String Theory II

Problem Set 2

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Th 15:00 Wk 2

$\boxed{1}$ 1) A torus $T^2_\tau = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus \tau\mathbb{Z}}$ means that

a coordinate w on the torus is periodically identified as

$$(*) \quad \boxed{w \cong w + 2\pi(m + n\tau)} \; ; m, n \in \mathbb{Z}$$

So $m \in \mathbb{Z}$, $n\tau \in \tau\mathbb{Z}$. $\mathbb{Z}$ and $\tau\mathbb{Z}$ being quotient out from $\mathbb{C}$ gives the periodicity $(*)$. Now we define two transformations

$$T: \tau \to \gamma\tau = \tau + 1$$

$$S: \tau \to \gamma\tau = -\frac{1}{\tau}$$

we see that:

- under T,

$$w \cong w + 2\pi(m + n\tau) = w + 2\pi((m-n) + n\underbrace{(\tau+1)}_{\gamma\tau})$$

$$= w + 2\pi((m-n) + n(\gamma\tau)) \in T^2_{\gamma\tau}$$

which gives the same set of identification as T^2_τ replacing $\underline{(m,n) \to (m-n, n)}$

And under S:

T_τ^2: $w' \cong w' + 2\pi(m + n\tau) \Rightarrow \frac{w'}{\tau} \cong \frac{w'}{\tau} + 2\pi(\frac{m}{\tau} + n)$

define $w' = w\tau$ so now

$w \cong w + 2\pi(\frac{m}{\tau} + n)$, now apply S,

$\because \frac{1}{\tau} = -\gamma\tau \quad \therefore w \cong w + 2\pi(n - m(\gamma\tau)) \in T_{\gamma\tau}^2$

which gives the same set of identification

as T_τ^2 replacing $(m, n) \to (n, -m)$

Hence, repeated applications of transformation

S and T describe the same space.

- Now we prove that repeated applications of

S and T gives the form $\gamma\tau = \frac{a\tau + b}{c\tau + d}$

with $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$. $(SL_2\mathbb{Z})$

proof:

① $T \subset SL_2\mathbb{Z}$ $\because$ $\gamma\tau = \tau + 1$ has

$a = 1, b = 1, c = 0, d = 1$, $ad - bc = 1$

(2) $S \subset SL_2\mathbb{Z}$: $\gamma\tau = -\frac{1}{\tau}$ has

$a = 0$, $b = 1$, $c = -1$, $d = 0$ $\therefore ad - bc = 1$

(3) for $\gamma\tau = \dfrac{a\tau + b}{c\tau + d} \in SL_2\mathbb{Z}$. apply T

gives $\gamma\gamma\tau = \dfrac{a\tau+b}{c\tau+d} + 1 = \dfrac{(a+c)\tau + (b+d)}{c\tau + d}$

$\underline{a + c \in \mathbb{Z}}$, $\underline{b + d \in \mathbb{Z}}$, and

$(a+c)d - (b+d)c = ad + \cancel{cd} - bc - \cancel{cd}$

$= ad - bc = \underline{1}$ $\Rightarrow$ $\gamma\gamma\tau \in SL_2\mathbb{Z}$

if apply S, $\gamma\gamma\tau = \dfrac{-c\tau - d}{a\tau + b}$, $\underline{-c \in \mathbb{Z}}$,

$\underline{-d \in \mathbb{Z}}$, and $(-c)(b) - (-d)a = \underline{ad - bc = 1}$

$\Rightarrow \gamma\gamma\tau \in SL_2\mathbb{Z}$

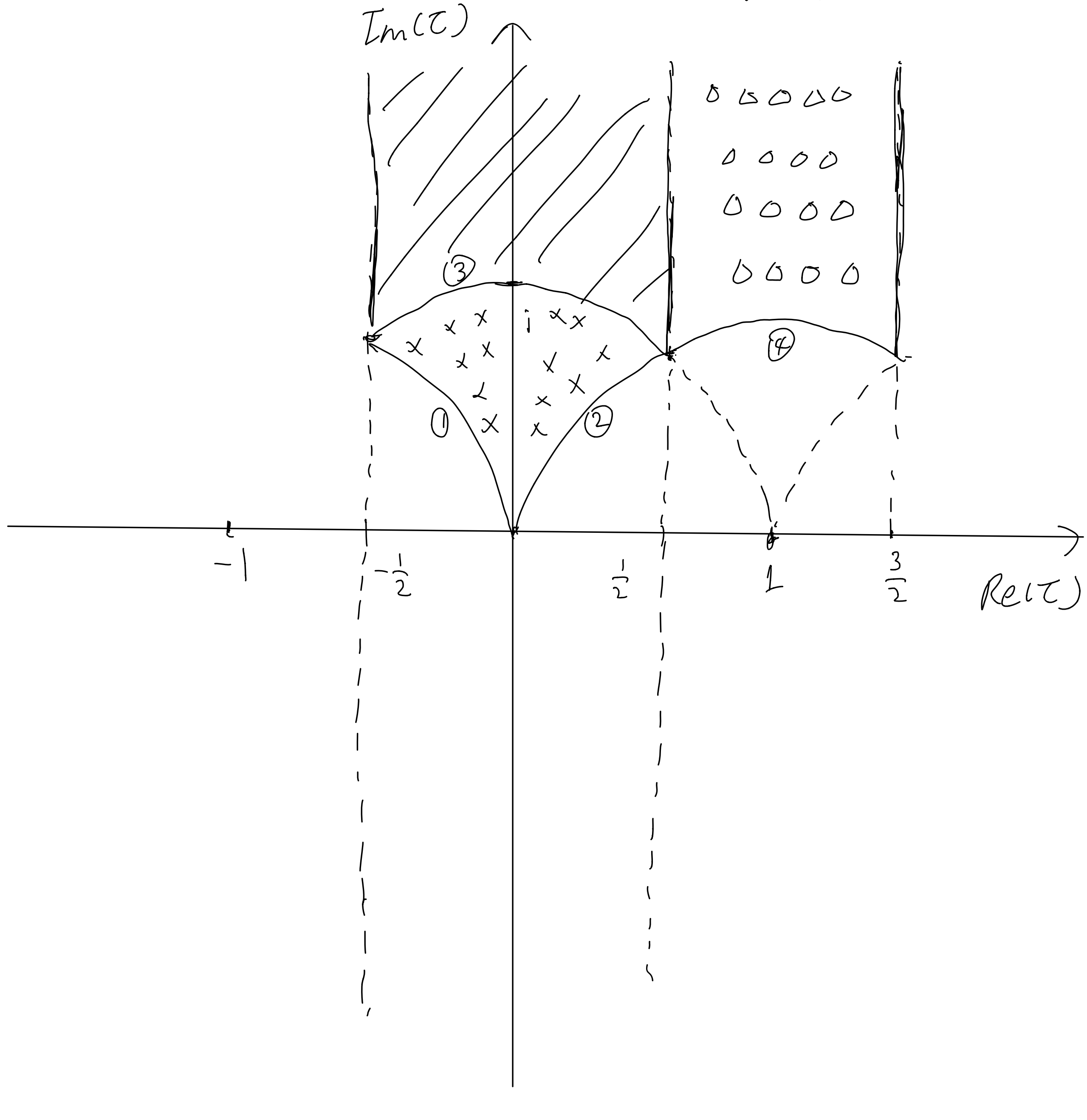
(1), (2), (3) $\Rightarrow$ repeated application of S, T gives $\gamma\tau = \dfrac{a\tau + b}{c\tau + d}$, $\gamma \in SL_2\mathbb{Z}$. $\square$

and since $T^2_\tau \to T^2_{\gamma\tau}$ under S or T has same identification on F, by above proven proposition we conclude that

$T^2_\tau \to T^2_{\gamma\tau}$ under $\gamma\tau = \dfrac{a\tau+b}{c\tau+d}$, $\gamma \in SL_2\mathbb{Z}$

has same identification on $\mathbb{C}$.

- Fundamental Domain: $-\frac{1}{2} \le Re(\tau) \le \frac{1}{2}$, $|\tau| \ge 1$



In the above diagram,

- the shaded region is the fundamental Domain D.
- the region filled with is the image of D under T
- the region filled with is the image of D under S

(1, 2, 3, 4 are segments of circles with centres at $\tau = -1, 1, 0, 1$ respectively).

2) The partition function

$$Z(\tau) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{c}{24}}\right)$$

where
$$L_0 = \frac{\alpha' k^2}{4} + \sum_{n,\mu} n N_{n,\mu}$$
$$\tilde{L}_0 = \frac{\alpha' k^2}{4} + \sum_{n,\mu} n \tilde{N}_{n,\mu}$$

the occupation numbers $N_{n,\mu}$, $\tilde{N}_{n,\mu}$ are such that $\alpha^{\mu}_{-n}\alpha_{\mu,n} = n N_{\mu,n}$ (no sum). and similar relation hold for $\tilde{N}$.

- The states of CFT Hilbert space are labeled by $|N_{n,\mu}, \tilde{N}_{n,\mu}, k\rangle$

$$\therefore Z(\tau) = (q\bar{q})^{-c/24}\, \mathrm{Tr}\left[q^{L_0}\bar{q}^{\tilde{L}_0}\right]$$

$$= (q\bar{q})^{-c/24}\, \mathrm{Tr}\left[(q\bar{q})^{\alpha' k^2/4}\, q^{\sum_n \sum_\mu n N_{n\mu}}\, \bar{q}^{\sum_n \sum_\mu n \tilde{N}_{n\mu}}\right]$$

$$= (q\bar{q})^{-c/24}\, \mathrm{Tr}\left[(q\bar{q})^{\alpha' k^2/4} \prod_{n=1}^{\infty}\prod_{\mu=0}^{D-1} q^{nN_{n\mu}}\, \bar{q}^{n\tilde{N}_{n\mu}}\right]$$

$\Rightarrow$ (trace over $N_{n\mu}$, $\tilde{N}_{n\mu}$, k)

$$= (q\bar{q})^{-c/24} \sum_{k}\sum_{N_{n\mu}}\sum_{\tilde{N}_{n\mu}} (q\bar{q})^{\alpha' k^2/4} \prod_{n=1}^{\infty}\prod_{\mu=1}^{D-1} q^{nN_{n\mu}}\, \bar{q}^{n\tilde{N}_{n\mu}}$$

Now in the limit of continuous momenta k, we have $\sum_k = V_D (2\pi)^{-D} \int d^D k$ where V_D is the spacetime volume of D-dimensions.

$$\rightarrow Z(\tau) = (q\bar{q})^{-c/24} V_D \int \frac{d^D k}{(2\pi)^D} (q\bar{q})^{\alpha' k^2/4} \times$$

$$\prod_{\mu=0}^{D-1} \prod_{n=1}^{\infty} \left(\sum_{N_{n\mu}=0}^{\infty} q^{nN_{n\mu}} \sum_{\tilde{N}_{n\mu}}^{\infty} \bar{q}^{\,n \tilde{N}_{n\mu}} \right)$$

use $\sum_{N=0}^{\infty} q^{nN} = (1-q^n)^{-1}$, Wick rotate the time component of k as $k^0 \rightarrow i k^D$

so $d^D k = dk^0 dk^1 \cdots dk^{D-1} \rightarrow i dk^1 \cdots dk^{D-1} dk^D = i d^D k_E$

$k^2 = -k_0^2 + \vec{k}\cdot\vec{k} = +k_D^2 + \vec{k}\cdot\vec{k} = +k_E^2$ in Euclidean signature, so Integral becomes.

$$Z(\tau) = (q\bar{q})^{-c/24} V_D \, i \int \frac{d^D k_E}{(2\pi)^D} |e^{2\pi i \tau}|^{+\alpha' k_E^2/2}$$

$$\prod_{n=1}^{\infty} (1-q^n)^{-D} (1-\bar{q}^n)^{-D}$$

where the $\prod_{\mu=0}^{D-1}$ turns into the power of D.

$\tau_1 = \mathrm{Re}(\tau)$, $\tau_2 = \mathrm{Im}(\tau)$ $\rightarrow$ $\tau = \tau_1 + i\tau_2$

$$|e^{2\pi i \tau}| = |e^{2\pi i \tau_1} e^{-2\pi \tau_2}| = e^{-2\pi \tau_2}$$

$\therefore$ the Gaussian k_E integral

$$= \int \frac{d^D k_E}{(2\pi)^D} e^{-\pi \tau_2 \alpha' k_E^2} = (4\pi^2 \alpha' \tau_2)^{-D/2}$$

$$\therefore Z(\tau) = i V_D (4\pi\alpha'\tau_2)^{-D/2} (q\bar{q})^{-D/24}$$
$$\times \prod_{n=1}^{\infty} (1-q^n)^{-D} (1-\bar{q}^n)^{-D}$$

The Dedekind η function is given by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \quad , \quad q = e^{2\pi i \tau}$$

$$\therefore Z(\tau) = i V_D (4\pi\alpha'\tau_2)^{-D/2} |\eta(\tau)|^{-2D}$$

$$= \frac{i V_D (4\pi\alpha')^{-D/2}}{\left(\sqrt{\tau_2}\, |\eta(\tau)|^2\right)^D} \quad \text{w/ } \tau_2 = \mathrm{Im}(\tau)$$

- To show that $Z(\tau)$ is modular invariant, it is suffice to show that the function $\sqrt{\tau_2}\,|\eta(\tau)|^2$ is invariant under the S and T transformations.

<u>under S</u> : $\tau \to -\frac{1}{\tau}$, $\eta(\tau) \to \sqrt{-i\tau}\,\eta(\tau)$

$\because$ $\tau = \tau_1 + i\tau_2$, $\bar{\tau} = \tau_1 - i\tau_2 \Rightarrow \tau_2 = \dfrac{\tau - \bar{\tau}}{2i}$

$$\therefore \sqrt{\tau_2}\,|\eta(\tau)|^2 \longrightarrow \sqrt{\mathrm{Im}(-\tfrac{1}{\tau})}\,|\eta(-\tfrac{1}{\tau})|^2$$

$$= \sqrt{\frac{\frac{1}{\bar{\tau}} - \frac{1}{\tau}}{2i}}\,\left|\sqrt{-i\tau}\,\eta(\tau)\right|^2 = \sqrt{\frac{\tau - \bar{\tau}}{2i\cancel{|\tau|^2}}}\,\cancel{\sqrt{|\tau|}^2}\,|\eta(\tau)|^2$$

$$= \sqrt{\frac{\tau - \bar{\tau}}{2i}}\,|\eta(\tau)|^2 = \underline{\sqrt{\tau_2}\,|\eta(\tau)|^2 \text{ (invariant)}}$$

<u>under T</u> : $\tau \to \tau + 1$, $\eta(\tau) \to e^{i\pi/12}\eta(\tau)$

$\because$ 1 is a real number $\therefore$ $\tau_2 \to \tau_2$ invariant.

$$\therefore \sqrt{\tau_2}\,|\eta(\tau)|^2 \to \sqrt{\tau_2}\,\left|e^{i\pi/12}\eta(\tau)\right|^2$$

$$= \underline{\sqrt{\tau_2}\,|\eta(\tau)|^2 \text{ (invariant)}}$$

So $Z(\tau)$ is modular invariant since S and T generates the full modular group.

- The relevance :

  Modular invariance helps in getting rid of some global anomalies in one-loop diagrams.

$\boxed{2}$ $$Z_{T^2} = V_{10}\int_D \frac{d^2\tau}{2\tau_2}\int\frac{d^{10}k}{(2\pi)^{10}}\, Tr_{H_k}\left[(-1)^F q^{\frac{\alpha'}{4}(k^2+M^2)}\, \bar{q}^{\frac{\alpha'}{4}(k^2+\tilde{M}^2)}\right]$$

we note that: $q = e^{2\pi i\tau}$, $z = e^{2\pi i\nu}$

(1) $$\theta_{00}(\nu,\tau) = \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-\frac{1}{2}})(1+z^{-1}q^{m-\frac{1}{2}}) = \theta_3$$

(2) $$\theta_{01}(\nu,\tau) = \prod_{m=1}^{\infty}(1-q^m)(1-zq^{m-\frac{1}{2}})(1-z^{-1}q^{m-\frac{1}{2}}) = \theta_4$$

(3) $$\theta_{10}(\nu,\tau) = 2q^{1/8}\cos\pi\nu\prod_{m=1}^{\infty}(1-q^m)(1+zq^m)(1+z^{-1}q^m) = \theta_2$$

at $\nu=0$, Jacobi's abstruse identity

(4) $$\theta_2^4(0,\tau) - \theta_3^4(0,\tau) + \theta_4^4(0,\tau) = 0$$

(5) $$\eta(\tau) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m)$$

- $(-1)^F$ is the spacetime Fermion operator

$$\frac{\alpha'}{4}M^2 = H^\perp, \quad \frac{\alpha'}{4}\tilde{M} = \tilde{H}^\perp$$

the transverse Hamiltonian $H^\perp, \tilde{H}^\perp$ are given by $H^\perp = \frac{\alpha'}{4}k^2 + \sum_{i=2}^{9}\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i - 1 + H^\perp_{R/NS}$

where $$H_R^\perp = \sum_{i=2}^{9}\sum_{m=1}^{\infty} m\, b_{-m}^i b_m^i + \frac{1}{3} \quad \text{in } R \text{ sector}$$

$$H_{NS}^\perp = \sum_{i=2}^{9}\sum_{r=\frac{1}{2}}^{\infty} r\, b_{-r}^i b_r^i - \frac{1}{6} \quad \text{in } NS \text{ sector}$$

similar expressions for $\tilde{H}$.

the normal ordering constants are obtained from the normal ordering constants of L_0.

- in R sector $a_R = -\frac{1}{24}$, so 8 transverse dimensions gives $\frac{1}{3}$
- in NS sector $a_{NS} = \frac{1}{48}$, so 8 transverse dimensions gives $-\frac{1}{6}$.

- Tr_{H_k} is trace over physical states. the k^2 in the expontial can be taken out of the trace because the summation of momentum states is dealt by the momentum integral, not by the trace.
- trace Tr_{H_k} include sums over left moving worldsheet fermions, right moving worldsheet fermions, worldsheet bosons, and spacetime fermion number operator.
- The sum over worldsheet bosons is identical to Question 1, and is not asked by this question, so we don4 deal with it.

- The sum over left moving worldsheet fermions gives a contribution that equals the complex conjugate of right moving ones.

Hence, we focus on the right moving world-sheet fermions. We should sum both the NS and the R sectors and include the GSO projections.

1) Fermion partition function Z_F is $Z_F = Z_{NS} + Z_R$ (its a sum here, not multiplication, because Z_{NS} and Z_R are two parts of the trace)

NS-Sector: $$Z_{NS} = Tr_{NS}\left(\underbrace{\tfrac{1}{2}(1-(-1)^F)}_{\text{GSO projector.}}\, q^{H^{\perp}_{NS}}\right)$$

where,

$$Tr_{NS}(q^{H^{\perp}_{NS}}) = q^{-\frac{1}{6}}\, Tr_{NS}\left(q^{\sum_{i=2}^{9}\sum_{r=\frac{1}{2}}^{\infty} r\, b^i_{-r} b^i_r}\right)$$

$$= q^{-\frac{1}{6}}\left(Tr_{NS}\left(q^{\sum_{r=\frac{1}{2}}^{\infty} r N_r}\right)\right)^8$$

where each $i = 2, \cdots, 9$ gives same contribution so a power of 8, and $b_{-r} b_r = N_r$ the occupation number operator. $\because$ b's are worldsheet fermions, $\therefore$ N_r can only take values 0 or 1 since $\{b_r, b_r\} = 0$ (Pauli Exclusion principle),

$$\rightarrow \mathrm{Tr}_{NS}(q^{H_{NS}^{\perp}}) = q^{-1/6}\left(\sum_{N_r=0,1}\left(q^{\sum_{r=\frac{1}{2}}^{\infty} rN_r}\right)\right)^8$$

$$= q^{-1/6} \prod_{r=\frac{1}{2}}^{\infty} \sum_{N_r=0,1} q^{rN_r}$$

change variable $n = r + \frac{1}{2}$

$$= q^{-1/6}\left(\prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}})\right)^8$$

$$= \underbrace{\left(q^{-1/24}\prod_{n=1}^{\infty}(1-q^n)^{-1}\right)^4}_{(5) \rightarrow \eta^{-4}(\tau)} \left(\prod_{n=1}^{\infty} \underbrace{(1-q^n)^4 (1+q^{n-\frac{1}{2}})^4 \times (1+q^{n-\frac{1}{2}})^4}_{(1) \rightarrow \theta_3^4(0,\tau)}\right)$$

$$= \underline{\frac{\theta_3^4(0,\tau)}{\eta^4(\tau)}}.$$

Similarly, keeping in mind that the world-sheet fermion number operator $(-1)^F$ gives

-1 factor to worldsheet fermions we have

$$\mathrm{Tr}\left((-1)^F q^{H^{\perp}_{NS}}\right) = q^{-\frac{1}{6}} \prod_{r=\frac{1}{2}}^{\infty} \left((-1)^F \sum_{\substack{N_r=0,1 \\ \swarrow \quad \searrow}} q^{rN_r} \right)^8$$

WS boson | WS fermion
+1 | -1

$$= q^{-\frac{1}{6}} \left(\prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}}) \right)^8 \qquad n = r + \tfrac{1}{2}$$

$$= \underbrace{\left(q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)^{-1} \right)^4}_{\eta^{-4}(\tau) \leftarrow (5)} \underbrace{\left(\prod_{n=1}^{\infty} (1-q^n)^4 (1-q^{n-\frac{1}{2}})^4 (1-q^{n-\frac{1}{2}})^4 \right)}_{\theta_4^4(0,\tau) \leftarrow (2)}$$

$$= \frac{\theta_4^4(0,\tau)}{\eta^4(\tau)}.$$

$$\Rightarrow Z_{NS}\Big|_{left} = \frac{1}{2\eta^4}(\theta_3^4 - \theta_4^4)$$

R sector : $Z_R = \mathrm{Tr}_R\left(\frac{1}{2}(1-(-1)^F)\, q^{H_R^{\perp}}\right)$

- First we note that $\mathrm{Tr}_R\left((-1)^F q^{H_R^{\perp}}\right) = 0$ identically because at each mass level we have the same number of worldsheet bosons and fermions in the R sector, and since we start with $(-1)^F|a\rangle = |a\rangle$ and $(-1)^F|\dot{a}\rangle = -|\dot{a}\rangle$, both sides cancel.

- Also, we need a ground state degeneracy for the R sector. Since the ground states of R sector are in $\underset{\sim}{8_s}$ and $\underset{\sim}{8_c}$ spinor representations, so we have a degeneracy of 16. This number is 1 in the NS sector case because the ground state of NS sector is a singlet ($\underset{\sim}{1}$ representation).

$$\therefore Z_R\Big|_{\text{left}} = 16\cdot\frac{1}{2}\mathrm{Tr}_R\left(q^{H_R^l}\right) = 16\cdot\frac{1}{2}q^{1/3}\prod_{m=1}^{\infty}\left(\sum_{N_m=0,1}q^{mN_m}\right)^8$$

$$= 16\cdot\frac{1}{2}q^{1/3}\left(\prod_{m=1}^{\infty}(1+q^m)\right)^8$$

$$= 16\cdot\frac{1}{2}q^{1/2}q^{-1/6}\left(\prod_{m=1}^{\infty}(1+q^m)\right)^8$$

$$= \frac{1}{2}\left(q^{-1/24}\prod_{m=1}^{\infty}(1-q^m)^{-1}\right)^4 \rightsquigarrow \eta^{-4}(\tau) \quad (5)$$

$$\times \left(2^4(q^{1/8})^4\prod_{m=1}^{\infty}(1-q^m)^4(1+q^m)^4(1+q^m)^4\right)$$

$$(3) \to \theta_2^4(0,\tau)$$

$= \frac{1}{2\eta^4}\theta_2^4$

2) Now we sum the R and NS sector partition functions. $\because$ the spacetime fermion number operator $(-1)^F$ gives $+1$ to NS states and -1 to R states

$$\therefore \; Z_F\big|_{\text{left}} = Z_{NS}\big|_{\text{left}} - Z_R\big|_{\text{left}}$$

$$= \frac{1}{2\eta^4}\left(\theta_3^4 - \theta_4^4 - \theta_2^4\right)$$

and similarly $Z_F\big|_{\text{right}} = \bar{Z}_F\big|_{\text{left}}$

since in $Z_F\big|_{\text{right}}$ q is replaced with $\bar{q}$

$$\Rightarrow \; Z_F = \frac{1}{4|\eta|^8}\left|\theta_3^4 - \theta_4^4 - \theta_2^4\right|^2$$

By (4), $\theta_3^4 - \theta_4^4 - \theta_2^4 = 0$

$$\therefore \; \underline{Z_F = 0} \quad , \because \; Z_{T^2} \propto \underset{\substack{\downarrow \\ \text{Boson}}}{Z_B} Z_F$$

$\therefore$ $\underline{Z_{T^2} = 0}$ partition funtion vanishes

$\square$

$\boxed{3}$

OPE's :

$$b(z)\, c(0) \sim \frac{1}{z} \qquad c(z)\, b(0) \sim \frac{1}{z}$$

$$\beta(z)\, \gamma(0) \sim -\frac{1}{z} \qquad \gamma(z)\, \beta(0) \sim \frac{1}{z}$$

1) weight h of an (primary) operator φ is given by

$$T(z)\, \varphi(0) = \frac{h\varphi}{z^2} + \cdots$$

(we don't show the terms like $\frac{\partial\varphi}{z}$ because we were only given the leading order OPE's as ingredients)

So,

$$T(z)\, b(0) = (\partial b(z))\, c(z)\, b(0) + \cancel{(\partial b(z))\, c(z)\, b(0)}$$

$$- \lambda (\partial b(z))\, c(z)\, b(0) \; \cancel{- \lambda (\partial b(z))\, c(z)\, b(0)}$$

$$- \lambda\, b(z)\, (\partial c(z))\, b(0) \; \cancel{- \lambda\, b(z)\, (\partial c(z))\, b(0)}$$

$$\sim \underbrace{\frac{1}{z}\partial b(z) - \frac{\lambda}{z}\partial b(z)}_{O(\frac{1}{z})} + (-\lambda)\left(-\frac{1}{z^2}\right) b(z)$$

$$\sim \frac{\lambda}{z^2} b(z) \quad \Rightarrow \quad \text{weight } \underline{\underline{h_b = \lambda}}$$

similarly, with simplified notation and note that leading order involves ∂

$$T(z)\, c(0) \sim (\partial b) c\, c - \lambda \partial(bc)\, c$$

$$\sim - c\, \partial b\, c + \lambda c\, \partial b\, c \quad \Big| \text{ anti-commuting } b, c$$

$$\sim -\left(-\frac{1}{z^2}\right) c(z) + \lambda\left(-\frac{1}{z^2}\right) c(z)$$

$$\sim \frac{1-\lambda}{z^2} c(z) \quad \Rightarrow \quad \text{weight } \underline{\underline{h_c = 1-\lambda}}$$

$$T(z)\, \beta(0) \sim \cancel{(\partial\beta)\gamma\, \beta} - \left(\lambda - \tfrac{1}{2}\right) \partial(\beta\gamma)\, \beta$$

$$\sim -\left(\lambda - \tfrac{1}{2}\right)\left(-\frac{1}{z^2}\right) \beta(z)$$

$$\sim \frac{\lambda - \frac{1}{2}}{z^2} \beta(z) \quad \Rightarrow \quad \text{weight } \underline{\underline{h_\beta = \lambda - \tfrac{1}{2}}}$$

$$T(z)\, \gamma(0) \sim (\partial\beta)\gamma\, \gamma - \left(\lambda - \tfrac{1}{2}\right) \partial(\beta\gamma)\, \gamma$$

$$\sim \left(+\frac{1}{z^2}\right) \gamma(z) - \left(\lambda - \tfrac{1}{2}\right)\left(+\frac{1}{z^2}\right) \gamma(z)$$

$$\sim \left(\frac{3}{2}-\lambda\right)\frac{1}{z^2}\gamma(z) \Rightarrow \text{weight}$$

$$h_\gamma = \frac{3}{2}-\lambda$$

2). Central charge is defined as

$$TT \sim \frac{c}{2z^4}, \quad c = \text{central charge.}$$

Calculate $T(z)T(w)$, the bc part is

$$TT\Big|_{bc} = \left[(\partial b)c - \lambda(\partial b)c - \lambda b\partial c\right]_z$$

$$\times \left[(\partial b)c - \lambda(\partial b)c - \lambda b\partial c\right]_0$$

$= (1-\lambda)^2 (\partial b)c\cdot(\partial b)c$	$(1-\lambda)^2\left(-\frac{1}{z^2}\right)\left(\frac{1}{z^2}\right)$
$-\lambda(1-\lambda)(\partial b)c\cdot b(\partial c)$	$-\lambda(1-\lambda)\left(-\frac{2}{z^3}\right)\left(\frac{1}{z}\right)$
$-\lambda(1-\lambda)\, b(\partial c)\cdot(\partial b)c$	$-\lambda(1-\lambda)\left(-\frac{2}{z^3}\right)\left(\frac{1}{z}\right)$
$+\lambda^2\, b(\partial c)\cdot b(\partial c)$	$+\lambda^2\left(\frac{1}{z^2}\right)\left(-\frac{1}{z^2}\right)$

$$= \frac{1}{2z^4}\left(-2(1-\lambda)^2 + 8(\lambda)(1-\lambda) - 2\lambda^2\right)$$

$$= \frac{1}{2z^4}\left(-2 + 4\lambda - 2\lambda^2 + 8\lambda - 8\lambda^2 - 2\lambda^2\right)$$

$$= \frac{1}{2z^4}\left(\underbrace{-12\lambda^2 + 12\lambda - 2}\right)$$

$$\rightarrow TT\Big|_{\beta\gamma} = \left[(\tfrac{3}{2}-\lambda)(\partial\beta)\gamma - (\lambda-\tfrac{1}{2})\beta\partial\gamma\right]_z$$

$$\left[(\tfrac{3}{2}-\lambda)\partial\beta\gamma - (\lambda-\tfrac{1}{2})\beta\partial\gamma\right]_0$$

$= (\frac{3}{2}-\lambda)^2(\partial\beta)\gamma \cdot \partial\beta\gamma$	$(\frac{3}{2}-\lambda)^2(\frac{1}{z^2})(\frac{1}{z^2})$
$-(\frac{3}{2}-\lambda)(\lambda-\frac{1}{2})(\partial\beta)\gamma \cdot \beta\partial\gamma$	$-(\frac{3}{2}-\lambda)(\lambda-\frac{1}{2})(\frac{2}{z^3})(\frac{1}{z})$
$-(\frac{3}{2}-\lambda)(\lambda-\frac{1}{2})\beta\partial\gamma \cdot \partial\beta\gamma$	$-(\frac{3}{2}-\lambda)(\lambda-\frac{1}{2})(-\frac{2}{z^3})(-\frac{1}{z})$
$+(\lambda-\frac{1}{2})^2\beta\partial\gamma \cdot \beta\partial\gamma$	$(\lambda-\frac{1}{2})^2(-\frac{1}{z^2})(-\frac{1}{z^2})$

$$= \frac{1}{2z^4}\left(2(\tfrac{3}{2}-\lambda)^2 + 4(\lambda-\tfrac{3}{2})(2\lambda-1) + 2(\lambda-\tfrac{1}{2})^2\right)$$

$$= \frac{1}{2z^4}\Big(\frac{9}{2} - 6\lambda + 2\lambda^2 + 8\lambda^2 - 12\lambda - 4\lambda$$
$$+ 6 + 2\lambda^2 - 2\lambda + \frac{1}{2} \Big)$$

$$= \frac{1}{2z^4}\big(\underbrace{11 - 24\lambda + 12\lambda^2}\big)$$

$\therefore$ central charge c is

$$c = \cancel{-12\lambda^2} + 12\lambda - 2 + 11 - 24\lambda + \cancel{12\lambda^2}$$

$$= \underline{\underline{9 - 12\lambda}}$$

3) For $\lambda = 2$, central charge

$C_g = 9 - 24 = -15$ for T_{ghost}.

For T_{RNS}, each boson contributes 1 and each fermion contributes $\frac{1}{2}$, so $C_{RNS} = \frac{3}{2}D$
(D = spacetime dimension)

The total central charge

$C = C_g + C_{RNS} = \frac{3}{2}D - 15$

for $C = 0$, need $\underline{\underline{D = 10}}$ ∎