QFT in Curved Spacetime

Problem Set 1

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$\boxed{1}$ i) Maxwell's equations :

$$g^{ac}\nabla_c F_{ab} = 0 \,, \quad \nabla_{[a}F_{bc]} = 0$$

under $\tilde{g}_{ab} = \Omega^2 g_{ab}$ (so $\tilde{g}^{ab} = \Omega^{-2} g^{ab}$)

If ∇_a is associated with g_{ab} and $\tilde{\nabla}_a$ is associated with $\tilde{g}_{ab}$, we denote for some covector ω_a

$$\tilde{\nabla}_a \omega_b = \nabla_a \omega_b - C^c{}_{ab}\,\omega_c \quad \text{so that}$$

$$C^c{}_{ab} = \frac{1}{2}\tilde{g}^{cd}\{\nabla_a \tilde{g}_{bd} + \nabla_b \tilde{g}_{ad} - \nabla_d \tilde{g}_{ab}\}$$

since $\nabla_a \tilde{g}_{bc} = \nabla_a(\Omega^2 g_{bc}) = 2\Omega g_{bc}\nabla_a\Omega$

so $C^c{}_{ab} = \Omega^{-1} g^{cd}\{g_{bd}\nabla_a\Omega + g_{ad}\nabla_b\Omega - g_{ab}\nabla_d\Omega\}$

$$= 2\delta^c{}_{(a}\nabla_{b)}\ln\Omega - g_{ab}\,g^{cd}\nabla_d\ln\Omega$$

Note that $C^c{}_{ab} = C^c{}_{ba}$, symmetric in lower indices

Assume we are in <u>n dimensions</u> and F_{ab} has <u>scaling weight s</u> (i.e. $F_{ab} \to \tilde{F}_{ab} = \Omega^s F_{ab}$)

$$\therefore g^{ac}\nabla_c F_{ab} \to \tilde{g}^{ac}\tilde{\nabla}_c(\Omega^s F_{ab})$$

$$= \Omega^{-2} g^{ac}\{\nabla_c(\Omega^s F_{ab}) - C^d{}_{ca}\Omega^s F_{db} - C^d{}_{cb}\Omega^s F_{ad}\}$$

$$= \Omega^{s-2} g^{ac} \nabla_c F_{ab} + (n-4+s)\Omega^{s-3} g^{ac} F_{ab} \nabla_c \Omega$$

$$\nabla_{[a} F_{bc]} \to \tilde{\nabla}_{[a} \Omega^s F_{bc]}$$

$$= \nabla_{[a} \Omega^s F_{bc]} - \underbrace{C^d_{[(ab)}}_{=0} \Omega^s F_{|d|c]} - \underbrace{C^d_{[(ac)}}_{=0} \Omega^s F_{b]d}$$

symmetrise then antisymmetrise

$$= \Omega^s \nabla_{[a} F_{bc]} + s\Omega^{s-1} (\nabla_{[a} \Omega) F_{bc]}$$

so in $n=4$, $s=0$

$$\left.\begin{aligned} g^{ac} \nabla_c F_{ab} &\to \Omega^{-2} g^{ac} \nabla_c F_{ab} \\ \nabla_{[a} F_{bc]} &\to \nabla_{[a} F_{bc]} \end{aligned}\right\} \Rightarrow \text{Conformally invariant}$$

$\square$

* 1) on the level of action, 4-dim

$$S = \int d^4x \sqrt{-g}\left(-\frac{1}{4}F_{ab}F^{ab}\right)$$

$$= \int d^4x \sqrt{-g}\left(-\frac{1}{4}\right) g^{ac}g^{bd}F_{ab}F_{cd}$$

for $g_{ab} \to \Omega^2 g_{ab}$ $\therefore$ inverse $g^{ab} \to \Omega^{-2} g^{ab}$

$g = \det(g_{ab}) \to \det(\Omega^2 g_{ab}) = (\Omega^2)^4 \det(g_{ab})$

$= \Omega^8 g$

$\therefore \sqrt{-g} \to \Omega^4 \sqrt{-g}$

$\therefore S \to -\frac{1}{4}\int d^4x\, \Omega^4 \sqrt{-g}\, \Omega^{-2} g^{ac}\, \Omega^{-2} g^{bd} F_{ab}F_{cd}$

$$= \int d^4x \sqrt{-g}\left(-\frac{1}{4}F_{ab}F^{ab}\right) = S$$

$\therefore$ Action of maxwell theory is invariant.

2) stress-energy tensor is defined by

$$T^{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{ab}} \Rightarrow \delta S = -\frac{1}{2}\int d^4x \sqrt{-g}\, T^{ab}\, \delta g_{ab}$$

infinitesimally $g_{ab} \to \Omega^2 g_{ab}$ becomes

$$g_{ab} \to (1+\epsilon)\, g_{ab} \quad \text{for small parameter } \epsilon(x)$$

$$\therefore \delta g_{ab} = \epsilon\, g_{ab} \Rightarrow \delta S = -\frac{1}{2}\int d^4x \sqrt{-g}\, T^a{}_a\, \epsilon(x)$$

$\therefore$ under conformal transformation $\delta S = 0$

$$\therefore \int d^4x \sqrt{-g}\, T^a{}_a\, \epsilon(x) = 0$$

true for arbitrary small $\epsilon(x)$

$$\Rightarrow \underline{T^a{}_a = 0} \quad \text{trace free.}$$

* 2) stress energy tensor $T^{ab} = T^{ba}$

conservation $\nabla_a T^{ab} = 0$ under conformal transformation $(T^{ab} \to \tilde{T}^{ab} = \Omega^s T^{ab}, \tilde{g}_{ab} = \Omega^2 g_{ab}$

in n dimensions:

$$\to \quad \tilde{\nabla}_a (\Omega^s T^{ab})$$

$$= \nabla_a (\Omega^s T^{ab}) + C^a{}_{ac} \Omega^s T^{cb} + C^b{}_{ac} \Omega^s T^{ac}$$

$$= \Omega^s \nabla_a T^{ab} + (s+n+2) \Omega^{s-1} T^{ab} \nabla_a \Omega - \Omega^{s-1} g^{ba} T \nabla_a \Omega$$

$(T = T^a{}_a = \text{trace of } T)$

choose scale weight $s = -n-2$.

For $n=4$, $s = -6$

$$\nabla_a T^{ab} \to \Omega^{-6} \nabla_a T^{ab} - \Omega^{-7} g^{ba} (\nabla_a \Omega) T$$

conformally invariant only if $\underline{T=0}$

$\Leftrightarrow$ $\underline{T^{ab} \text{ trace free}}$

3) for $d=n$ dimensional wave equation.

(massless) $\Rightarrow$ $(\Box + aR)\phi = 0$

under $\tilde{g}_{ab} = \Omega^2 g_{ab}$, $\phi \to \Omega^s \phi$

$$\Box \to \tilde{\Box} = \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\phi = \Omega^{-2} g^{ab}\tilde{\nabla}_a[\nabla_b(\Omega^s\phi)]$$

$$= \Omega^{-2} g^{ab}[\nabla_a\nabla_b(\Omega^s\phi) - C^c{}_{ab}\nabla_c(\Omega^s\phi)]$$

$$= \Omega^{s-2} g^{ab}\nabla_a\nabla_b\phi + (2s+n-2)\Omega^{s-3} g^{ab}\nabla_a\Omega\nabla_b\phi$$

$$+ s\Omega^{s-3}\phi\, g^{ab}\nabla_a\nabla_b\Omega$$

$$+ s(n+s-3)\Omega^{s-4}\phi\, g^{ab}\nabla_a\Omega\nabla_b\Omega$$

$$R_{abc}{}^d \to \tilde{R}_{abc}{}^d = R_{abc}{}^d - 2\nabla_{[a}C^d{}_{b]c} + 2C^e{}_{c[a}C^d{}_{b]e}$$

$$= R_{abc}{}^d + 2\delta^d{}_{[a}\nabla_{b]}\nabla_c\ln\Omega - 2g^{de}g_{c[a}\nabla_{b]}\nabla_e\ln\Omega$$

$$+ 2(\nabla_{[a}\ln\Omega)\delta^d{}_{b]}\nabla_c\ln\Omega - 2(\nabla_{[a}\ln\Omega)g_{b]c}g^{df}\nabla_f\ln\Omega$$

$$- 2g_{c[a}\delta^d{}_{b]}g^{ef}(\nabla_e\ln\Omega)\nabla_f\ln\Omega$$

So $\tilde{R}_{ac} = R_{ac} - (n-2)\nabla_a\nabla_c\ln\Omega - g_{ac}g^{de}\nabla_d\nabla_e\ln\Omega$

$$+ (n-2)(\nabla_a\ln\Omega)\nabla_c\ln\Omega - (n-2)g_{ac}g^{de}(\nabla_d\ln\Omega)\nabla_e\ln\Omega$$

contracting with $\tilde{g}^{ac}$

$$\Rightarrow \quad \tilde{R} = \Omega^{-2}\big(R - 2(n-1) g^{ac} \nabla_a \nabla_c \ln\Omega$$
$$- (n-2)(n-1) g^{ac} (\nabla_a \ln\Omega) \nabla_c \ln\Omega \big)$$

choose $\quad a = -\dfrac{n-2}{4(n-1)}$, $\quad \underline{s = 1 - \frac{n}{2}}$, then

$$\left(\Box - \frac{n-2}{4(n-1)}\right)\phi \longrightarrow \left(\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b - \frac{n-2}{4(n-1)}\tilde{R}\right)\left[\Omega^{1-\frac{n}{2}}\phi\right]$$

$$= \Omega^{-1-\frac{n}{2}}\left(g^{ab}\nabla_a\nabla_b - \frac{n-2}{4(n-1)}R\right)\phi$$

$\rightarrow$ conformally invariant

scale weight (for $n=d$) , is

$$\underline{\underline{s = 1 - \frac{d}{2}}} \qquad \square$$

For $d = n = 4$, $\quad a = -\dfrac{4-2}{4(4-1)} = -\dfrac{1}{6}$

$\therefore$ in $d=4$, $\left(\Box - \frac{R}{6}\right)\phi = 0$ is conformally invariant.

$\boxed{2}$ conventions: $\psi^A \varepsilon_{AB} = \psi_B$, $\psi^A = \varepsilon^{AB}\psi_B$

identities: $\varepsilon_{A'C'}\varepsilon_{B'}{}^{C'} = -\varepsilon_{B'}{}^{C'}\varepsilon_{C'A'} = -\varepsilon_{B'A'}$

$= \varepsilon_{A'B'}$

$$\varepsilon_B{}^C\phi_{AC} = -\varepsilon_{BC}\phi_A{}^C = \phi_A{}^C\varepsilon_{CB} = \phi_{AB}$$

$$\varepsilon_{A'C'}\bar\phi_{B'}{}^{C'} = -\bar\phi_{B'}{}^{C'}\varepsilon_{C'A'} = -\bar\phi_{B'A'} = -\bar\phi_{A'B'}$$

$\varepsilon_{C'D'}\varepsilon^{C'D'} = +2$, $\phi_{AC}\phi_B{}^C = +\frac{1}{2}\varepsilon_{AB}\phi_{CD}\phi^{CD}$ for symmetric ϕ

$\varepsilon_{AC}\phi_B{}^C = -\phi_{AB}$, $\varepsilon_{B'}{}^{C'}\bar\phi_{A'C'} = \bar\phi_{A'B'}$

These will be used in the calculation

maxwell theory: $F_{ab} = \varepsilon_{A'B'}\phi_{AB} + \varepsilon_{AB}\bar\phi_{A'B'}$

$= F_{AA'BB'}$ (ϕ_{AB}, $\bar\phi_{A'B'}$ symmetric)

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}$$

$$= -F_{AA'CC'}F_{BB'}{}^{CC'} + \frac{1}{4}\varepsilon_{AB}\varepsilon_{A'B'}F_{CC'DD'}F^{CC'DD'}$$

The quadratic terms:

$$F_{AA'CC'}F_{BB'}{}^{CC'} = (\varepsilon_{A'C'}\phi_{AC} + \varepsilon_{AC}\bar\phi_{A'C'})(\varepsilon_{B'}{}^{C'}\phi_B{}^C + \varepsilon_B{}^C\bar\phi_{B'}{}^{C'})$$

$$= \varepsilon_{A'C'}\varepsilon_{B'}{}^{C'}\phi_{AC}\phi_B{}^C + \varepsilon_{A'C'}\varepsilon_B{}^C\phi_{AC}\bar\phi_{B'}{}^{C'}$$

$$+ \varepsilon_{AC}\varepsilon_{B'}{}^{C'}\bar\phi_{A'C'}\phi_B{}^C + \varepsilon_{AC}\varepsilon_B{}^C\bar\phi_{A'C'}\bar\phi_{B'}{}^{C'}$$

$$= \varepsilon_{A'B'}\phi_{AC}\phi_B{}^C - \bar{\phi}_{A'B'}\phi_{AB} - \phi_{AB}\bar{\phi}_{A'B'}$$

$$+ \varepsilon_{AB}\bar{\phi}_{A'C'}\bar{\phi}_{B'}{}^{C'}$$

$$= -2\phi_{AB}\bar{\phi}_{A'B'} + \varepsilon_{AB}\bar{\phi}_{A'C'}\bar{\phi}_{B'}{}^{C'} + \varepsilon_{A'B'}\phi_{AC}\phi_B{}^C$$

$$= -2\phi_{AB}\bar{\phi}_{A'B'} + \frac{1}{2}\varepsilon_{AB}\varepsilon_{A'B'}\bar{\phi}_{C'D'}\bar{\phi}^{C'D'}$$

$$+ \frac{1}{2}\varepsilon_{AB}\varepsilon_{A'B'}\phi_{CD}\phi^{CD}$$

$$\frac{1}{4}F_{CC'DD'}F^{CC'DD'}$$

$$= \frac{1}{4}(\varepsilon_{C'D'}\phi_{CD} + \varepsilon_{CD}\bar{\phi}_{C'D'})(\varepsilon^{C'D'}\phi^{CD} + \varepsilon^{CD}\bar{\phi}^{C'D'})$$

$$= \frac{1}{4}(\underbrace{\varepsilon_{C'D'}\varepsilon^{C'D'}}_{+2}\phi_{CD}\phi^{CD} + \varepsilon_{C'D'}\underbrace{\varepsilon^{CD}\phi_{CD}}_{=0,\ \text{sym} \times \text{antisym}}\bar{\phi}^{C'D'}$$

$$\varepsilon_{CD}\underbrace{\varepsilon^{C'D'}\bar{\phi}_{C'D'}}_{=0\ \text{sym} \times \text{antisym}}\phi^{CD} + \underbrace{\varepsilon_{CD}\varepsilon^{CD}}_{+2}\bar{\phi}_{C'D'}\bar{\phi}^{C'D'})$$

$$= +\frac{1}{2}\phi_{CD}\phi^{CD} + \frac{1}{2}\bar{\phi}_{C'D'}\bar{\phi}^{C'D'}$$

$$\Rightarrow T_{ab} = 2\phi_{AB}\bar{\phi}_{A'B'} + \frac{1}{2}\varepsilon_{AB}\varepsilon_{A'B'}(-\cancel{\bar{\phi}_{C'D'}\bar{\phi}^{C'D'}}$$

$$-\cancel{\phi_{CD}\phi^{CD}} + \cancel{\phi_{CD}\phi^{CD}} + \cancel{\bar{\phi}_{C'D'}\bar{\phi}^{C'D'}}) = \boxed{2\phi_{AB}\bar{\phi}_{A'B'}}$$

→ scalar massless wave equation with $a=0$

$$T_{ab} = \partial_a\phi\,\partial_b\phi - \frac{1}{2}g_{ab}\,\partial_c\phi\,\partial^c\phi$$

$$= \nabla_{AA'}\phi\,\nabla_{BB'}\phi - \frac{1}{2}\varepsilon_{AB}\,\varepsilon_{A'B'}\nabla_{CC'}\phi\,\nabla^{CC'}\phi$$

$$= \nabla_{AA'}\phi\,\nabla_{BB'}\phi - \varepsilon_{A'B'}\nabla_{AC'}\phi\,\nabla_B{}^{C'}\phi$$

(we used $\nabla_{AC'}\phi\,\nabla_B{}^{C'}\phi = +\frac{1}{2}\varepsilon_{AB}\nabla_{CC'}\phi\,\nabla^{CC'}\phi$)

$\qquad\qquad\qquad\qquad\parallel$

and notice that $\varepsilon_{A'B'}\nabla_{AC'}\phi\,\varepsilon^{C'D'}\nabla_{BD'}\phi$

for $\underline{A'=1 \quad B'=2}$

$$\begin{array}{cccccccc} 1 & 2 & A & 1 & 1 & 2 & B & 2 \\ 1 & 2 & A & 2 & 2 & 1 & B & 1 \end{array}$$

$(\varepsilon_{12}=+1 \quad \varepsilon^{12}=+1 \quad \varepsilon^{21}=-1)$

so this $= +\nabla_{AA'}\phi\,\nabla_{BB'}\phi - \nabla_{AB'}\phi\,\nabla_{BA'}\phi$

$$\therefore T_{ab} = \cancel{\nabla_{AA'}\phi\,\nabla_{BB'}\phi} - (\cancel{+\nabla_{AA'}\phi\,\nabla_{BB'}} - \nabla_{AB'}\phi\,\nabla_{BA'}\phi)$$

$$\Rightarrow \underline{T_{ab} = \nabla_{AB'}\phi\,\nabla_{BA'}\phi}$$

- when contracted with $l^a l^b$ with l being time-like, we can write $\phi_{AB} = \alpha_A \alpha_B$ for some α

and $l^a = l^{AA'} = \sigma^A \bar{\sigma}^{A'}$ for some σ

so with $T_{ab} = \phi_{AB} \bar{\phi}_{A'B'}$

$$T_{ab} l^a l^b = \phi_{AB} \bar{\phi}_{A'B'} \sigma^A \bar{\sigma}^{A'} \sigma^B \bar{\sigma}^{B'}$$

$$= \alpha_A \alpha_B \bar{\alpha}_{A'} \bar{\alpha}_{B'} \sigma^A \bar{\sigma}^{A'} \sigma^B \bar{\sigma}^{B'}$$

$$= |\alpha_A \sigma^A \alpha_B \sigma^B|^2 \geq 0$$

similarly, with $T^{ab} = \nabla_{AB'} \phi \nabla_{BA'} \phi$

and ϕ is real scalar

$$T_{ab} l^a l^b = \nabla_{AB'} \phi \nabla_{BA'} \phi \sigma^A \bar{\sigma}^{A'} \sigma^B \bar{\sigma}^{B'}$$

$$= |\sigma^A \nabla_{AB'} \phi \bar{\sigma}^{B'}|^2 \geq 0$$

Hence the dominant energy condition

D.

$\boxed{4}$ 1) $\nabla_l = l^a \nabla_a$ for null vector l^a.

we know $\nabla_l \zeta = -\rho\zeta - \sigma\bar{\zeta}$, $X^a = \zeta\bar{m}^a + \bar{\zeta}m^a$

For null congruences X^a, $\underline{\nabla_l X^a = X^b \nabla_b l^a}$

so $\nabla_l \nabla_l X^d = \nabla_l (X^b \nabla_b l^d)$

$$= \nabla_l \nabla_X l^d = (\nabla_X \nabla_l + \nabla_{\underbrace{[l,X]}_{\text{Lie bracket } = 0}}) l^d + R_{abc}{}^d l^a X^b l^c$$

$$= X^a \nabla_a (\underbrace{\nabla_l l^d}_{=0 \ \because \text{ geodesic equation}}) + R_{abc}{}^d l^a X^b l^c$$

$$\Rightarrow \quad \nabla_l \nabla_l X^d = R_{abc}{}^d l^a X^b l^c \quad \text{(geodesic deviation)}$$

Given that $\Psi_0 = \underbrace{C_{abcd}}_{\text{Weyl curvature tensor}} l^a m^b l^c m^d$

and $\Phi_{00} = -\frac{1}{2} R_{ab} l^a l^b$

$$\left\{ \begin{array}{l} X^a = \zeta\bar{m}^a + \bar{\zeta}m^a , \quad m^a m_a = 0 , \bar{m}^a \bar{m}_a = 0 , \\ \nabla_l m^a = 0 , \quad \nabla_l \bar{m}^a = 0 , \\ m^a \bar{m}_a = o^A \bar{\iota}^{A'} \bar{o}_{A'} \iota_A = |\underbrace{o^A \iota_A}_{1}|^2 = \underline{1} \end{array} \right.$$

use these we have

$$\bar{m}^d (D_1 D_1 \xi) + m^d (D_1 D_1 \xi)$$

$$= R_{abc}{}^{d}\, l^a (\bar{m}^b \xi + m^b \bar{\xi})\, l^c$$

contract with m_d gives

$$\underbrace{(m_d \bar{m}^d)}_{1} (D_1 D_1 \xi) + \underbrace{m_d m^d}_{0} (D_1 D_1 \xi)$$

$$= \underbrace{R_{abcd}\, l^a l^c \bar{m}^b m^d \xi}_{(1)} + \underbrace{R_{abcd}\, l^a m^b l^c m^d}_{(2)} \bar{\xi}$$

$$(1) \underbrace{=}_{R_{abcd} = R_{cdab}} R_{abcd}\, l^a l^c \bar{m}^{(b} m^{d)} \underbrace{=}_{\text{use } \bar{m}^{(b} m^{d)} = \frac{1}{2} g^{bd}} \frac{1}{2} R_{ac}\, l^a l^c$$

$$= -\left(-\frac{1}{2} R_{ac}\, l^a l^c\right) = -\Phi_{00}$$

(2) $\because\ l^a m^b l^c m^d$ is trace free (l, m both null) $\therefore$ it only picks up the trace free part of R_{abcd}, which is $-C_{abcd}$

$$\therefore\ (2) = -C_{abcd}\, l^a m^b l^c m^d \bar{\xi} = -\Psi_0 \bar{\xi}$$

Hence $\quad D_1 D_1 \xi = -\Phi_{00} \xi - \Psi_0 \bar{\xi} \quad (*)$

in matrix form, $\nabla_l \xi = -\rho\xi - \sigma\bar{\xi}$ and its conjugate becomes

$$\nabla_l \underbrace{\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}}_{\vec{z}} = -\underbrace{\begin{pmatrix} \rho & \sigma \\ \bar{\sigma} & \bar{\rho} \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}}_{\vec{z}}$$

$$\Rightarrow \quad \nabla_l \vec{z} = -P\vec{z}$$

and (*) becomes (including its conjugate)

$$\nabla_l \nabla_l \vec{z} = -Q\vec{z} \quad \text{where}$$

$$Q = \begin{pmatrix} \Phi_{00} & \Psi_0 \\ \bar{\Psi}_0 & \Phi_{00} \end{pmatrix}$$

So $\quad \nabla_l \nabla_l \vec{z} = \underline{-Q\vec{z}} = \nabla_l(-P\vec{z})$

$$= -(\nabla_l P)\vec{z} - P \underbrace{\nabla_l \vec{z}}_{-P\vec{z}} = \underline{(-\nabla_l P + P^2)\vec{z}}$$

$$\Rightarrow \quad \nabla_l P = P^2 + Q$$

$$\Rightarrow \quad \begin{cases} \nabla_l \rho = \rho\bar{\rho} + \sigma\bar{\sigma} + \Phi_{00} \\ \nabla_l \sigma = (\rho + \bar{\rho})\sigma + \sigma\bar{\sigma} + \Psi_0 \end{cases} \quad \text{(Sach's equation)}$$

2) vanishing twist $\Rightarrow \rho = \bar{\rho} \Rightarrow \rho\bar{\rho} = \rho^2$

vanishing shear $\Rightarrow \sigma = 0$, and that

$l^a \partial_a = \partial_s = \frac{d}{ds}$ for some affine s.

flat space $\Rightarrow$ $\underline{\Phi_{00} = 0}$, $\underline{D_l = l^a \partial_a}$

so $D_l \rho = \rho^2 \Rightarrow \underline{\frac{d\rho}{ds} = \rho^2}$

$$\therefore \int_{\rho = \rho_0}^{\rho} \frac{d\rho'}{\rho'^2} = \int_{s=0}^{s} ds' \Rightarrow \frac{1}{\rho_0} - \frac{1}{\rho} = s$$

$$\therefore \frac{1}{\rho} = \frac{1}{\rho_0} - s = \frac{1 - \rho_0 s}{\rho_0}$$

$$\Rightarrow \underline{\underline{\rho = \frac{\rho_0}{1 - \rho_0 s}}} \quad \square$$

$\rho = \mathrm{Re}(\rho)$ is the <u>contraction</u>, so the area of horizon shrinks faster and faster ($\rho_0 > 0$) and the rate of shrinking goes to ∞ at time $s = \frac{1}{\rho_0}$

3) Assume still vanishing twist and shear, but in curved space so $\Phi_{00} \neq 0$, $\Phi_{00} = -\frac{1}{2} R_{ab} l^a l^b$.

But Einstein equation gives

$$R_{ab} - \frac{1}{2} g_{ab}(R-2\Lambda) = -8\pi G T_{ab}$$

$\therefore$ $R_{ab} l^a l^b = -\frac{1}{2}(R-2\Lambda) l_a l^a = -8\pi G T_{ab} l^a l^b$

for future pointing null l^a, $l^a l_a = 0$

dominant energy condition $T_{ab} l^a l^b \geq 0$

$\therefore R_{ab} l^a l^b = -8\pi G T_{ab} l^a l^b \leq 0$

$\Rightarrow \quad -2\Phi_{00} \leq 0 \quad \therefore \Phi_{00} \geq 0$

Sach's equation in this case is

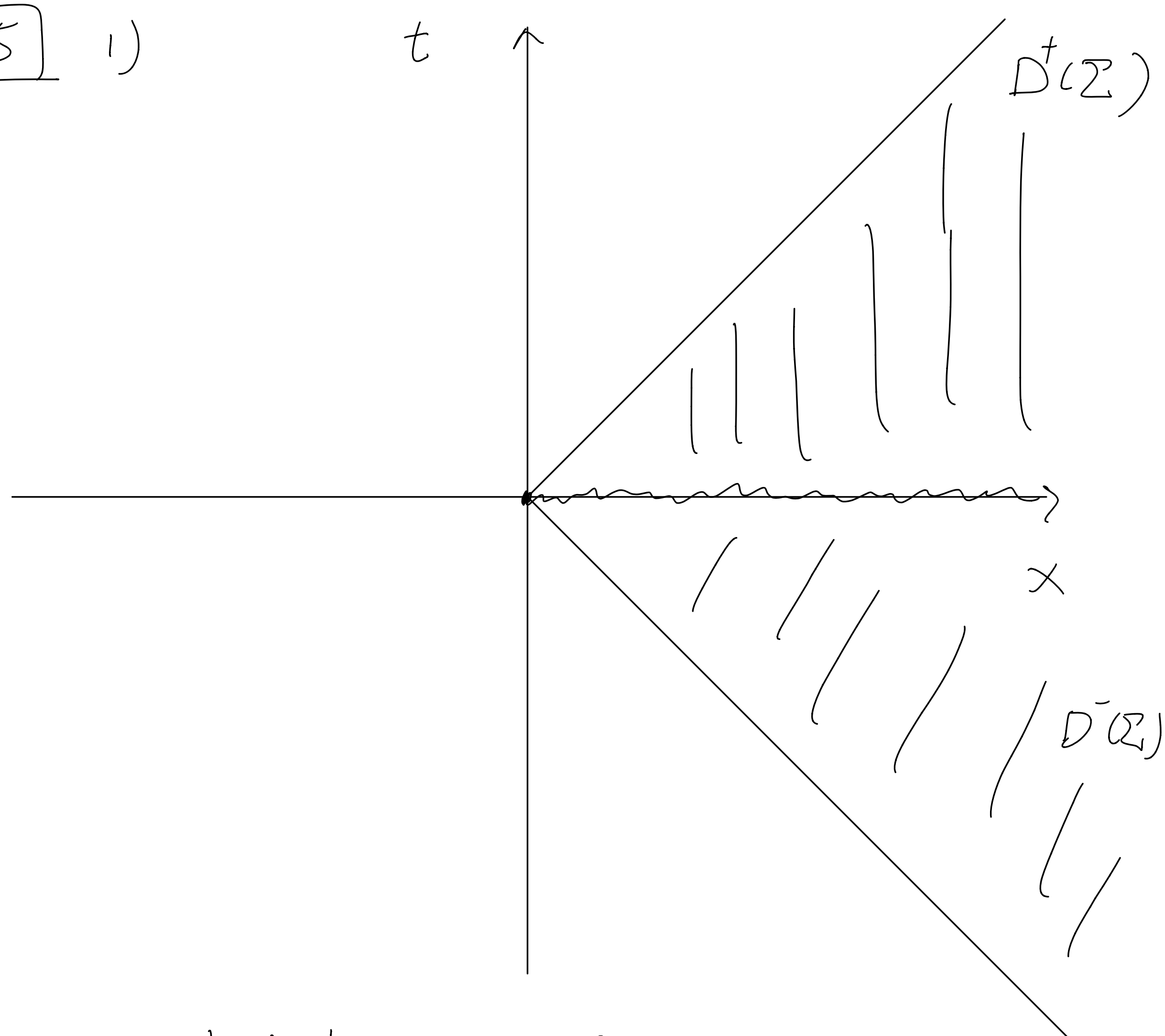
$\frac{d\rho}{ds} = \rho^2 + \Phi_{00}$ $\because \Phi_{00} \geq 0 \quad \therefore \frac{d\rho}{ds} \geq \rho^2$

we see previously that if $\frac{d\rho}{ds} = \rho^2$ then with $\rho_0 > 0$, ρ blows up at proper time $s = \frac{1}{\rho_0}$ = finite , since with curvature

$\frac{d\rho}{ds} \geq \rho^2$, so ρ grows faster than in the $\frac{d\rho}{ds} = \rho^2$ case, and blows up <u>within</u> proper time $\frac{1}{\rho_0}$.

$\Rightarrow$ blows up in finite time.

$\boxed{5}$ 1)



The shaded area $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$ is the Domain of dependence of hyper-surface $\Sigma: t=0, x>0$ in Minkowski spacetime.

(〰〰 represents Σ)

2) $ds^2 = a^2\xi^2 d\tau^2 - d\xi^2 - dy^2 - dz^2$

consider coordinates change

$$\boxed{\begin{aligned} t &= \xi \sinh(a\tau) \\ x &= \xi \cosh(a\tau) \end{aligned}} \qquad (*)$$

$\Rightarrow \quad dt = \sinh(a\tau)\, d\xi + a\xi \cosh(a\tau)\, d\tau$

$\quad\;\; dx = \cosh(a\tau)\, d\xi + a\xi \sinh(a\tau)\, d\tau$

$\therefore\; dt^2 - dx^2 = \sinh^2(a\tau)\, d^2\xi + a^2\xi^2 \cosh^2(a\tau)\, d^2\tau$

$\qquad - \cosh^2(a\tau)\, d^2\xi - a^2\xi^2 \sinh^2(a\tau)\, d^2\tau$

$\qquad \cancel{+2a\xi \sinh(a\tau) \cosh(a\tau)\, d\xi d\tau}$

$\qquad \cancel{-2a\xi \sinh(a\tau) \cosh(a\tau)\, d\xi d\tau}$

$= (a^2\xi^2 d^2\tau - d\xi^2)\underbrace{(\cosh^2(a\tau) - \sinh^2(a\tau))}_{1}$

$= a^2\xi^2 d\tau^2 - d\xi^2$

so $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ under $(*)$

$\therefore \Rightarrow$ Minkowski

3) The domain of dependence above has boundary:

$$x = \pm t \implies x^2 = t^2$$

$$\because \begin{cases} x = \xi \cosh(a\tau) \\ t = \xi \sinh(a\tau) \end{cases} \implies \xi^2 \cosh^2(a\tau) = \xi^2 \sinh^2(a\tau)$$

$$\implies \xi^2 (1 + \cancel{\sinh^2(a\tau)}) = \xi^2 \cancel{\sinh^2(a\tau)}$$

$$\implies \xi^2 = 0 \quad \therefore \quad \underline{\underline{\xi = 0}}$$

The domain of dependence (interior) is

$\underline{x^2 > t^2 \quad , x > 0}$, this corresponds

to $\underline{\xi > 0}$

$\therefore$ $D(\Sigma)$ has been mapped to the half plane $\underline{\xi \geq 0}$

4) for $(\xi, y, z) =$ const, $d\xi = dy = dz = 0$

$\therefore ds^2 = a^2 \xi^2 dT^2 \Rightarrow \frac{ds}{dT} = \pm a\xi$

there is gravitational redshift between different ξ coordinates, so a is the gravitational acceleration □

5) Milne universe :

$$ds^2 = d\tau^2 - \tau^2 d\chi^2 - dy^2 - dz^2$$

consider $\boxed{\begin{aligned} x &= \tau \sinh(\chi) \\ t &= \tau \cosh(\chi) \end{aligned}}$, we have

$$dt^2 - dx^2 = (d\tau \cdot \cosh(\chi) + \tau \sinh(\chi) \cdot d\chi)^2$$

$$- (d\tau \sinh(\chi) + \tau \cosh(\chi) \cdot d\chi)^2$$

$$= (d\tau^2 - \tau^2 d\chi^2)\underbrace{(\cosh^2(\chi) - \sinh^2(\chi))}_{=1}$$

$$= d\tau^2 - \tau^2 d\chi^2$$

i, Milne metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \Rightarrow \text{Minkowski (Flat)}$$

It is Globally hyperbolic