2}F_{ij}\Gamma^{ij} + D_iA_p\Gamma^{ip} - \frac{1}{2}[A_p, A_q]\Gamma^{pq}\right)\epsilon}$$

But we are not done yet, we should also decompose spinors under $\underline{SO(1,9) \rightarrow SO(d) \times SO(1,9-d)}$

Our Lagrangian $\mathcal{L}_{10}$ has λ a Majarana-~~Wely~~ Weyl spinor in the $\underset{\sim}{16}$ representation of $SO(1,9)$. This is a $\mathcal{N} = 1$ SUSY theory (supergravity) with 16 supersymmetries.

This can only be decomposed ~~into~~ as

$SO(1,9) \rightarrow SO(6) \times$ ~~[illegible]~~ $SO(1,3)$, which is a

$\mathcal{N} = 4$ SUSY gauge theory, ~~or~~ or to

$SO(1,9) \rightarrow SO(4) \times$ ~~SO(5,1)~~ $SO(1,5)$, which has

$\mathcal{N}=2$ supersymmetry.

- First case $d=6$, $\mathcal{N}=4$, $SO(1,9) \to SO(6)\times SO(1,3)$

the spinor representation decomposes like

$$\underset{SO(1,9)}{\underset{\sim}{16}^+} \to (\underset{SO(1,3)}{\underset{\sim}{2}^+} \otimes \underset{SO(6)}{\underset{\sim}{4}^+}) \oplus (\underset{SO(1,3)}{\underset{\sim}{2}^-} \otimes \underset{SO(6)}{\underset{\sim}{4}^-})$$

+ and − are chirality.

under this, we decompose the 10-d Gamma matrices Γ^μ as

$$\Gamma^i = \gamma^i \otimes \mathbb{1}_8 \qquad i=0,1,2,3$$

$$\Gamma^p = \gamma_5 \otimes \rho^p \qquad p=4,5,\cdots,9. \text{ on } T^6.$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Γ^μ is 32×32 matrix (10d spinors are 32-dimensional, Majorana-Weyl conditions make them ~~to~~ have 16 degrees of freedom), so we tensor with $\mathbb{1}_8$.

γ^i are 4d (4×4) gamma matrices (Minkowski)

ρ^p are 6d (8×8) gamma matrices (Euclidean)

we ~~we~~ then decompose spinors. we write λ_α as

$$\lambda_\alpha = \lambda_{AM}$$

where $A = 1,2,3,4 \in \underset{\sim}{2}^+ \oplus \underset{\sim}{2}^-$ in $SO(1,3)$

$M = 1,2,3,\cdots,8 \in \underset{\sim}{4}^+ \oplus \underset{\sim}{4}^-$ in $SO(6)$

Now we write A as A and $\dot{A}$ to turn Dirac spinors into Weyl spinors, and similarly for $M \to m$ and $\dot{m}$. We then have

~~$\lambda_{\dot{A}}$~~ ~~$\lambda_A = (4$~~

$$\lambda_m = \begin{pmatrix} 4^m_A \\ 0 \end{pmatrix} , \quad \lambda_{\dot{m}} = \begin{pmatrix} 0 \\ \chi^{\dot{A}}_{\dot{m}} \end{pmatrix}$$

To clarify, each of ~~we~~ λ_m and $\lambda_{\dot{m}}$, $m = 1,2,3,4$, $\dot{m} = \dot{1},\dot{2},\dot{3},\dot{4}$ is a ~~4d~~ Weyl spinor in 4d (2 components, $A = 1,2$, $\dot{A} = \dot{1},\dot{2}$)

with ~~the~~ the Majorana condition gives

$$\lambda_m = \begin{pmatrix} \chi^m_A \\ 0 \end{pmatrix} , \quad \lambda_{\dot{m}} = \begin{pmatrix} 0 \\ \chi^{\dot{A}}_{\dot{m}} \end{pmatrix}$$

So the spinor terms.

$$-\quad \bar{\lambda}\Gamma^i D_i \lambda = \bar{\lambda}_{AM} \Gamma^i_{ABMN} D_i \lambda_{BN}$$

$$= \bar{\lambda}_{AM} \gamma^i_{AB} \mathbb{1}_{MN} D_i \lambda_{BN} = \bar{\lambda}_{AM} \gamma^i_{AB} D_i \lambda_{BM}$$

$$-\quad \bar{\lambda}\Gamma^P [A_P, \lambda] = \bar{\lambda}_{AM} \gamma^J_{AB} \rho^P_{MN} [A_p, \lambda_{BM}]$$

So Lagrangian

$$\boxed{\begin{aligned} \mathcal{L}_4 = -\frac{1}{4g_{YM}^2} \Big(& \mathrm{Tr}(F_{ij}F^{ij}) + 2\,\mathrm{Tr}\left((D_i A_p)(D^i A^P)\right) \\ & - \mathrm{Tr}\left([A_p, A_q][A^P, A^q]\right) + 2i\,\mathrm{Tr}\left[\bar{\lambda}_{AM}\gamma^i_{AB} D_i \lambda_{BM}\right] \\ & + 2\,\mathrm{Tr}\left(\bar{\lambda}_{AM} \gamma^J_{AB} \rho^P_{MN} [A_p, \lambda_{BM}]\right) \Big) \end{aligned}}$$

Where $\lambda_{AM} = \begin{pmatrix} \chi_{AM} \\ 0 \end{pmatrix}$ $A=1,2$, $M=1,2,3,4$

$\lambda_{AM} = \begin{pmatrix} 0 \\ \chi^{\dot{A}\dot{M}} \end{pmatrix}$ $\dot{A}=\dot{1},\dot{2}$, $\dot{M}=\dot{1},\dot{2},\dot{3},\dot{4}$

SUSY transformation is then: (turn ϵ_α into ϵ_{AM} as just as λ)

$$\delta A_i = -i\,\bar{\epsilon}_{AM}\,\gamma_{i\,AB}\,\lambda_{BM}$$

$$\delta A_p = -i\,\bar{\epsilon}_{AM}\,\gamma_{5,AB}\,\rho_{p,MN}\,\lambda_{BN}$$

$$\delta\lambda = \frac{1}{2}\left(F_{ij}\gamma^{ij} + D_i A_p \left(\frac{1}{2}\gamma^{[i}\rho^{p]}\right) - \frac{1}{2}[A_p, A_q]\,\rho^{pq}\right)\epsilon$$

- The other case is $\mathcal{N}=2$, $d=4$.
compactification on T^4.

$$SO(1,9) \to SO(4) \times SO(1,5)$$

$$\underset{\sim}{16}^{+} \to (\underset{\sim}{4}^{+}_{\,SO(1,5)} \otimes \underset{\sim}{2}^{+}_{\,SO(4)}) \oplus (\underset{\sim}{4}^{-}_{\,SO(1,5)} \otimes \underset{\sim}{2}^{-}_{\,SO(4)})$$

decompose ~~γ matr~~ Γ^N matrices as

$$\Gamma^i = \gamma^i \otimes \mathbb{1}_8 \qquad i = 0, 1, 2, \cdots, 5.$$

$$\Gamma^p = \gamma_7 \otimes \cancel{R}\,\rho^p \qquad p = 6, 7, 8, 9.$$

$$\gamma_7 = i\gamma^0\gamma^1 \cdots \gamma^5.$$

The ~~decomposition~~ Lagrangian and SUSY transformation after reduction is the same as $d=6$, $\mathcal{N}=4$ case, except γ_5 replaced by γ_7 and the range of i and p have changed to the above.

$\boxed{2}$

1) $X^\mu(z,\bar z)\, X^\nu(0,0) \sim -\eta^{\mu\nu}\frac{\alpha'}{2}\left(\underbrace{\ln z + \ln\bar z}_{=\ln z\bar z = \ln|z|^2}\right)$

$\lambda^A(z)\,\lambda^B(0) \sim \delta^{AB}\frac{1}{z}.$

$\tilde\psi^\mu(\bar z)\,\tilde\psi^\nu(0) \sim \eta^{\mu\nu}\frac{1}{\bar z}$

$X\lambda \sim \lambda\psi \sim X\psi \sim 0$

$T(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\lambda^A\partial\lambda^A$

$\bar T(\bar z) = -\frac{1}{\alpha'}\bar\partial X^\mu\bar\partial X_\mu - \frac{1}{2}\tilde\psi^\mu\bar\partial\tilde\psi_\mu$

<u>$T T$, OPE</u>: calculate term by term

we first focus on leading terms $\sim \frac{1}{z^4}$

~~$\partial X^\mu(z)\partial X_\mu(z)\partial X^\nu$~~

leading term has double contractions

① $= \partial X^\mu(z,\bar z)\,\partial X_\mu(z,\bar z)\,\partial X^\nu(0,0)\,\partial X_\nu(0,0)$

$= \partial X^\mu(z,\bar z)\,\partial X_\mu(z,\bar z)\,\partial X^\nu(0,0)\,\partial X_\nu(0,0).$

$+\,\partial X^\mu(z,\bar z)\,\partial X_\mu(z,\bar z)\,\partial X^\nu(0,0)\,\partial X_\nu(0,0)$

$= 2\,\partial X^\mu(z,\bar z)\,\partial X_\mu(z,\bar z)\,\partial X^\nu(0,0)\,\partial X_\nu(0,0)$

$\because\ X^\mu(z,\bar z)\,X^\nu(0,0) = -\eta^{\mu\nu}\frac{\alpha'}{2}(\ln z + \ln\bar z)$

$\therefore\ \partial X^\mu(z,\bar z)\cdot X^\nu(0,0) = -\eta^{\mu\nu}\frac{\alpha'}{2}\frac{\partial}{\partial z}\left(\ln(z-w) + \ln(\bar z - \bar w)\right)\Big|_{w=0}$

$= -\eta^{\mu\nu}\frac{\alpha'}{2}\frac{1}{z-w}\Big|_{w=0}$

and

$$\partial X^\mu(z,\cancel{\bar z})\,\partial X^\nu(0,\cancel{0}) = -\eta^{\mu\nu}\frac{\alpha'}{2}\frac{\partial}{\partial w}\left(\frac{1}{z-w}\right)\Big|_{w=0.}$$

$$= -\eta^{\mu\nu}\frac{\alpha'}{2}\frac{-1}{(z-w)^2}(-1)\Big|_{w=0.}$$

$$= -\eta^{\mu\nu}\frac{\alpha^2}{2}\frac{1}{z^2}$$

So

$$① = 2\,\overbrace{\partial X^\mu(z,\bar z)\,\partial X^\nu(\cancel{0},\cancel{0})}\;\overbrace{\partial X_\mu(z,\bar z)\,\partial X_\nu(0,0)}$$

$$= 2\cdot\left(-\eta^{\mu\nu}\frac{\alpha'}{2}\frac{1}{z^2}\right)\left(-\eta_{\mu\nu}\frac{\alpha'}{2}\frac{1}{z^2}\right)$$

$$= 2D\cdot\frac{(\alpha')^2}{4}\frac{1}{z^4} = \frac{D}{2}(\alpha')^2\frac{1}{z^4} \underset{D=10.}{=} \frac{5(\alpha')^2}{z^4}$$

~~② = $\lambda^A\partial\lambda$ $\lambda^A(z)\partial\lambda$~~

$$② = \lambda^A(z)\,\partial\lambda^A(z)\,\cancel{\partial}\lambda^B(0)\,\partial\lambda^B(0)$$

$$= \overbrace{\lambda^A(z)\,\underbrace{\partial\lambda^A(z)\,\cancel{\partial}\lambda^B(0)\,\partial\lambda^B(0)}}$$

$$+ \underbrace{\lambda^A(z)\,\underbrace{\partial\lambda^A(z)\,\cancel{\partial}\lambda^B(0)}\,\partial\lambda^B(0)}$$

$$\lambda^A(z)\,\lambda^B(w) = \delta^{AB}\frac{1}{z-w}$$

$$\therefore\ \partial\lambda^A(z)\,\lambda^B(w) = \delta^{AB}\left(-\frac{1}{(z-w)^2}\right)$$

$$\lambda^A(z)\,\partial\lambda^B(w) = \delta^{AB}\left(\frac{1}{(z-w)^2}\right)$$

~~$\partial^A(z)\,\partial^B$~~ $\partial\lambda^A(z)\,\partial\lambda^B(w) = \delta^{\cancel{AB}}\left(-\frac{2}{(z-w)^3}\right)$

1 swap −

$$\therefore\ ② = -\lambda^A(z)\, \lambda^B(0)\, \partial\lambda^A(z)\, \partial\lambda^B(0)$$

λ anti commuting

$$+\ \lambda^A(z)\, \partial\lambda^B(0)\, \partial\lambda^A(z)\, \lambda^B(0)$$

2 swaps +

$$= \delta^{AB}\frac{1}{z}\cdot\left(+\delta^{AB}\frac{2}{(z)^3}\right)$$

$$+\ \left(\delta^{AB}\frac{1}{z^2}\right)\left(\delta^{AB}\left(-\frac{1}{z^2}\right)\right)$$

$$= +\underbrace{\delta^{AA}}_{=32}\left(\frac{2}{z^4}-\frac{1}{z^4}\right) = +\frac{32}{z^4}$$

So, $\underline{T(z)\,T(0)} = \left(-\frac{1}{\alpha'}\right)^2 ① + \left(-\frac{1}{2}\right)^2 ②$

$$= \frac{1}{(\alpha')^2}\frac{J}{z^4}(\alpha')^2 - \frac{32}{z^4}\times\frac{1}{4}\times 3'$$

$$= \frac{J}{z^4} - \frac{8}{z^4} = ~~-\frac{3}{z^4}~~ \quad \underline{\frac{13}{z^4}} \text{ (leading term)}$$

③ = ~~$\bar\partial X^\mu(z,\bar z)\, \bar\partial X_\mu(z,\bar z)\, \bar\partial X^\nu(0,0)\, \bar\partial X_\nu(0,0)$~~

$\bar\partial X^\mu(\bar z)\, \partial X_\mu(\bar z)\, \bar\partial X^\nu(0)\, \bar\partial X_\nu(0)$

is exactly the same as ① except $z \leftrightarrow \bar z$

So $③ = \dfrac{J(\alpha')^2}{\bar z^4}$

④ = ~~$\tilde\psi^\mu(\bar z)\bar\partial\psi_\mu(\bar z)$~~ $\tilde\psi^\mu(\bar z)\, \bar\partial\tilde\psi_\mu(\bar z)\, \tilde\psi^\nu(0)\, \bar\partial\tilde\psi_\nu(0)$.

$$= \tilde\psi^\mu(\bar z)\, \bar\partial\tilde\psi_\mu(\bar z)\, \tilde\psi^\nu(0)\, \bar\partial\tilde\psi_\nu(0)$$

$$+\ \tilde\psi^\mu(\bar z)\, \bar\partial\tilde\psi_\mu(\bar z)\, \tilde\psi^\nu(0)\, \bar\partial\tilde\psi_\nu(0)$$

~~wt~~ we have $\tilde\psi^\mu(\bar z)\,\tilde\psi^\nu(\bar w) \sim \eta^{\mu\nu}\frac{1}{\bar z-\bar w}$

∴ $\bar\partial\tilde\psi^\mu(\bar z)\,\tilde\psi^\nu(\bar w) = -\eta^{\mu\nu}\frac{1}{(\bar z-\bar w)^2}$

$\tilde\psi^\mu(\bar z)\,\bar\partial\tilde\psi^\nu(\bar w) = \eta^{\mu\nu}\frac{1}{(\bar z-w)^2}$

$\bar\partial\tilde\psi^\mu(\bar z)\,\bar\partial\tilde\psi^\nu(\bar w) = -2\eta^{\mu\nu}\frac{1}{(\bar z-\bar w)^3}$

so ④ $= -\tilde\psi^\mu\underbrace{(\bar z)\,\tilde\psi^\nu}(0)\,\bar\partial\tilde\psi_\mu\underbrace{(\bar z)\,\bar\partial\tilde\psi_\nu}(0)$ 1 swap. "−" sign

$+\,\tilde\psi^\mu\underbrace{(\bar z)\,\bar\partial\tilde\psi^\nu}(0)\,\bar\partial\tilde\psi^\mu\underbrace{(\bar z)\,\tilde\psi^\nu}(0)$ 2 swaps

∵ $\tilde\psi$ anticommutes.

$= \eta^{\mu\nu}\frac{1}{\bar z}\cdot\eta_{\mu\nu}\left(+2\frac{1}{\bar z^3}\right)$

$+\left(\eta^{\mu\nu}\frac{1}{\bar z^2}\right)\left(-\eta^{\mu\nu}\frac{1}{\bar z^2}\right)$

$= +\underbrace{\eta^{\mu\nu}\eta_{\mu\nu}}_{10}(2-1)\frac{1}{\bar z^4} = \cancel{-\frac{30}{\bar z^4}}\ \frac{10}{\bar z^4}.$

∴ $\bar T(z)\,\bar T(0) \sim \left(-\frac{1}{\alpha'}\right)^2$ ③ $+\left(-\frac{1}{2}\right)^2$ ④

$= \frac{1}{(\alpha')^2}\frac{5(\alpha')^2}{\bar z^4} + \frac{1}{4}\times\frac{10}{\bar z^4}$

$= \cancel{\frac{8}{2\bar z^4}}\ \frac{30}{4\bar z^4}$ (leading terms)

Now we compute the subleading terms.

subleading terms have single contractions.

$$\textcircled{J} = \partial X^\mu(z,\bar z)\,\partial X_\mu(z,\bar z)\,\partial X^\nu(0,0)\,\partial X_\nu(0,0)$$

$$= 4 \times \left(-\frac{1}{\alpha'^2}\right) \overbrace{\partial X^\mu(z,\bar z)\,\partial X^\nu(0,0)}^{} \,\partial X_\mu(z,\bar z)\,\partial X_\nu(0,0)$$

4 possible single contractions: $-\eta^{\mu\nu}\frac{\alpha'}{2}\frac{1}{z^2}$

$$= 4\,\frac{1}{\alpha'^2}\left(-\eta^{\mu\nu}\right)\frac{\alpha'}{2}\frac{1}{z^2}\,\partial X_\mu(z,\bar z)\,\partial X_\nu(0,0).$$

$$= \cancel{\frac{2}{\alpha'} + 2\left(\frac{-1}{\alpha'}\right)} \;\frac{2}{z^2}\left(-\frac{1}{\alpha'}\partial X^\mu(z,\bar z)\,\partial X_\mu(0,0)\right)$$

use $\partial X^\mu(z,\bar z) = \cancel{\partial X^\mu(0,0) + \partial^2 X^\mu(0,0)\, z}$

$$\partial\left(X^\mu(z) + \bar X^\mu(\bar z)\right) = \partial X^\mu(z)$$

$$= \partial X^\mu(0) + \partial^2 X^\mu(0)\, z + \dots$$

$$= \partial X^\mu(0,0) + \partial^2 X^\mu(0,0)\, z + \dots$$

$$\therefore\ \textcircled{J} = \frac{2}{z^2}\left(-\frac{1}{\alpha'}\partial X^\mu(0,0)\,\partial X_\mu(0,0)\right.$$

$$\left.\left. -\frac{1}{\alpha'}\partial_z^2 X^\mu(0,0)\,\partial_w X_\mu(0,0)\right)\right|_{w=0}.$$

$$\cancel{= \frac{2}{z^2}\left(-\frac{1}{\alpha'}\partial\right.}$$

$$= \frac{2}{z^2}\left(-\frac{1}{\alpha'}\partial X^\mu(0,0)\,\partial X_\mu(0,0)\right)$$

$$+ \frac{1}{z}\left(\partial\left(-\frac{1}{\alpha'}\partial X^\mu(0,0)\,\partial X_\mu(0,0)\right)\right).$$

5)

⑥ $= \bar\partial X^\mu \bar\partial X_\mu{}_{(\bar z)}\, \bar\partial X^\nu \bar\partial X_\nu{}_{(0)}$ is the same as ⑤ with $z \leftrightarrow \bar z$

$$\therefore ⑥ = \frac{2}{\bar z^2}\left(-\frac{1}{\alpha'}\bar\partial X^\mu(0.0)\bar\partial X_\mu(0.0)\right)$$

$$+ \frac{1}{\bar z}\left(\bar\partial\left(-\frac{1}{\alpha'}\bar\partial X^\mu(0,0)\bar\partial X_\mu(0.0)\right).\right)$$

⑦ ∴ = subleading of $\frac{1}{4}.\tilde\psi^\mu(\bar z)\bar\partial\tilde\psi_\mu(\bar z)\bar\partial\tilde\psi^\nu(0)\bar\partial\tilde\psi_\nu(0).$

~~$= -\tilde\psi^\mu(\bar z)\partial\tilde\psi^\nu(0)\bar\partial\tilde\psi_\mu(\bar z)$~~

$$= \Big(-\overbrace{\tilde\psi^\mu(\bar z)\tilde\psi^\nu(0)}\,\bar\partial\tilde\psi_\mu(\bar z)\bar\partial\tilde\psi_\nu(0).$$

$$+\overbrace{\tilde\psi^\mu(\bar z)\bar\partial\tilde\psi^\nu(0)}\,\bar\partial\tilde\psi_\mu(\bar z)\tilde\psi_\nu(0)$$

$$+\overbrace{\bar\partial\tilde\psi^\mu(\bar z)\tilde\psi^\nu(0)}\,\tilde\psi_\mu(\bar z)\bar\partial\tilde\psi_\nu(0)$$

$$-\overbrace{\bar\partial\tilde\psi^\mu(\bar z)\bar\partial\tilde\psi^\nu(0)}\,\tilde\psi_\mu(\bar z)\tilde\psi_\nu(0)\Big)\times\frac{1}{4}.$$

$$= \Big(-\eta^{\mu\nu}\frac{1}{\bar z}\bar\partial\tilde\psi_\mu(\bar z)\bar\partial\tilde\psi_\nu(0)$$

$$+\eta^{\mu\nu}\frac{1}{\bar z^2}\bar\partial\tilde\psi_\mu(\bar z)\tilde\psi_\nu(0)$$

$$-\eta^{\mu\nu}\frac{1}{\bar z^2}\tilde\psi_\mu(\bar z)\bar\partial\tilde\psi_\nu(0)$$

$$+2\eta^{\mu\nu}\frac{1}{\bar z^3}\tilde\psi_\mu(\bar z)\tilde\psi_\nu(0)\Big)\times\frac{1}{4}.$$

$$\tilde\psi_\mu(\bar z) = \tilde\psi_\mu(0) + \bar\partial\tilde\psi_\mu(0)\,\bar z + \frac{1}{2}\bar\partial\bar\partial\tilde\psi_\mu(0)\,\bar z^2$$

so $\text{⑦} = -\Big(-\frac{1}{\bar z}\cancel{\bar\partial\tilde\psi_\mu(0)\,\underbrace{\partial\tilde\psi^\mu(0)}_{=0}} + \cdots$ regular

antisym.

anti commutes

$$+\underline{\frac{1}{\bar z^2}\bar\partial\tilde\psi^\mu(0)\,\tilde\psi_\mu(0)} + \frac{1}{\bar z}\underbrace{\bar\partial(\bar\partial\tilde\psi^\mu(0))\tilde\psi_\mu(0))}.$$

$$-\underline{\frac{1}{\bar z^2}\tilde\psi^\mu(0)\,\bar\partial\tilde\psi_\mu(0)} - \frac{1}{\bar z}\cancel{\bar\partial(\tilde\psi^\mu(0))\underbrace{\bar\partial\tilde\psi_\mu(0)}_{=0}})$$

antisym

$$+2\,\frac{1}{\bar z^3}\cancel{\tilde\psi^\mu(0)\,\underbrace{\tilde\psi_\mu(0)}_{=0}} + \underline{\frac{2}{\bar z^2}\bar\partial\tilde\psi^\mu(0)\,\tilde\psi_\mu(0)}$$

antisym.

$$+\underbrace{\frac{1}{\bar z}\,\bar\partial\bar\partial\tilde\psi^\mu(0)\,\tilde\psi_\mu(0)}\Big)\times\frac{1}{4}.$$

$$= -\frac{4}{\bar z^2}\tilde\psi^\mu(0)\bar\partial\tilde\psi_\mu(0) - \frac{2}{\bar z}\bar\partial(\tilde\psi^\mu(0)\bar\partial\tilde\psi_\mu(0))$$

$$= \frac{2}{\bar z^2}\Big(-\frac{1}{2}\tilde\psi^\mu(0)\bar\partial\tilde\psi_\mu(0)\Big) + \frac{1}{\bar z}\bar\partial\Big(-\frac{1}{2}\tilde\psi^\mu(0)\bar\partial\tilde\psi_\mu(0)\Big)$$

∴ overall:

$$\underline{\bar T(\bar z)\bar T(0)} = \frac{30}{4\bar z^4} + \frac{2}{\bar z^2}\Big(\overset{⑥}{-\frac{1}{\alpha'}\bar\partial X^\mu(0,0)\bar\partial X_\mu(0,0)}$$

$$-\frac{1}{2}\tilde\psi^\mu(0)\bar\partial\tilde\psi_\mu(0)\Big) + \frac{1}{\bar z}\Big(\overset{⑥}{\bar\partial\Big(-\frac{1}{\alpha'}\bar\partial X^\mu(0,0)\bar\partial X_\mu(0,0)\Big)}$$

$$\overset{⑦}{+\bar\partial\Big(-\frac{1}{2}\tilde\psi^\mu(0)\bar\partial\tilde\psi_\mu(0)\Big)}\Big).$$

$$= \boxed{\frac{30}{4\bar z^4} + \frac{2}{\bar z^2}\bar T(0) + \frac{1}{\bar z}\bar\partial\bar T(0)}$$

7)

Apply exactly same argument of $\tilde{\psi}$ to λ, we get the overall TT OPE

$$\boxed{T(z)\,T(0) \sim \frac{13}{z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0)}$$

8)

2. To do bosonization, note that $\lambda^A(z)$ has $A = 1, 2, \dots, 32$. we ~~wite~~ write

$$\lambda^{K\pm} = 2^{-1/2}(\lambda^{2K-1} \pm i\lambda^{2K}) \quad , \underline{K = 1, \dots, 16}$$

~~key~~ since $\lambda^A(z)\lambda^B(0) \sim \frac{1}{z}$, we have

$$\cancel{\lambda^{K+}\lambda^{K-} \sim \frac{1}{2}(}$$

$$\lambda^{K+}(z)\lambda^{K-}(0) \sim \frac{1}{2}\left(\frac{1}{z} + \frac{1}{z}\right) = \frac{1}{z}$$

$$\lambda^{K+}(z)\lambda^{K+}(z) \sim \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z}\right) = 0$$

$$\lambda^{K-}(z)\lambda^{K-}(0) \sim 0$$

the holomorphic ~~of~~ part of a scalar field $H(z)$ has OPE

$$H(z)H(0) \sim \cancel{\ln z} - \ln z$$

$$\text{so} \quad \left.\begin{array}{l} e^{iH(z)}e^{-iH(0)} \sim \frac{1}{z} \\ e^{iH(z)}e^{iH(0)} \sim 0 \\ e^{-iH(z)}e^{-iH(z)} \sim 0 \end{array}\right\} \begin{array}{l} \text{same OPE} \\ \text{as } \lambda^{K\pm} \end{array}$$

we thus can write

$$\lambda^{K+}(z) = e^{iH^K(z)} \quad , \quad \lambda^{K-} = e^{-iH^K(z)}$$

where $K = 1, 2, \dots, 16$. , $H^K(z)$ is a scalar bosonic field.

We turned a pair of fermions ~~the~~ into a boson, this gives the name bosoni~~z~~ation.

∵ χ^A's obey periodic boundary conditions, so do the $H^K(z)$'s.

The periodic boundary condition on $H^K(z)$ (16 bosons) is equivalent to ~~compactify.~~ compactifying them on a T^{16} Torus.

Consider this 16 compact bosons $X_L^m(z)$ $m=1,2,\cdots,16$. It only has z, not $\bar{z}$, so ~~the to~~ these bosons are Chiral

the vertex operater for these bosons is then given by

$V(z) = e^{ik_L\cdot X_L(z)}$ and two vertex operators gives an OPE

~~$V(z)V'(0) \sim z^{l_L\cdot l_L'} e^{ik_L\cdot X_L+k_R}$~~

$$V(z)V'(0) = e^{ik_L\cdot X_L}\cdot e^{ik_L'\cdot X_L}$$

$$\sim z^{l_L\cdot l_L'} e^{i(k_L+k_L')\cdot X_L}$$

where l_L is the rescaled momentum of k_L.
$l_L^m = \sqrt{\frac{\alpha'}{2}}k_L^m$

Single-valuedness requires that (monodromy).

$$l_L\circ l_L' = \langle l_L, l_L'\rangle \in \mathbb{Z}$$

and modular invariance gives $l_L \circ l_L = 2\mathbb{Z}$ (even) and this lattice formed by l_L's is self dual

So, since l_L^m has $m = 1, 2, \cdots, 16$, we have a even, self-dual, Euclidean lattice of momenta l_L^m. The dimension of this lattice is 16.

there are only 2 such lattices.

$$\Gamma_{16} \quad \text{and} \quad \Gamma_8 \oplus \Gamma_8$$

where Γ_{16} is the set of points.

$$(n_1, \cdots, n_{16}) \quad \text{or} \quad (n_1 + \tfrac{1}{2}, \cdots, n_{16} + \tfrac{1}{2})$$

$$\sum_i n_i \in 2\mathbb{Z}$$

and Γ_8 is the set of points

$$(n_1, \cdots, n_8) \quad \text{or} \quad (n_1 + \tfrac{1}{2}, \cdots, n_8 + \tfrac{1}{2})$$

$$\sum_i n_i \in 2\mathbb{Z}$$

since we impose periodic boundary conditions on all 32 λ^A fermions and thus on all 16 $H^K(z)$ bosons, this corresponds to the Γ_{16} lattice

For Γ_{16}, the points of length squared 2 ~~are jus~~

are just the SO(32) roots, so the CFT of λ^A's is equivalent to a 10d SO(32) heterotic string

Q.

3. Fermionic desuription.

the fermions λ^A, $A=1,2,\cdots 32$ satisfy periodic or antiperiodic conditions

$$\lambda^A(z+2\pi) = \begin{cases} +\lambda^A(z), & P \text{ sector} \\ -\lambda^A(z), & A \text{ sector} \end{cases}$$

we can separate the 32 fermions λ^A into 2 groups, one has $A=1,2,\cdots,n$, the other has $A=n+1,\cdots,32$. The two groups can obey different boundary conditions (P or A)

Then there are 4 sectors in total, namely AA, AP, PA, PP for the two groups of λ^A.

we compute the normal ordering constants for the 4 sectors. Recall that normal ordering constant for a boson is $\frac{1}{24}$, for a periodic fermion is $-\frac{1}{24}$, and for an antiperiodic fermion is $\frac{1}{48}$.

8 transverse bosonic dimensions gives the normal ordering constants as

$$a_{AA} = \frac{8}{24} + \frac{n}{48} + \frac{32-n}{48} = 1$$

$$a_{AP} = \frac{8}{24} + \frac{n}{48} - \frac{32-n}{24} = \frac{n}{16} - 1$$

$$a_{PA} = \frac{8}{24} - \frac{n}{24} + \frac{32-n}{48} = 1 - \frac{n}{16}$$

$$a_{PP} = \frac{8}{24} - \frac{n}{24} - \frac{32-n}{24} = -1$$

- The PP sector has $a > 0$, by mass shell condition there is no massless states in this sector

- The AA sector has $a = 1$, so massless states are $\underline{\alpha^i_{-1}|0\rangle_A \ , \ \lambda^A_{-\frac{1}{2}} \lambda^B_{-\frac{1}{2}} |0\rangle_A}$

Now impose GSO projection $(-1)^F$ on bosons and $(-1)^{F_1}$ on fermions λ^A

with $A=1,\cdots,n$ and $(-1)^{F_2}$ on fermions λ^A with $A=n+1,\cdots,32$. and projects to $(-1)^F=(-1)^{F_1}=(-1)^{F_2}=1$. Since $(-1)^{F_1}$ anticommutes with λ^A $(A=1,\cdots,n)$ but commutes with λ^A $(A=n+1,\cdots,32)$, and $(-1)^{F_2}$ commutes with λ^A $(A=1,\cdots,n)$ and anticommutes with λ^A $(A=n+1,\cdots,32)$, we have that states $\lambda^A_{-\frac{1}{2}}\lambda^B_{-\frac{1}{2}}|0\rangle_A$ with $A=1,\cdots,n$ and $B=n+1,\cdots,32$ or vice versa are projected out by GSO. So massless states are $\lambda^A_{-\frac{1}{2}}\lambda^B_{-\frac{1}{2}}|0\rangle_A$ with A, B in the same group of indices.

- If $\underline{n=16}$, then $a_{PA}=a_{AP}=0$, there are additional massless states in

PA and AP sectors, given by the ground states ($\because a=0$). Making 8 raising and lowering operators out of the 16 λ^A fermion zero modes gives a 256 dimensional spinor representation of $SO(16)$, but GSO projection cuts it to two 128 representations $128 + 128'$, one for each group of indices $A = 1, \cdots, 16$ and $A = 16, \cdots, 32$.

4. we consider the $n=16$ case. From 3. by GSO projection the massless states $\lambda^A_{1/2}\lambda^B_{1/2}|0\rangle_A$ have

① A, B both $= 1, 2, \cdots, 16$ or

② A, B both $= 17, \cdots, 32$

in ①, we have A, B both take 16 values but antisymmetrised, so it gives a $\frac{16 \times 15}{2} = \underset{\sim}{120}$, the adjoint representation of $SO(16)$. of first 16 indices, and a singlet of last 16 indices.

② is similar to ①. ② gives a singlet for first 16 indices and a $\underset{\sim}{120}$ representation of SO(16).

So togather $\lambda^{A}_{1/2}\lambda^{B}_{1/2}|0\rangle_A$ states gives

a $(\underset{\sim}{120}, \underset{\sim}{1}) \oplus (\underset{\sim}{1}, \underset{\sim}{120})$ representation

of $\underline{SO(16) \times SO(16)}$

The massless ground states in PA and AP sectors give, as mentioned in 3),

a $(128, 1) \oplus (1, 128)$ representation of

SO(16) × SO(16).

So massless fields realize a $\underline{SO(16) \times SO(16)}$ algebra in the representation

$$(120, 1) + (1, 120) + (128, 1) + (1, 128)$$

for the left movers

17) - The right movers (regular RNS string)

has massless $\alpha^i_{-1}|0\rangle_A$ states in $\underset{\sim}{8_v}$ representation of $SO(8)_{spin}$ internal group. so for each $SO(16)$, massless vector bosons transform as $\underset{\sim}{120}+\underset{\sim}{128}$ representation of $SO(16)$, which gives the adjoint $\underset{\sim}{248}$ of the E_8 group. so overall we have $\underline{E_8 \times E_8}$ symmetry.

The momentum operator that extends $SO(16) \times SO(16)$ to $E_8 \times E_8$ is

$$\underline{\exp\left[i \sum_{k=1}^{16} q_k H^k(z)\right]}$$

where $H^k(z)$ is the scalar boson after bosonization in 2).

for first E_8 $\quad q_k = \begin{cases} \pm\frac{1}{2}, & k=1,\cdots,8 \\ 0, & k=9,\cdots,16 \end{cases}$

for second E_8 $\quad q_k = \begin{cases} 0, & k=1,\cdots,8 \\ \pm\frac{1}{2}, & k=9,\cdots,16 \end{cases}$

18)

and we have $\sum_{k=1}^{16} q_k \in 2\mathbb{Z}$ (an even, self-dual Euclidean lattice as shown in 2)) .