Quantum Field Theory in

Curved Spacetime

Problem Sheet 3

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$\boxed{1}$

The integral $\int_{\mathbb{R}^n} d^n x\, e^{-x^2} = \int dx_1\, e^{-x_1^2} \int dx_2\, e^{-x_2^2} \cdots \int dx_n\, e^{-x_n^2}$

$(\ x = (x_1, x_2, \cdots, x_n)\ )$

$$= \left(\int dx\, e^{-x^2}\right)^n = \left(\int dx\, dy\, e^{-x^2+y^2}\right)^{\frac{n}{2}}$$

$$\underbrace{=}_{\substack{\cancel{r^2 = x^2+y^2} \\ x = r\sin\theta \\ y = r\cos\theta}} \left(\int r\, dr\, d\theta\, e^{-r^2}\right)^{\frac{n}{2}} = \left(\underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^r r e^{-r^2}}_{\frac{1}{2}}\right)^{\frac{n}{2}} = \underline{\underline{(\pi)^{n/2}}}$$

On the other hand

$$\int_{\mathbb{R}^n} d^n x\, e^{-x^2} = \int \underbrace{dx_1 \cdots dx_n}_{= A_{n-1} r^{n-1} dr}\, e^{-\underbrace{(x_1^2 + x_2^2 + \cdots + x_n^2)}_{r^2 = x_1^2 + \cdots + x_n^2}}$$

$$= A_{n-1} \int_0^\infty dr\, r^{n-1} e^{-r^2} \underbrace{=}_{\substack{\cancel{dr =} \\ y = r^2,\ dy = 2r\, dr}} A_{n-1} \int_0^\infty \frac{dy}{2r} r^{n-1} e^{-r^2}$$

$$= \frac{A_{n-1}}{2} \underbrace{\int_0^\infty dy\, y^{\frac{n-2}{2}} e^{-y}}_{\equiv \Gamma\left(\frac{n}{2}\right)} = \underline{\underline{\frac{A_{n-1}}{2} \Gamma\left(\frac{n}{2}\right)}}$$

so we must have

$$\frac{A_{n-1}}{2}\Gamma\left(\frac{n}{2}\right) = \pi^{n/2} \Rightarrow A_{n-1} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

$$\Rightarrow A_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}$$

Green's function $(\square - m^2)\, G(x,x') = -g^{-1/2}\,\delta(x-x')$

in Minkowski space time $\square = \partial_\mu \partial^\mu$, $g = \eta = 1$

so $(\partial_\mu \partial^\mu - m^2)\, G(x,x') = -\delta^n(x-x')$ $\left(\partial_\mu = \frac{\partial}{\partial x^\mu}\right)$

~~So $G(x,x') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')}$~~

So $G(x,x') = \int \frac{d^n k}{(2\pi)^n} \frac{e^{ik\cdot(x-x')}}{k^2+m^2}$

(check: $(\partial^2 - m^2)\, G = \int \frac{d^n k}{(2\pi)^n} \frac{-k^2-m^2}{k^2+m^2} e^{ik(x-x')}$

$= -\int \frac{d^n k}{(2\pi)^n} e^{ik(x-x')} = -\delta^n(x-x')$)

use $\frac{1}{k^2+m^2} = i\int_0^\infty ds\, e^{-is(k^2+m^2)}$ we have

$$G(x,x') = \int \frac{d^n k}{(2\pi)^n}\, i\int_0^\infty ds\; e^{i(k(x-x') - s(k^2+m^2))}$$

look at the exponent $i(k(x-x') - s(k^2+m^2))$

$$k(x-x') - s(k^2+m^2) = -s\left(k^2 - \frac{k(x-x')}{s} + \frac{(x-x')^2}{4s^2}\right) + \frac{(x-x')^2}{4s}$$
$$- m^2 s$$

$$= -s(\underbrace{k - \frac{(x-x')}{s}}_{=p})^2 + \frac{(x-x')^2}{4s} - m^2 s$$

$$= -sp^2 + \frac{(x-x')^2}{4s} - m^2 s$$

p and k is related by a traslation, so

$d^n k = d^n p$ (Jacobian = 1)

$$G(x,x') = \int \frac{d^n p}{(2\pi)^n} (i) \int_0^\infty ds\, e^{i\left(-sp^2 + \frac{(x-x')^2}{4s} - m^2 s\right)}.$$

$$= i\int_0^\infty ds\, e^{i\left(\frac{(x-x')^2}{4s} - m^2 s\right)} \int \frac{d^n p}{(2\pi)^n} e^{-isp^2}$$

$$\cancel{= i\int_0^\infty ds\, e^{i\left(\frac{(x-x')}{4s} - m^2 s\right)} \int \frac{dp_0}{(2\pi)} e^{+isp_0^2} \underbrace{\int \frac{d^{n-1}p_i}{(2\pi)^{n-1}} e^{-isp_i^2}}}$$

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~~use~~ $\cancel{\int \frac{dp_0}{(2\pi)} e^{isp_0^2} = \frac{1}{2\sqrt{-i\pi s}}}$ $\cancel{\int \frac{dp_i}{(2\pi)} = e}$

$\cancel{\int \frac{dp_i}{2\pi} e^{-isp_i^2} = \frac{1}{2\sqrt{i\pi s}}}$ (~~Fresnel's~~ integrals)

~~So we get~~

$\cancel{G(x,x') = i}$

for $\int \frac{d^n p}{(2\pi)^n} e^{-isP}$, $\quad P = -p_0^2 + p_1^2 + \cdots + p_{n-1}^2$

wick rotate $\quad p_0 \to i p_n \quad$ so $P = p_1^2 + \cdots + p_{n-1}^2 + p_n^2 = p_E^2$

so $\int \frac{d^n p_E}{(2\pi)^n} e^{-isp_E} = \left(\int \frac{dp}{2\pi} e^{-isp} \right)^n$

$= \left(\frac{1}{2\sqrt{i\pi s}} \right)^n = \left(\frac{1}{4\pi i s} \right)^{n/2}$

use fresnel's formula.

$\therefore G(x,x') = \cancel{i \left(\frac{1}{4\pi i s}\right)^{n/2} \int_0 ds}$

$= i \int_0^\infty ds \left(\frac{1}{4\pi i s} \right)^{n/2} e^{i\left(\frac{(x-x')^2}{4s} - m^2 s\right)}$

change variable $\quad s = \frac{1}{4w} \quad ds = -\frac{1}{4w^2} dw$

$G(x,x') = i \int_0^\infty dw \frac{1}{4w^2} \left(\frac{1}{4\pi i} \right)^{n/2} (4w)^{n/2} e^{i\left((x-x')^2 w - \frac{m^2}{4w}\right)}$

$= i \frac{1}{4} \left(\frac{4}{4\pi i} \right)^{n/2} \int_0^\infty dw\, w^{\frac{n}{2}-2} e^{i\left((x-x')^2 w - \frac{m^2}{4w}\right)}$

use $\int_0^\infty \frac{dw}{w} w^\nu e^{i\left(aw + \frac{b}{4w}\right)} = 2\left(\frac{b}{4a}\right)^{\nu/2} i\frac{\pi}{2} e^{i\pi\nu/2} \times H_\nu^{(1)}(\sqrt{ab})$

where $H_\nu^{(1)}$ is the Hankel function of the first kind.

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$$G(x,x') = i\frac{1}{4}\left(\frac{1}{\pi i}\right)^{n/2}\int_0^\infty \frac{d\omega}{\omega}\,\omega^{\frac{n}{2}-1} e^{i\left((x-x')^2\omega - \frac{m^2}{4\omega}\right)}.$$

$$\underbrace{=}_{\substack{\nu=\frac{n}{2}-1=\frac{n-2}{2}\\ a=(x-x')^2\\ b=-m^2}} i\frac{1}{4}\left(\frac{1}{\pi i}\right)^{n/2}\cdot 2\cdot\left(\frac{-m^2}{4(x-x')^2}\right)^{\frac{n-2}{4}} H^{(1)}_{\frac{n-2}{2}}\left(\sqrt{-m^2(x-x')^2}\right).$$

$$\times\, i \times \frac{\pi}{2}\times\big(\underbrace{e^{-i\pi/2}}_{-i}\big)^{\frac{n-2}{2}}$$

$$\theta$$

$$\cancel{= i\frac{1}{4}\left(\frac{1}{\pi}\right)^{n/2}\frac{(i)^{\frac{n-2}{2}}}{(i)^{\frac{n}{2}}}\cdot 2\cdot i\cdot\frac{\pi}{2}}$$

$$\theta = \left(\frac{\frac{\pi i}{2}H^{(1)}_{\frac{n-2}{2}}\left(i\sqrt{m^2(x-x')^2}\right)}{\left(i\sqrt{m^2(x-x')^2}\right)^{\frac{n-2}{2}}} \times \cancel{i^{+\frac{n-2}{2}}}\times\left(m^2(x-x')^2\right)^{+\frac{n-2}{4}}\right)$$

$$\times\, i\frac{1}{4}\left(\frac{1}{\pi}\right)^{n/2}\underbrace{i^{-n/2}}_{\text{from }\left(\frac{1}{i\pi}\right)^{n/2}}\underbrace{i^{\frac{n-2}{2}}}_{\cancel{i^{-1}}}\cdot 2\cdot\underbrace{\cancel{i^{\frac{n-2}{2}}}}_{\text{from }(e^{i\pi/2})^\nu}\cdot\underbrace{(m^2)^{\frac{n-2}{4}}\left(i(4(x-x')^2)\right)^{-\frac{n-2}{4}}}_{\text{from }(-1)^{\frac{m-2}{4}}}$$

$-1=(-i)^2$

$$\underbrace{=}_{\text{use }\frac{\frac{\pi i}{2}H^{(1)}_\nu(iz)}{(iz)^\nu}=\frac{K_\nu(z)}{z^\nu}} \frac{K_{\frac{n-2}{2}}\left(\sqrt{m^2(x-x')^2}\right)}{\left(\sqrt{m^2(x-x')^2}\right)^{\frac{n-2}{2}}}\times\left(m^2(x-x')^2\right)^{+\frac{n-2}{4}}$$

$$\times\frac{1}{2\pi}\left(\frac{1}{\pi}\right)^{n/2-1}\left(\frac{m^2}{4(x-x')^2}\right)^{\frac{n-2}{4}}$$

$$\times K_{\frac{n-2}{2}}\left(\sqrt{m^2(x-x')^2}\right)$$

$$= \frac{1}{2\pi}\left(\frac{m^2}{4\pi^2(x-x')^2}\right)^{\frac{n-2}{4}} K_{\frac{n-2}{2}}\left(\sqrt{m^2(x-x')^2}\right) \qquad (*)$$

$K_\nu(z)$ is the modified Bessel function.

Now, when we integrated $\int d^n p$ we need to justify that $\int ds$ and $\int d^n p$ can be interchanged,

~~this~~ the $e^{i(x-x')/4s}$ term under this exchange is only integrable ~~only~~ if we include a $i\epsilon$ prescription, so we replace all $(x-x')^2$ with

$(x-x')^2 + i\epsilon(2)$ and define $\underline{\sigma = \frac{1}{2}(x-x')^2 + i\epsilon}$

- so replace all $(x-x')^2$ in $(*)$ by 2σ

we get

$$G(x, x') = \frac{1}{2\pi}\left(\frac{m^2}{8\pi^2 \sigma}\right)^{\frac{n-2}{4}} K_{\frac{n-2}{2}}\left(\sqrt{2m^2\sigma}\right)$$

For $\nu = \frac{n-2}{2}$, at small $|z|$ we have

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu$$

$\therefore$ as $m^2 \to 0$

$$K_\nu(\sqrt{2m^2\sigma}) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{\sqrt{2m^2\sigma}}\right)^\nu = \frac{\Gamma(\nu)}{2}\left(\frac{4}{2m^2\sigma}\right)^{\nu/2}$$

$$\therefore G(x, x') \sim \frac{1}{2\pi}\left(\frac{\cancel{m^2}}{8\pi^2\sigma} \times \frac{4}{2\cancel{m^2}\sigma}\right)^{\nu/2} \cdot \frac{\Gamma(\nu)}{2}$$

$$= \frac{1}{4\pi}\left(\frac{1}{4\pi^2\sigma^2}\right)^{\nu/2}\Gamma(\nu) \qquad \text{when } n=4,\ \nu=1$$

this $= \frac{1}{4\pi}\left(\frac{1}{2\pi\sigma}\right)(1) = \frac{1}{8\pi^2\sigma}$

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$$f(\nu) = \int_0^\infty d\xi\, \tilde{f}(\xi) K_{i\nu}(\xi) \qquad ①$$

$$\tilde{f}(\xi) = \frac{2}{\pi^2 \xi} \int_0^\infty d\nu\, \nu \sinh(\pi\nu)\, K_{i\nu}(\xi) f(\nu) \qquad ②$$

so, sub ② into ① $\Rightarrow$

$$f(\nu) = \int_0^\infty d\xi \frac{2}{\pi^2 \xi} \int_0^\infty d\nu' \nu' \sinh(\pi\nu')\, K_{i\nu}(\xi) K_{i\nu'}(\xi) f(\nu')$$

$$= \cancel{\int_0^\infty d} \int_{-\infty}^\infty d\nu'\, \delta(\nu-\nu') f(\nu') = \underbrace{\int_0^\infty d\nu'\, \delta(\nu-\nu') f(\nu')}_{\nu > 0}$$

$$\Rightarrow \int_0^\infty d\xi \frac{2}{\pi^2 \xi} \nu' \sinh(\pi\nu')\, K_{i\nu}(\xi) K_{i\nu'}(\xi)$$

$$= \delta(\nu-\nu')$$

$$\Rightarrow \int_0^\infty \frac{d\xi}{\xi} K_{i\nu}(\xi) K_{i\nu'}(\xi) = \frac{\pi^2}{2} \frac{\delta(\nu-\nu')}{\nu' \sinh(\pi\nu')}$$

$$= \frac{\pi^2}{2} \frac{\delta(\nu-\nu')}{\nu \sinh(\pi\nu)} \qquad \square.$$

this $\neq 0$ only if $\nu = \nu'$.
so can freely change ν, ν'

sub ① into ② $\Rightarrow$

$$\tilde{f}(\xi) = \frac{2}{\pi^2 \xi} \int_0^\infty d\nu\, \nu \sinh(\pi\nu)\, K_{i\nu}(\xi) \int_0 d\xi' \tilde{f}(\xi') K_{i\nu}(\xi')$$

$$= \int_{-\infty}^\infty \delta(\xi - \xi')\, d\xi'\, \tilde{f}(\xi') = \underbrace{\int_0^\infty d\xi'\, \delta(\xi-\xi') \tilde{f}(\xi')}_{\xi > 0}$$

$$\Rightarrow \frac{2}{\pi^2 \zeta}\int_0^\infty d\nu\, \nu \sinh(\pi\nu)\, K_{i\nu}(\zeta)\, K_{i\nu}(\zeta') = \delta(\zeta-\zeta').$$

$$\Rightarrow \int_0^\infty d\nu\, \nu \sinh(\pi\nu)\, K_{i\nu}(\zeta)\, K_{i\nu}(\zeta') = \frac{\pi^2}{2}\zeta\, \delta(\zeta-\zeta').$$

□.

Then, $V_{\underline{k}\nu}(x) = \frac{1}{2\pi^2}\sqrt{\sinh \nu\pi}\, e^{-i\nu\tau} K_{i\nu}(\mu\zeta)\, e^{i\underline{k}\cdot\underline{y}}$

$$(V_{\underline{k}\nu}, V_{\underline{k}'\nu'}) = i\int d^2\underline{y}\int_0^\infty \frac{d\zeta}{\zeta}\, V^*_{\underline{k}\nu}\overleftrightarrow{\frac{\partial}{\partial\tau}} V_{\underline{k}'\nu'}$$

where $f\overleftrightarrow{\partial_\tau} g \equiv f\partial_\tau g - (\partial_\tau f)g$

$\underline{k} = (0, 0, k_y, k_z)$ $\quad\quad$ $\underline{y} = (0, 0, y, z)$

So

$$(V_{\underline{k}\nu}, V_{\underline{k}'\nu'}) = i\int d^2\underline{y}\int_0^\infty \frac{d\zeta}{\zeta}\Big[\frac{1}{2\pi^2}\sqrt{\sinh\pi\nu}\, e^{+i\nu\tau} K_{i\nu}(\mu\zeta)\, e^{-i\underline{k}\cdot\underline{y}}$$

$$\times (-i\nu')\frac{1}{2\pi^2}\sqrt{\sinh\pi\nu'}\, e^{-i\nu'\tau} K_{i\nu'}(\mu\zeta)\, e^{+i\underline{k}'\cdot\underline{y}}$$

$$- \frac{1}{2\pi^2}(i\nu)\sqrt{\sinh\pi\nu}\, e^{+i\nu\tau} K_{i\nu}(\mu\zeta)\, e^{-i\underline{k}\cdot\underline{y}}$$

$$\times \frac{1}{2\pi^2}\sqrt{\sinh\pi\nu'}\, e^{-i\nu'\tau} K_{i\nu'}(\mu\zeta)\, e^{+i\underline{k}'\cdot\underline{y}}$$

Note $\displaystyle\int_0^\infty \frac{d\zeta}{\zeta} K_{i\nu}(\underbrace{\mu\zeta}_{a}) K_{i\nu'}(\underbrace{\mu\zeta}_{a}) = \int_0^\infty \frac{da}{a} K_{i\nu}(a) K_{i\nu'}(a)$

$\displaystyle = \frac{\pi^2}{2}\frac{\delta(\nu-\nu')}{\nu\sinh\pi\nu}$ still

$a = \mu\zeta$

$da = \mu\, d\zeta$

$\displaystyle\frac{da}{a} = \frac{\mu\, d\zeta}{\mu\zeta} = \frac{d\zeta}{\zeta}$

$$(V_{\underline{k}\nu}, V_{\underline{k}'\nu'}) = i \int d^2\underline{y} \underbrace{\int_0^\infty \frac{d\xi}{\xi} K_{i\nu}(\mu\xi) K_{i\nu'}(\mu\xi)}_{\frac{\pi^2}{2}\frac{\delta(\nu-\nu')}{\nu \sinh(\nu\pi)}} \left(\frac{1}{2\pi^2}\right)^2 \times \sqrt{\sinh\pi\nu \sinh\pi\nu'}$$

$$\times e^{i\tau(\nu-\nu')} \times e^{-i\underline{y}\cdot(\underline{k}-\underline{k}')} (-i)(\nu+\nu')$$

$$= \Bigg(\underbrace{\int \frac{d^2 y}{(2\pi)^2} e^{-i\underline{y}\cdot(\underline{k}-\underline{k}')}}_{\delta(\underline{k}-\underline{k}')} \Bigg) \frac{\cancel{\pi^2}}{\cancel{2}} \frac{\delta(\nu-\nu')}{\cancel{\nu} \sinh(\nu\pi)} \frac{1}{\cancel{\pi^2}} \times \sinh(\pi\nu) \times 1 \times \cancel{2\nu}$$

$\imath \delta(\nu-\nu')$ set $\nu=\nu'$ for all other terms

$$= \delta(\underline{k}-\underline{k}')\,\delta(\nu-\nu') \qquad \text{Orthonormal.}$$

we know $\quad -i G(x,x') = \langle 0 | T \varphi(x) \varphi(x') | 0 \rangle$

and we expand in modes

$$\varphi(x) = a V(x) + a^* V^*(x) \qquad \varphi(x') = a V(x') + a^* V(x')$$

where an implicit sum over $\underline{k}$ and ν indices in the product $\varphi\varphi$ is implied.

and a, a^* are the annihilation / creation operators such that $a|0\rangle = 0$ $\quad \langle 0| a^* = 0$ and $[a, a^*] = 1$

So ~~$\langle 0 | T$~~ $\langle 0 | \varphi(x) \varphi(x') | 0 \rangle$

$$= \langle 0 | (a V(x) + a^* V^*(x))(a V(x') + a^* V^*(x')) | 0 \rangle.$$

but only $\langle 0| a a^* V(x) V^*(x') |0\rangle$ term remains after annihilation

this $= V(x) V^*(x') \langle 0| a a^* |0\rangle$

$= V(x) V^*(x') \langle 0| \underbrace{a^* a}_{\text{gives } 0} + \underbrace{[a, a^*]}_{\text{gives } -i} |0\rangle$

$= V(x) V^*(x') \underbrace{\langle 0|0\rangle}_{1} (-i) = -i V_{(x)} V^*_{(x')}$

$V(x)_{\underline{k}, \nu} = \frac{1}{2\pi^2} \sqrt{\sinh \nu\pi}\, e^{-i\nu\tau} K_{i\nu}(\mu\xi) e^{i\underline{k}\cdot\underline{y}}$

$V'(x')_{\underline{k}, \nu} = \frac{1}{2\pi^2} \sqrt{\sinh \nu\pi}\, e^{+i\nu\tau'} K_{i\nu}(\mu'\xi') e^{-i\underline{k}\cdot\underline{y}'}$

so $V(x) V^*(x')$ with indices $\underline{k}, \nu$ ~~stam~~ integrated is

$G(x, x') = i \langle 0| \hat{T}(\varphi(x), \varphi(x')) |0\rangle = V(x) V^*(x')$

$= \hat{T} \Big(\int_0^\infty d\nu \int d^2k \left(\frac{1}{2\pi^2}\right)^2 \sinh \nu\pi\, e^{-i(\tau - \tau')} K_{i\nu}(\nu\xi) K_{i\nu}(\mu'\xi') e^{i\underline{k}\cdot(\underline{y} - \underline{y}')} \Big)$

$= \int_0^\infty \frac{d\nu}{\pi^2} \sinh \nu\pi \int \frac{d^2k}{(2\pi)^2} K_{i\nu}(\mu\xi) K_{i\nu}(\mu'\xi') e^{-i|\tau - \tau'| + i\underline{k}\cdot(\underline{y} - \underline{y}')}$

where $|\tau - \tau'|$ represents the time ordering $\hat{T}$.