Quantum Field Theory in Curved Spacetime

Problem Sheet 2

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$\boxed{1}$ FRW : $ds^2 = dT^2 - a(T)^2(d\psi^2 + \sin^2\psi\, d\Omega^2)$

$(d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2\ )$

Schwarzschild $ds^2 = (1-\frac{2m}{r})dt^2 - \frac{dr}{(1-\frac{2m}{r})} - r^2 d\Omega$

hypersurface $\psi = \psi_0 \Rightarrow \sin^2\psi = \sin^2\psi_0\ ,\ d\psi = 0$

So FRW $\rightarrow$ $ds^2 = dT^2 - a(T)^2 \sin^2\psi_0\, d\Omega^2 \quad (*)$

3-surface $(t, r) = (t(T), R(T))$, so

Schwarzschild : $ds^2 = (1-\frac{2m}{R})dt^2 - \frac{dR}{1-\frac{2m}{R}} - R^2 d\Omega$

$\because dt = \frac{dt}{dT} dT\ ,\quad dR = \frac{dR}{dT} dT$,

So Schwarzschild

$$\Rightarrow ds^2 = \left((1-\frac{2m}{R})(\frac{dt}{dT})^2 - \frac{(dR/dT)^2}{(1-\frac{2m}{R})}\right)dT^2 - R^2 d\Omega^2 \quad (**)$$

match $(**)$ with FRW $(*)$ gives

$$\begin{cases} R^2(T) = a(T)^2 \sin^2\psi_0 & (1) \\ (1-\frac{2m}{R(T)})(\frac{dt}{dT})^2 - \frac{1}{1-\frac{2m}{R(T)}}(\frac{dR}{dT})^2 = 1 & (2) \end{cases}$$

$(1) \Rightarrow R(T) = \pm a(T) \sin\psi_0$. consider "+" sign

$$\frac{dR}{dT} = \sin\psi_0 \frac{da}{dT} \Rightarrow \left(\frac{dR}{dT}\right)^2 = \sin^2\psi_0 \left(\frac{da}{dT}\right)^2$$

use $\left(\frac{da}{dT}\right)^2 + 1 = \frac{8\pi\rho_0}{3a}$, we have

$\left(\frac{dR}{dT}\right)^2 = \sin^2\psi_0 \left(\frac{8\pi\rho_0}{3a} - 1\right)$, sub this into ②

gives $\left(1-\frac{2m}{R}\right)\left(\frac{dt}{dT}\right)^2 - \frac{\sin^2\psi_0}{\left(1-\frac{2m}{R}\right)}\left(\frac{8\pi\rho_0}{3a} - 1\right) = 1$

$\because R = a\sin\psi_0$

$$\therefore \left(1-\frac{2m}{a\sin\psi_0}\right)\left(\frac{dt}{dT}\right)^2 - \frac{\sin^2\psi_0}{1-\frac{2m}{a\sin\psi_0}}\left(\frac{8\pi\rho}{3a} - 1\right) = 1$$

$$\therefore \left(\frac{dt}{dT}\right)^2 = \frac{\left(1-\frac{2m}{a\sin\psi_0}\right) + \sin^2\psi_0\left(\frac{8\pi\rho}{3a} - 1\right)}{\left(1-\frac{2m}{a\sin\psi_0}\right)^2} \quad ③$$

② $\Rightarrow g_{tt}\dot{t}^2 - g_{rr}\dot{r}^2 = 1 \Rightarrow$ geodesic time-like

differential equation ③ can be solved and togather with ① gives continuous metric across the hypersurface

$\boxed{2}$ $\underline{k^a \nabla_a k_b = \kappa k_b}$

So $\nabla_a k^b k_b = 2k^b \underbrace{\nabla_a k_b}_{\substack{k \text{ killing vector} \\ \nabla_a k_b = -\nabla_b k_a}} = -2k^b \underbrace{\nabla_b k_a}_{\kappa k_a}$

$= -2\kappa k_a$

On $\mathcal{H}$ k_a is hypersurface orthogonal (HSO)

$\Rightarrow$ $k_{[a} \nabla_b k_{c]} = 0$, and together with

$\nabla_{(b} k_{c)} = 0$ (Killing equation) gives

$k_a \nabla_b k_c = -k_b \nabla_c k_a - k_c \nabla_a k_b$

So $k_c (\nabla_a k_b)(\nabla^a k^b)$

$= -(k_a \nabla_b k_c + k_b \nabla_c k_a) \nabla^a k^b$

$= -(\underbrace{k_a \nabla^a k^b}_{\kappa k^b})(\nabla_b k_c) - (k_b \underbrace{\nabla^a k^b}_{=-\nabla^b k^a})(\underbrace{\nabla_c k_a}_{=-\nabla_a k_c})$

$= -\kappa \underbrace{k^b \nabla_b k_c}_{\kappa k_c} + k_b \nabla^b k^a (-\nabla_a k_c)$

$= -\kappa^2 k_c - \kappa k^a \nabla_a k_c = -2\kappa^2 k_c$ on $\mathcal{H}$.

$\Rightarrow$ Hence $\underline{\kappa^2 = -\frac{1}{2} (\nabla_a k_b)(\nabla^a k^b)}$ on $\mathcal{H}$.

Now, to proceed we show $\nabla_a \nabla_b k_c = -R_{bcad} k^d$

<u>Proof:</u> Killing equation $\nabla_\nu k_\rho + \nabla_\rho k_\nu = 0$ for Killing vector k, take covariant derivative

$$\nabla_\mu \nabla_\nu k_\rho + \nabla_\mu \nabla_\rho k_\nu = 0 \Rightarrow$$

$$0 = [\nabla_\mu \nabla_\nu k_\rho + \nabla_\mu \nabla_\rho k_\nu] - [\nabla_\rho \nabla_\mu k_\nu + \nabla_\rho \nabla_\nu k_\mu]$$
$$+ [\nabla_\nu \nabla_\rho k_\mu + \nabla_\nu \nabla_\mu k_\rho] - [\nabla_\mu \nabla_\nu k_\rho + \nabla_\mu \nabla_\rho k_\nu]$$

$$= \nabla_\mu \nabla_\nu k_\rho - \nabla_\mu \nabla_\rho k_\nu + [\nabla_\mu, \nabla_\rho] k_\nu - [\nabla_\rho, \nabla_\nu] k_\mu$$
$$+ [\nabla_\nu, \nabla_\mu] k_\rho$$

$\Rightarrow$ use $[\nabla_\mu, \nabla_\nu] V^\sigma = R^\sigma{}_{\beta\mu\nu} V^\beta$

and $\nabla_\nu k_\rho = -\nabla_\rho k_\nu$

we have

$$2 \nabla_\mu \nabla_\nu k_\rho = [-R_{\nu\sigma\mu\rho} + R_{\mu\sigma\rho\nu} - R_{\rho\sigma\nu\mu}] k^\sigma$$

$$\rightarrow \nabla_\mu \nabla_\nu k_\rho = \frac{1}{2} [-R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} - R_{\sigma\rho\mu\nu}] k^\sigma$$

$$\because R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} + R_{\sigma\rho\mu\nu} = 0$$

$$\therefore \nabla_\mu \nabla_\nu k_\rho = R_{\sigma\mu\nu\rho} k^\sigma = R_{\rho\nu\mu\sigma} k^\sigma$$

$$\Rightarrow \underline{\nabla_a \nabla_b k_c = R_{cbad} k^d = -R_{bcad} k^d} \quad \square \quad (*)$$

Now consider $k^a \nabla_a k_b = \kappa k_b$, act both sides with $k^c \nabla_c$ gives

$$RHS = k^c \nabla_c (\kappa k_b) = \kappa k^c \nabla_c k_b + k^c k_b \nabla_c \kappa$$
$$= \kappa^2 k_b + k^c k_b \nabla_c \kappa$$

$$LHS = k^c \nabla_c (k^a \nabla_a k_b)$$
$$= (\nabla_a k_b) k^c \nabla_c k^a + k^c k^a \nabla_c \nabla_a k_b$$
$$= \kappa^2 k_b - k^c k^a R_{abcd} k^d \quad \leftarrow \text{used } (*)$$

$$\Rightarrow \quad k^c k_b \nabla_c \kappa = - k^c k^a R_{abcd} k^d$$
$$= - k^a R_{ab(cd)} k^{(c} k^{d)}$$
$$= 0$$

$$\Rightarrow k^c \nabla_c \kappa = k^c \partial_c \kappa = \boxed{\mathcal{L}_k \kappa = 0} \quad \text{so}$$

κ is constant up the generators.

Now on the bifurcation surface $\mathcal{F}$ of $\mathcal{H}$ where $\underline{k_a = 0}$. Consider $\kappa^2 = -\frac{1}{2} (\nabla_a k_b)(\nabla^a k^b)$ and take the Lie derivative with respect to a direction Z^μ tangent to $\mathcal{F}$.

So $\kappa \mathcal{L}_Z \kappa = -\frac{1}{2} Z^a (\nabla_a \nabla_b k_c)(\nabla^b k^c)$

$$= \frac{1}{2} k^d Z^a R_{bcad} (\nabla^b k^c) \quad \leftarrow \text{used } (*)$$

$= 0 \quad \because \ k^d = 0 \quad \text{on } \mathcal{F}. \Rightarrow \boxed{\mathcal{L}_\xi \kappa = 0}$

So we know:

① κ constant along generators.

② on $\mathcal{F}$, κ constant across generators.

①, ② $\Rightarrow$ κ constant everywhere on horizon

$\boxed{3}$ Reissner - Nordstrom metric

$$ds^2 = \frac{\Delta}{r^2} dt^2 - \frac{r^2}{\Delta} dr^2 - r^2 d\Omega^2$$

change to Eddington - Finkelstein coordinates

$$ds^2 = + \frac{\Delta}{r^2} dv^2 - 2\, dv\, dr - r^2 d\Omega^2 \quad \text{with}$$

$$dv = dt + \frac{r^2 dr}{\Delta} \quad , \quad \Delta = (r - r_+)(r - r_-)$$
$$= r^2 - 2mr + Q^2$$

The stationary Killing vector field is

$k = \frac{\partial}{\partial v}$. At $r = r_+$, we have $\Delta = 0$

so the hypersurface $f = r_+ - r$ has

orthogonal $\nabla_a f = (0, -1, 0, 0)$

$\because k^a = (1, 0, 0, 0)$ and on f, $\Delta = 0$

$$\therefore k_v = \underbrace{g_{vv}}_{=0 \because \Delta = 0} k^v = 0 \qquad k_r = \underbrace{g_{rv}}_{=-1} k^v = -1$$

$\rightarrow k_a = (0, -1, 0, 0) = \nabla_a f$ on f

so Killing vector k_a orthogonal to

$f = r_+ - r \implies f$ a Killing horizon.

use $\nabla_a k^b k_b = -2\kappa k_a$, take $a = r$, $k_r = (-1)$

$$\therefore \nabla_r (k^b k_b) = -2\kappa(-1) = 2\kappa \Big|_{r=r_+}$$

$$\therefore \kappa = \frac{1}{2} \nabla_r (k^b k_b)\Big|_{r=r_+} = \frac{1}{2} \partial_r (k^b k_b)\Big|_{r=r_+}$$

$$\because k^b = (1,0,0,0) \quad \therefore k^b k_b = k_v = g_{vv} k^v = \frac{\Delta}{r^2}$$

$$\therefore \kappa = \frac{1}{2} \partial_r \left(\frac{\Delta}{r}\right)\Big|_{r=r_+} = \frac{1}{2}\left(\frac{\Delta'}{r^2} - \frac{2\Delta}{r^3}\right)\Big|_{\substack{r=r_+ \\ \Delta=0}}$$

$$= \frac{1}{2} \frac{1}{r^2} \frac{d}{dr}\left(r^2 - (r_+ + r_-) r - r_+ r_-\right)\Big|_{r=r_+}$$

$$= \frac{1}{2} \frac{1}{r_+^2} (2r_+ - r_+ - r_-) = \boxed{\frac{r_+ - r_-}{2r_+^2}}$$

$\boxed{4}$ It has been proven in $\boxed{2}$ that

$$\underline{\nabla_a \nabla_b k_c = R_{cbad} k^d}$$

$$\therefore \quad \nabla^b \nabla_b k_c = -R^b{}_{cbd} k^d = -R_{cd} k^d$$

$$\rightarrow \underline{\nabla^b \nabla_b k_a = -R_{ab} k^b} \quad (**)$$

$$\therefore (*d*dk)_a = -\nabla^b (dk)_{ab} = -\nabla^b \nabla_a k_b + \nabla^b \nabla_b k_a$$

$$\underbrace{=}_{\text{Killing eqn.}} 2\nabla^b \nabla_b k_a = -2R_{ab} k^b$$

$$\Rightarrow \quad *d*dk = R_{ab} k^a dx^b$$

$$\Rightarrow \quad \underline{d*dk = R_{ab} k^a * dx^b} \qquad \square$$

Now consider Bianchi identity $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$

so $\frac{1}{2}k^\nu \nabla_\nu R = k^\nu \nabla^\mu R_{\mu\nu} = \nabla^\mu (k^\nu R_{\mu\nu}) - \underbrace{R_{\mu\nu}\nabla^\mu k^\nu}_{=0}$

$\because$ symmetric $\times$ antisymmetric

$= -\nabla^\mu(\underbrace{\nabla^\nu \nabla_\nu k_\mu}_{\text{Killing}}) = \nabla_\mu \nabla_\nu \underbrace{\nabla^\mu k^\nu}_{\text{Killing}} = \nabla_{[\mu}\nabla_{\nu]}\nabla^{[\mu}k^{\nu]}$

(**)

$= \frac{1}{2}[\nabla_\mu, \nabla_\nu]\nabla^{[\mu}k^{\nu]}$

Now use $[\nabla_\rho, \nabla_\sigma]T^{\mu\nu} = R^\mu{}_{\lambda\rho\sigma}T^{\lambda\nu} + R^\nu{}_{\lambda\rho\sigma}T^{\mu\lambda}$

$= \frac{1}{2}\left(R^\mu{}_{\lambda\mu\nu}\nabla^{[\lambda}k^{\nu]} - R^\nu{}_{\lambda\nu\mu}\nabla^{[\mu}k^{\lambda]}\right)$

$= \frac{1}{2}\left(\underbrace{R_{\lambda\nu}\nabla^\lambda k^\nu}_{0} - \underbrace{R_{\lambda\mu}\nabla^\mu k^\lambda}_{0}\right)$ (sym $\times$ antisym)

$= 0 \implies \underline{k^a \nabla_a R = 0}$ (***)

$J_a = T_{ab}k^b - \frac{1}{2}T k_a = (T_{ab} - \frac{1}{2}T g_{ab})k^b$

Einstein equation $R_{ab} - \frac{1}{2}R g_{ab} = G T_{ab}$ $\left(G \equiv \frac{8\pi G}{c^4}\right)$

$\implies R - 2R = GT \quad \therefore \quad -R = TG$

$R_{ab} = G(T_{ab} - \frac{1}{2}T g_{ab})$

$\rightarrow R_{ab}k^b = G(T_{ab} - \frac{1}{2}T g_{ab})k^b = G J_a$

$$\therefore \quad \nabla^a J_a = \frac{1}{G} \nabla^a (R_{ab} k^b)$$

$$= \frac{1}{G} \left(\underbrace{R_{ab} \nabla^a k^b}_{=0} + k^b \nabla^a R_{ab} \right)$$

sym × antisym

$$= \frac{1}{2G} k^b \nabla_b R \qquad \text{(Bianchi)}$$

$$= 0 \qquad \text{by } (\ast\ast\ast)$$