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Quantum Field Theory

Problem set 4

~~TA:~~

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Wk 8

(11) ϕ^3 theory

Lagrangian (Minkowski) $\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0 \partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{3!}\phi_0^3$

(subscript 0 → bare)

(a) We consider the mass dimensions of various quantities.

$[m] = [p] = 1 \qquad [x] = -1 \qquad [S] = 0$

↳ action

vertex function

$$\Gamma_0^{(N)}(p_1, \cdots, p_N) = \left[\prod_{j=1}^{N} \tilde{G}_0^{(2)}(p_j)^{-1} \tilde{G}_0^{(N)}(p_1, \cdots, p_N)\right]\Bigg|_{1PI}$$

1 point irreducible diagrams

$$= \sum_{n=0}^{\infty} \Gamma_{0,n}^{(N)} \lambda_0^n \quad (1)$$

↳ given by a sum of 1PI Feynman diagrams of n ~~verte~~ vertices.

From lecture notes:

$$[\Gamma_0^{(N)}] = N + D - N\frac{D}{2} = D + N(1 - \frac{D}{2}) \quad (2)$$

(this derivation is general with respect to the dimension of λ_0).

$\because$ $[S] = 0$ and $S = \int d^D x \, \mathcal{L}$.
$[x] = -1$

$\therefore [\mathcal{L}] = D$

For ϕ^3 theory : $[\lambda_0 \phi^3] = D$

$$[m^2\phi^2] = D \underbrace{\Rightarrow}_{[m]=1} [\phi] = \frac{D-2}{2}$$

$$\therefore [\lambda_0][\phi^3] = D = [\lambda_0] + \left(\frac{3}{2}D - 3\right)$$

$$\therefore [\lambda_0] = \cancel{\tfrac{3}{2}} \; 3 - \frac{D}{2}$$

①, ② $\Rightarrow$ $\delta = [\Gamma^{(N)}_{0,n}] = D + N\left(1 - \frac{D}{2}\right) + n\left(\frac{D}{2} - 3\right)$

for this ϕ^3 theory.

~~Now For $D < 6$:~~

~~If $N=2$, $[\Gamma^{(2)}_{0,n}] = 2 + n\left(\frac{D}{2} - 3\right)$, we know that $\tilde{G}^{(N)} = 0$ for odd N, $\therefore$ next for $N = 4, 6, 8 \ldots$~~

~~$[\Gamma^{(N)}_{0,n}] = \cancel{D + N - \frac{ND}{2} + n\frac{D}{2} - 3n} = D + N\left(1 - \frac{D}{2}\right) + n\left(\frac{D}{2} - 3\right)$~~

~~$< D + N\left(1 - \frac{D}{2}\right)$~~

~~$D < 6$~~
~~$\rightarrow \frac{D}{2} - 3 < 0$~~

we see that $[\lambda_0] = 0$ when $\underline{\underline{D_c = 6.}}$

this is when the theory is exactly renormalisable.

b) – For $D < D_c = 6$, $n(\frac{D}{2} - 3)$ term is negative $\therefore \delta < 0$ for sufficiently large order n ~~for~~ all N

For $n = 1$, if we demand $D + N(1 - \frac{D}{2}) + \frac{D}{2} - 3 < 0$

$$\Rightarrow -\frac{N}{2}(D-2) + \frac{3}{2}(D-2) < 0$$

$$\therefore \quad N(D-2) > 3(D-2)$$

If ~~$D > 2$~~ $2 < D < 6 \Rightarrow \underline{N > 3 \Leftrightarrow \delta < 0}$

$\therefore$ There are finite divergences for $N \leq 3$, and no divergence for $N > 3$ in the case of $2 < D < 6$. $\rightarrow$ <u>super-renormalisable.</u>

– If $D > 6$, $n(\frac{D}{2} - 3) > 0$, so all $\Gamma_{0,n}^{(N)}$ are primitively divergent up to sufficient high order $\rightarrow$ <u>non-renormalisable.</u>

– If $D = D_c = 6$. $n(\frac{D}{2} - 3) = 0$ <u>exactly renormalisable</u>

$$\delta = D_c + N(1 - \frac{D_c}{2}) = 6 + N(1-3) = 6 - 2N$$

for ~~$\delta =$~~ $\delta \geq 0$, $N = 1, 2, 3$ ~~are~~ contains divergences ~~an~~

~~So~~ ~~$\Gamma^{(N)}$~~ up to any order. that means all $\Gamma_{0,n}^{(N)}$ are divergent for $N = 1, 2, 3$.

$$\Rightarrow \underline{\Gamma_0^{(1)}, \Gamma_0^{(2)}, \Gamma_0^{(3)} \text{ are divergent}}$$

- These should be made finite by regularising our theory, either by introducing a UV cut-off Λ or by changing D_c to $D_c - 2\varepsilon$ (dimensional regularisation).

c) The (bare) Lagrangian in Euclidean space is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{6}\phi_0^3$$

Renormalisation & counterterms.

$$\phi_0 = (\underbrace{1+\delta Z_\phi}_{Z})^{\frac{1}{2}}\phi$$

$$\therefore \mathcal{L} = \frac{1}{2}Z\,\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}Zm_0^2\phi^2 + \frac{1}{6}Z^{3/2}\lambda_0\phi^3$$

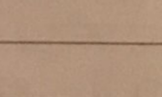
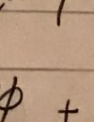
Define $Z = 1+\delta Z_\phi$, $Zm_0^2 = m^2 + \delta m^2$

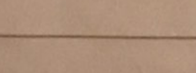
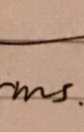
$$Z^{3/2}\lambda_0 = \lambda + \delta\lambda$$

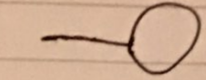
then

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{6}\phi^3 + \underbrace{\left(\frac{\delta Z_\phi}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{6}\phi^3\right)}_{\text{counterterms.}}$$

New Feynman rules

 $= \dfrac{1}{p^2+m^2}$  $= (-\lambda)$

 $= -p^2\delta Z_\phi - \delta m^2$  $= (-\delta\lambda)$

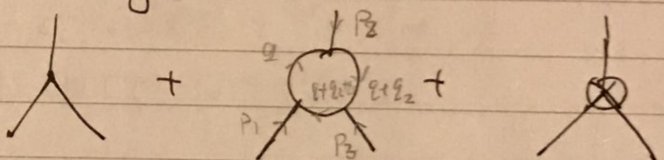
For $\Gamma_{0,n}^{(1)}$, tadpole diagrams like 
will be cancelled completely by their counter-terms after renormalisation $\rightarrow$ ignore $\Gamma_0^{(1)}$

~~Focuse~~ Focus on $\Gamma_0^{(2)}$, $\Gamma_0^{(3)}$ only. with m and later take $m \to 0$

good

Start with $\Gamma_0^{(3)}$:

We only consider 1 loop contributions, so the division of $\tilde{G}_0^{(2)}$ in the expression of $\Gamma_0^{(N)}$ is just the action of removing the external propagators (amputating)

Hence $\Gamma_0^{(3)} =$ 

tree

1-loop

counter term

amputated.

$$\cancel{= (-\lambda_0) + (-\lambda_0)^3 \int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2+m^2}}$$

$$= (-\lambda) + \frac{(-\lambda)^3}{I} \underbrace{\int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2+m^2} \frac{1}{(q+P_2)^2+m^2} \frac{1}{(q+P_2+P_3)^2+m^2}}_{I_1}$$

$$+ (-\delta\lambda).$$

Integral $$I_1 = \int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2+m^2} \frac{1}{(q+P_2)^2+m^2} \frac{1}{(q+P_2+P_3)^2+m^2}$$

$$= 2 \int \frac{d^Dq}{(2\pi)^D} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\delta(x+y+z-1)}{\left[x(q^2+m^2) + y((q+P_2)^2+m^2) + z((q+P_2+P_3)^2+m^2)\right]^3}$$

Feynman parametrisation

$$= 2\int_0^1 dx \int_0^{1-x} dy \int \frac{d^Dq}{(2\pi)^D} \frac{1}{[x(q^2+m^2)+y((q+p_2)^2+m^2)+(1-x-y)((q+p_2+p_3)^2 ...}$$

the denominator ~~cubed~~ cube-rooted is.

$$x(q^2+m^2)+y((q+p_2)^2+m^2)+(1-x-y)((q+p_2+p_3)^2+m^2)$$

$$= m^2(\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y}) + q^2x + y(q^2+2qp_2+p_2^2)$$
$$+ (1-x-y)(q^2+2q\cdot(p_2+p_3)+(p_2+p_3)^2).$$

$$= m^2 + (\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y})q^2 + q(2p_2y+2(1-x-y)(p_2+p_3))$$
$$+ p_2^2y + (1-x-y)(p_2+p_3)^2$$

$$= m^2 + q^2 + (2p_2+2p_3-2p_2x-2p_3x-2p_3y).q.$$
$$+ (p_2^2y+(1-x-y)(p_2+p_3)^2).$$

$$= m^2 + q^2 + 2(p_2y+(1-x-y)(p_2+p_3))q + (p_2y+(1-x-y)(p_2+p_3))^2$$
$$- (p_2y+(1-x-y)(p_2+p_3))^2 + (p_2^2y+(1-x-y)(p_2+p_3)^2).$$

$$= [\underbrace{q+(p_2y+(1-x-y)(p_2+p_3))}_{\bar{q}}]^2 + W$$

~~where~~

$$= \bar{q}^2 + W(x,y)$$

where $\bar{q} = q + (p_2y + (1-x-y)(p_2+p_3))$ $\qquad d\bar{q} = dq$

$$W = m^2 - (p_2y+(1-x-y)(p_2+p_3))^2 + (p_2^2y+(1-x-y)(p_2+p_3)^2)$$

$$\Rightarrow I_1 = 2\int_0^1 dx \int_0^{1-x} dy \underbrace{\int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+W)^3}}_{J_1}$$

use schwinger parametrisation:

$$J_1 = \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+w)^3} = \int \frac{d^D q}{(2\pi)^D} \cdot \frac{1}{\Gamma(3)} \int_0^\infty dt \cdot t^2 e^{-(q^2+w)t}$$

$$= \frac{1}{\Gamma(3)} \int_0^\infty dt \cdot t^2 e^{-wt} \underbrace{\int \frac{d^D q}{(2\pi)^D} e^{-tq^2}}_{(4\pi t)^{-\frac{D}{2}}}$$

$$= \frac{1}{\Gamma(3)} \frac{1}{(4\pi)^{D/2}} \int_0^\infty dt\, t^{2-\frac{D}{2}} e^{-wt}$$

$u = wt$.
$t = \frac{1}{w} u$.
$dt = \frac{1}{w} du$.

$$= \frac{1}{\Gamma(3)} \frac{1}{(4\pi)^{D/2}} w^{\frac{D}{2}-3} \underbrace{\int_0^\infty du\, u^{(3-\frac{D}{2})-1} e^{-u}}_{\Gamma(3-\frac{D}{2})}$$

$$= \frac{\Gamma(3-\frac{D}{2})}{\Gamma(3)} \frac{w^{\frac{D}{2}-3}}{(4\pi)^{D/2}}$$

$\therefore$ Dimensional regularisation $\underline{D = 6-2\varepsilon}$ $(\varepsilon \to 0)$

$3 - \frac{D}{2} = 3 - \frac{1}{2}(6-2\varepsilon) = \varepsilon$, $\frac{D}{2} - 3 = -\varepsilon$, $\frac{D}{2} = 3-\varepsilon$

$$\therefore J_1 = \frac{\Gamma(\varepsilon)}{\Gamma(3)} \frac{w^{-\varepsilon}}{(4\pi)^{3-\varepsilon}} = \frac{1}{\underbrace{\Gamma(3)}_{=2!=2}(4\pi)^3} \Gamma(\varepsilon) \left(\frac{4\pi}{w}\right)^\varepsilon$$

$$= \frac{1}{2(4\pi)^3} \left(\frac{1}{\varepsilon} - \gamma + O(\varepsilon) + \cdots\right) \left(1 + \varepsilon \log\left(\frac{4\pi}{w}\right) + \cdots\right)$$

$$= \frac{1}{2(4\pi)^3} \left(\frac{1}{\varepsilon} + \text{finite}\right)$$ ok

$$I_1 = 2\int_0^1 dx \int_0^{1-x} dy\, J_1$$

$$= 2\int_0^1 dx \int_0^{1-x} dy \left(\frac{1}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon} + \text{finite.} \right)$$

$$= \frac{1}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon} + \text{finite.}$$

$$\Gamma_0^{(3)} = (-\lambda) + \frac{(-\lambda)^3}{1} I_1 - \delta\lambda$$

$$= -\lambda \cancel{+} - \frac{\lambda^3}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon} + \text{finite} - \delta\lambda$$

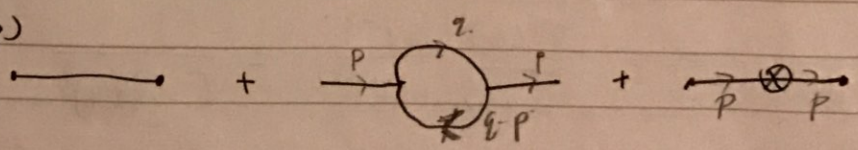
$$= -\delta\lambda - \frac{\lambda^3}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon} + \text{finite.}$$

Apply minimal subtraction, $\delta\lambda$ contains the divergent terms of this order but no more:

$$\underline{\delta\lambda = -\frac{\lambda^3}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon}}$$ (Note, independent of m)

The remaining δZ_ϕ and δm are fixed by $\Gamma_0^{(2)}$ renormalisation.

(1 loop)

$\Gamma_0^{(2)} =$ 

$$= -p^2 \delta Z_\phi - \delta m^2 + \frac{(-\lambda)^2}{2} \underbrace{\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+m^2} \frac{1}{(q-p)^2+m^2}}_{I_2}$$

amputated

$$I_2 = \int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2+m^2} \frac{1}{(q-p)^2+m^2}$$

$$= \int \frac{d^Dq}{(2\pi)^D} \int_0^1 dx \int_0^1 dy \frac{\delta(x+y-1)}{\left(x(q^2+m^2)+y((q-p)^2+m^2)\right)^2}.$$

Feynman parametrisation.

~~Γ(2)~~ $(2-1)! = 1! = 1$

$$= \cancel{\int_0^1 dx \int}$$

$$= \int_0^1 dx \int \frac{d^Dq}{(2\pi)^D} \frac{1}{\left(x(q^2+m^2)+(1-x)((q-p)^2+m^2)\right)^2}.$$

Denominator square-rooted is

$$x(q^2+m^2)+(1-x)((q-p)^2+m^2)$$

$$= m^2 + xq^2 + (1-x)(q-p)^2$$

$$= m^2 + xq^2 + (1-x)q^2 - 2(1-x)p\cdot q + (1-x)p^2$$

$$= (q^2 - 2(1-x)pq) + m^2 + (1-x)p^2.$$

$$= (q^2 - 2(1-x)pq + (1-x)^2p^2) + m^2 + (1-x)p^2 - (1-x)^2p^2$$

$$= (\underbrace{q-(1-x)p}_{\tilde{q}})^2 + \underbrace{m^2 + x(1-x)p^2}_{S}$$

$$= \tilde{q}^2 + S$$

$$\therefore I_2 = \int_0^1 dx \int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+S)^2} \underbrace{=}_{\text{shwinger } (\Gamma(2)=1)} \int_0^1 dx \int \frac{d^Dq}{(2\pi)^D} \int dt \cdot t \cdot e^{-t(q^2+S)}$$

$$= \int_0^1 dx \int dt \cdot t \cdot e^{-tS} \underbrace{\int \frac{d^D q}{(2\pi)^D} e^{-tq^2}}_{(4\pi t)^{-\frac{D}{2}}}$$

$$= \frac{1}{(4\pi)^{D/2}} \int_0^1 dx \int dt \cdot t^{1-\frac{D}{2}} e^{-tS}$$

$u = tS$

$t = \frac{u}{S}$

$dt = \frac{du}{S}$

$$= \frac{1}{(4\pi)^{D/2}} \int_0^1 dx\, S^{\frac{D}{2}-2} \int du\, u^{1-\frac{D}{2}} e^{-u}$$

$$= \frac{1}{(4\pi)^{D/2}} \int_0^1 dx\, S^{\frac{D}{2}-2} \int du \underbrace{u^{(2-\frac{D}{2})-1} e^{-u}}_{\Gamma(2-\frac{D}{2})}$$

$$= \frac{\Gamma(2-\frac{D}{2})}{(4\pi)^{D/2}} \int_0^1 dx\, S^{\frac{D}{2}-2}$$

$\because D = 6-2\varepsilon \Rightarrow 2-\frac{D}{2} = 2-(3-\varepsilon) = -1+\varepsilon$

$\frac{D}{2} = 3-\varepsilon \quad \frac{D}{2}-2 = 1-\varepsilon$

$$= \frac{\Gamma(-1+\varepsilon)}{(4\pi)^3} \int_0^1 dx \cdot \left(\frac{4\pi}{S}\right)^{\varepsilon} \cdot S$$

$$\therefore \Gamma_0^{(2)} = -p^2 \delta Z_\phi - \delta m^2 + \frac{\lambda^2}{2} I_2$$

$$= -p^2 \delta Z_\phi - \delta m^2 + \frac{\lambda^2}{2} \frac{\Gamma(-1+\varepsilon)}{(4\pi)^3} \int_0^1 dx\, S \left(\frac{4\pi}{S}\right)^{\varepsilon}$$

$$\because \Gamma(-1+\varepsilon) = (-1)\left(\frac{1}{\varepsilon} - \gamma + 1 + O(\varepsilon)\right)$$

$$= -p^2 \delta Z_\phi - \delta m^2 + \frac{\lambda^2}{2}\frac{1}{(4\pi)^3}\left[-\frac{1}{\varepsilon} + \gamma - 1 + O(\varepsilon)\right]\int_0^1 dx \cdot S \cdot \left(\frac{4\pi}{S}\right)^\varepsilon$$

$$= -p^2 \delta Z_\phi - \delta m^2 - \frac{\lambda^2}{2}\frac{1}{(4\pi)^3}\left[\frac{1}{\varepsilon} + 1 - \gamma\right]\int_0^1 dx\, S \cdot \left[1 + \varepsilon \log\left(\frac{4\pi}{S}\right)\right]$$

$$= -p^2 \delta Z_\phi - \delta m^2 - \frac{\lambda^2}{2}\frac{1}{(4\pi)^3}\left[\frac{1}{\varepsilon} + 1 - \gamma\right]\int_0^1 dx \cdot S \cdot \left[1 - \varepsilon \log\left(\frac{S}{4\pi}\right)\right]$$

$$= -p^2 \delta Z_\phi - \delta m^2 - \frac{\lambda^2}{2}\frac{1}{(4\pi)^3}\left[\frac{1}{\varepsilon} + 1 - \gamma\right]\int_0^1 dx\, (m^2 + (1-x)x p^2)\cancel{\left[1 - \varepsilon \log\left(\frac{S}{4\pi}\right)\right]}$$

$$\times \left[1 - \varepsilon \log\left(\frac{m^2 + (1-x)x p^2}{4\pi}\right)\right]$$

$$= -p^2 \delta Z_\phi - \delta m^2 - \frac{\lambda^2}{2}\frac{1}{(4\pi)^3}\frac{1}{\varepsilon}\int_0^1 dx\, (m^2 + x(1-x)p^2) + \text{finite}$$

$$= -p^2 \delta Z_\phi - \delta m^2 - \frac{\lambda^2}{2(4\pi)^3}\frac{1}{\varepsilon} m^2 - \frac{\lambda^2}{12}\frac{1}{(4\pi)^3}\frac{1}{\varepsilon} p^2 + \text{finite}$$

$\Rightarrow$ By minimal subtraction,

$$\delta Z_\phi = -\frac{\lambda^2}{12(4\pi)^3}\frac{1}{\varepsilon}, \quad \delta m^2 = -\frac{\lambda^2}{2(4\pi)^3}\frac{1}{\varepsilon} m^2$$

Now ~~use~~ we remind ourselves that we are doing a mass-less theory.

take $m \to 0$, $\delta m^2 \to 0$, $m \to 0$. ~~$m_0 \to 0$~~

~~$m_0^2 : Z m_0^2 = m^2 + \delta m^2$~~

we do not have infrared divergence due to $m \to 0$ in this case, because

when $m=0$,

$$W = -(P_2 y + (1-x-y)(P_2+P_3))^2 + (P_3^2 y + (1-x-y)(P_2+P_3)^2)$$

$$S = m^2 + x(1-x)p^2$$

~~are finite.~~

so $\int_0^1 dx \int_0^{1-x} dy \; W^{\varepsilon}(x,y)$ is finite $\because \varepsilon \ll 1$

~~when $m=0$~~ $\because \varepsilon \ll 1$

and $\int_0^1 dx \; S \log(S)$ is finite

when $m=0$. $\because$ $S\log(S)$ finite as $S \to 0$.

$$\therefore \; Z = 1 + \delta Z_\phi = 1 - \frac{\lambda^2}{12(4\pi)^3}\frac{1}{\varepsilon}$$

and $Z^{3/2}\lambda_0 = \lambda + \delta\lambda$

$$\therefore \left(1 - \frac{\lambda^2}{12(4\pi)^3}\frac{1}{\varepsilon}\right)^{3/2} \lambda_0 = \lambda - \frac{\lambda^3}{2(4\pi)^3}\frac{1}{\varepsilon}$$

$$\Rightarrow \lambda_0 = \left(\lambda - \frac{\lambda^3}{2(4\pi)^3}\frac{1}{\varepsilon}\right)\left(1 - \frac{\lambda}{12(4\pi)^3}\frac{1}{\varepsilon}\right)^{-3/2}$$

$$\simeq \left(\lambda - \frac{\lambda^3}{2(4\pi)^3}\frac{1}{\varepsilon}\right)\left(1 + \frac{\lambda^2}{8(4\pi)^3}\frac{1}{\varepsilon} + \cdots\right)$$

In fact we need to fix the dimensions of all the coupling constants.

$\because [\lambda] = 3 - \frac{D}{2}$ and $D = 6 - 2\varepsilon$

$\therefore [\lambda] = \varepsilon$

$\delta\lambda = -\frac{\lambda^3}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon}$ has dimensional problem

$\because [\delta\lambda] = [\lambda] \neq [\lambda^3]$

But we can introduce $g_0 = \lambda_0 \mu^{-\varepsilon}$ and ~~$g = \lambda_0 \mu^{-\varepsilon}$~~
$g = \lambda \mu^{-\varepsilon}$

such that $[g] = 0$, $[g_0] = 0$ ok

$$\therefore \delta\lambda = -\frac{\lambda}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon} g^2 \mu^{+2\varepsilon}$$

$$= -\frac{g^3 \mu^{\varepsilon}}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon} \left(1 + 2\varepsilon \log(\mu) + \cdots\right)$$

$$= -\frac{g^3 \mu^{\varepsilon}}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon} \left(1 + \varepsilon \log(\mu^2) + \cdots\right)$$

this $+ \varepsilon \log(\mu^2)$ combine with $\varepsilon \log\left(\frac{1}{W}\right)$
$\underset{m^2}{\wr}$

gives a log that has dimensionless ~~argu.~~ argument $\rightarrow$ which is good.

$$\therefore \delta\lambda = -\frac{g^3 \mu^{\varepsilon}}{2} \frac{1}{(4\pi)^3} \frac{1}{\varepsilon}$$

Similarly $\delta Z_\phi = -\frac{g^2}{12(4\pi)^3}\frac{1}{\varepsilon}$ $\quad \because [\delta Z_\phi] = 1$

$$\therefore \; Z = 1 + \delta Z_\phi = 1 + \frac{g^2}{12(4\pi)^3}\frac{1}{\varepsilon}$$

$$\therefore \left(1 - \frac{g^2}{12(4\pi)^3}\right)^{3/2} \lambda_0 = \lambda - \frac{\lambda g^2}{2(4\pi)^3}\frac{1}{\varepsilon}$$

$$\therefore \; g_0 = \left(g - \frac{g^3}{2(4\pi)^3}\frac{1}{\varepsilon}\right)\left(1 - \frac{g^2}{12(4\pi)^3}\right)^{-3/2}$$

$$= \left(g - \frac{g^3}{2(4\pi)^3}\frac{1}{\varepsilon}\right)\left(1 + \frac{g^2}{8(4\pi)^3} + \dots\right)$$

$$= g - \frac{g^3}{2(4\pi)^3}\frac{1}{\varepsilon} + \frac{g^3}{8(4\pi)^3}\frac{1}{\varepsilon} + O(g^4)$$

$$= g - \frac{3g^3}{8(4\pi)^3}\frac{1}{\varepsilon} + O(g^4)$$

We want to invert the above perturbation

try $g = g_0 + a g_0^2 + b g_0^3 + \dots$

$$\Rightarrow g_0 = g_0 + a g_0^2 + \left(b - \frac{3}{8(4\pi)^3 \varepsilon}\right) g_0^3 + \dots$$

Equating orders:

$O(g_0) \Rightarrow \quad g_0 = g_0$

$O(g_0^2) \Rightarrow \quad a g_0^2 = 0$

$O(g_0^3) \Rightarrow b - \frac{3}{8(4\pi)^3 \varepsilon} = 0 \Rightarrow b = \frac{+3}{8(4\pi)^3 \varepsilon}$

$\therefore g = g_0 + \frac{3}{8(4\pi)^3 \varepsilon} g_0^3 + O(g_0^4)$

$\therefore \cancel{\lambda = \lambda_0 + \frac{3}{8(4\pi)^3 \varepsilon} \lambda_0^3}$

$\therefore g = \lambda_0 \mu^{-\varepsilon} + \frac{3}{8(4\pi)^3 \varepsilon} \lambda_0^3 \mu^{-3\varepsilon} \quad (*)$

$\therefore \lambda = \lambda_0 + \frac{3}{8(4\pi)^3 \varepsilon} \lambda_0^3 \mu^{-2\varepsilon}$ ✓ is the

Renormalised coupling constant.

d) Beta function at 1-loop:

$\beta(g) = \mu \frac{\partial g}{\partial \mu}\Big|_{\lambda_0}$, use $(*)$

$\beta = \mu \frac{\partial}{\partial \mu} \left(\lambda_0 \mu^{-\varepsilon} + \frac{3 \lambda_0^3}{8(4\pi)^3 \varepsilon} \mu^{-3\varepsilon} \right)$ ✓

$= -\varepsilon \lambda_0 \mu^{-\varepsilon} - \frac{9 \lambda_0^3}{8(4\pi)^3 \varepsilon} \mu^{-3\varepsilon} \times \varepsilon$

$\xrightarrow{\varepsilon \to 0}$ $= \cancel{-\varepsilon \lambda_0} \quad \cancel{-\frac{9\lambda_0^3 \mu^{-3\varepsilon}}{8(4\pi)^3}} \quad - \frac{9 \overbrace{\lambda_0^3 \mu^{-3\varepsilon}}^{g_0^3}}{8(4\pi)^3}$ ✓

good!

(12) In Euclidean space.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1^0 \partial^\mu \phi_1^0 + \partial_\mu \phi_2^0 \partial^\mu \phi_2^0) + \frac{\lambda^0}{4!}(\phi_1^{0\,4} + \phi_2^{0\,4}) + \frac{2\chi_0}{4!}(\phi_1^{0\,2}\phi_2^{0\,2})$$

We are interest in the corrections to the propagator and the vertex. We use minimal subtraction because ~~there~~ the theory is massless.

~~The propagator~~

Renormalisation! $\cancel{\phi_1^0 = (1+\delta Z\phi}$

$$\phi_1^0 = \underbrace{(1+\delta Z_1)}_{Z_1}^{\frac{1}{2}} \phi_1 \qquad \phi_2^0 = \underbrace{(1+\delta Z_2)}_{Z_2}^{\frac{1}{2}} \phi_2$$

$$Z_1 = Z_2 = Z, \qquad m_0 = m = 0 \qquad Z^2 m_0^2 = m^2 + \delta m^2$$

$$\delta Z_1 = \delta Z_2 = \delta Z$$

$$Z^2 \lambda_0 = \lambda + \delta\lambda \qquad Z^2 \chi_0 = \chi + \delta\chi$$

then

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2) + \frac{\lambda}{4!}(\phi_1^4 + \phi_2^4)$$

$$+ \frac{2\chi}{4!}(\phi_1^2 \phi_2^2) + \frac{1}{2}\delta Z(\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2)$$

$$+ \frac{\delta\lambda}{4!}(\phi_1^4 + \phi_2^4) + \frac{2\delta\chi}{4!}(\phi_1^2 \phi_2^2).$$

~~To one loop, the propagators and their~~

~~corrections are.~~

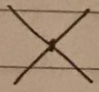
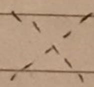
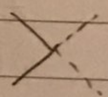
~~Same~~ Similar argument as (11) gives

Critical dimension of the theory to be 4 and we need to renormalise $\Gamma_0^{(2)}$ and $\Gamma_0^{(4)}$.

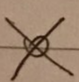
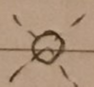
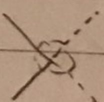
In particular, renormalising $\Gamma_0^{(2)}$ gives us δZ and δm^2, and renormalising $\Gamma_0^{(4)}$ gives $\delta\lambda$ and $\delta\chi$.

Now consider $\Gamma_0^{(2)}$: which gives corrections to propagators. ~~we have~~.

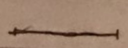
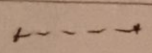
The vertices are:

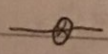
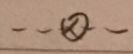
 $-\lambda$  $-\lambda$  $-\frac{\chi}{3}$

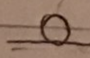
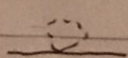
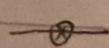
vertices ~~cont~~ counterterms:

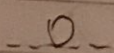
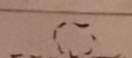
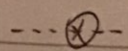
 $-\delta\lambda$  $-\delta\lambda$  $-\frac{\delta\chi}{3}$

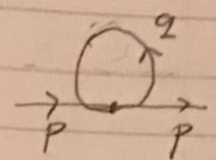
propagators:

 $\frac{1}{p^2}$  $\frac{1}{p^2}$

 $-p^2\delta Z - \delta m^2$  $-p^2\delta Z - \delta m^2$.

$\Gamma_0^{(2)} =$  +  +  for ϕ_1

and  +  +  for ϕ_2

this kind of diagrams like 

gives loop contribution that is independent of p. so there is no term to be cancelled by $-p^2\delta Z$ $\Rightarrow \delta Z = 0 \Rightarrow \underline{Z = 1}$

Also $\because$ $$\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+m^2} = \frac{\pi^{D/2}}{(2\pi)^D} \Gamma\left(1-\frac{D}{2}\right) m^{D-2}$$

$$\propto m^{D-2}$$

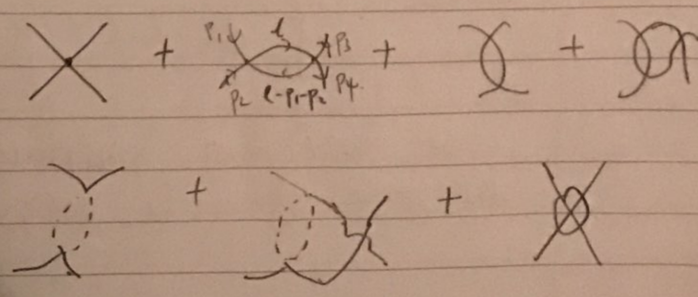
and the infinity arises will be proportional to m^{D-2}. and $\because \delta Z = 0$, δm^2 should be responsible for cancelling all infinities. $\Rightarrow \delta m^2 \propto m^{D-2}$. But in our <u>massless theory</u>, $m=0$ $\therefore \delta m^2 = 0$.

$\Rightarrow$ $\underline{\delta Z = 0, \quad \delta m^2 = 0, \quad Z = 1}$

Trivial propagator correction.

Now consider the vertex correction ($\Gamma_0^{(4)}$):

Case is the same for ϕ_1 and ϕ_2, so let's take ϕ_1:

$\Gamma_0^{(4)}(1\text{ loop}, \phi_1) =$ 

permutation of similar ~~same~~ graphs.

$$= -\lambda + \frac{3(-\lambda)^2}{2}\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2}$$

$$+\frac{3(-\frac{\kappa}{3})^2}{2}\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2}$$

$$-\delta\lambda.$$

$$= -\lambda + \frac{3}{2}\left(\lambda^2+\frac{\kappa^2}{9}\right)\underbrace{\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2}}_{I_1} - \delta\lambda$$

Dimensional regularisation $D = 4-2\varepsilon$

using result from Problem set 3 . Q9 ,

and $\lambda = g_\lambda \mu^{2\varepsilon}$ $\kappa = g_\kappa \mu^{2\varepsilon}$

$$I_1 = \frac{3\mu^{2\varepsilon}}{32\pi^2}\left(g_\lambda^2+\frac{g_\kappa^2}{9}\right)\left[\frac{1}{\varepsilon}-\gamma-\int_0^1 dx \ln\left(\frac{(p_1+p_2)^2 x(1-x)}{4\pi\mu^2}\right)\right]$$

(m is taken to be 0)

No extra divergence coming from $\int dx$ integral

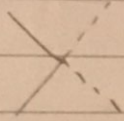
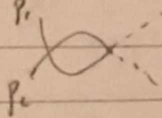
because p_1 & p_2 are finite external momenta and

$$\int_0^1 dx \ln(x(1-x)) = -2 \text{ is finite.}$$

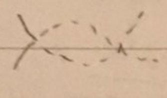
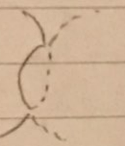
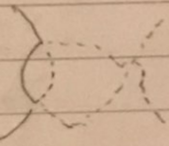
Use minimal subtraction to cancel the divergence of I_1 using $\delta\lambda$, this gives.

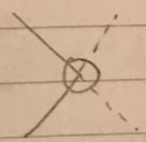
$$\delta\lambda = \cancel{\frac{3}{32}\mu^{2\varepsilon}} \; \frac{3\mu^{2\varepsilon}}{32\pi^2\varepsilon}\left(g_\lambda{}^2 + \frac{g_\kappa^2}{9}\right) = \frac{\mu^{2\varepsilon}}{32\pi^2}\left[3g_\lambda^2 + \frac{g_\kappa^2}{3}\right]$$

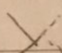
Now consider the correction to κ vertices.

$\Gamma_0^{(4)}(1\text{ loop}, \phi_1\phi_2) =$  + 

p_1, p_2

+  +  + 

+ 

[ gives $\frac{\kappa}{3}$ because it has $\frac{2!\cdot 2!\cdot 2}{4!} = \frac{2}{6} = \frac{1}{3}$]

$$= -\frac{\kappa}{3} - \frac{\delta\kappa}{3} + \frac{2(-\lambda)(-\frac{\kappa}{3})}{2}\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2}\Bigg|_{m=0}$$

$$+ \frac{2(-\frac{\kappa}{3})^2}{1}\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2}\Bigg|_{m=0}$$

$$= -\frac{\kappa}{3} - \frac{\delta\kappa}{3} + \frac{\left[2(\frac{\lambda\kappa}{3}) + 4\frac{\kappa^2}{9}\right]}{2}\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2}\Bigg|_{m=0}$$

$$= -\frac{\kappa}{3} - \frac{\delta\kappa}{3} + \frac{\mu^{4\varepsilon}}{2}\left(\frac{2}{3}g_\lambda g_\kappa + \frac{4}{9}g_\kappa^2\right)\int \frac{d^D l}{(2\pi)^D}\frac{1}{l^2+m^2}\frac{1}{(l-p_1-p_2)^2+m^2}\Bigg|_{m=0}$$

Same integral.

$(D = 4-2\varepsilon)$

$$= -\frac{\kappa}{3} - \frac{\delta\kappa}{3} + \cancel{\mu^{4\varepsilon}}\;\frac{\mu^{2\varepsilon}}{32\pi^2}\left[\frac{2}{3}g_\lambda g_\kappa + \frac{4}{9}g_\kappa^2\right]\left[\frac{1}{\varepsilon} - \gamma - \int dx \log\left(\frac{(p_1+p_2)^2 x(1-x)}{4\pi\mu^2}\right)\right]$$

minimal subtraction

$$\frac{\delta x}{3} = \frac{\mu^{2\varepsilon}}{32\pi^2\varepsilon}\left[\frac{2}{3} g_\lambda g_x + \frac{4}{9} g_x^2\right]$$

$$\therefore \delta x = \frac{\mu^{2\varepsilon}}{32\pi^2\varepsilon}\left[2 g_\lambda g_x + \frac{4 g_x^2}{3}\right]$$

$$\lambda_0 = \lambda + \delta\lambda = \lambda + \frac{\mu^{2\varepsilon}}{32\pi^2\varepsilon}\left[3 g_\lambda^2 + \frac{g_x^2}{3}\right]$$

$$x_0 = x + \delta x = x + \frac{\mu^{2\varepsilon}}{32\pi^2\varepsilon}\left[2 g_\lambda g_x + \frac{4 g_x^2}{3.}\right]$$

$$\lambda_0 = g_\lambda \mu^{2\varepsilon} + \frac{3\mu^{2\varepsilon}}{32\pi^2\varepsilon} g_\lambda^2 + \frac{\mu^{2\varepsilon}}{3(32\pi^2)\varepsilon} g_x^2$$

$$x_0 = g_x \mu^{2\varepsilon} + \frac{2\mu^{2\varepsilon}}{32\pi^2\varepsilon} g_\lambda g_x + \frac{4\mu^{2\varepsilon}}{3(32\pi^2)\varepsilon} g_x^2$$

$$\therefore g_{\lambda 0} = g_\lambda + \frac{3}{32\pi^2\varepsilon} g_\lambda^2 + \frac{1}{3(32\pi^2)\varepsilon} g_x^2$$

$$g_{x0} = g_x + \frac{2}{32\pi^2\varepsilon} g_\lambda g_x + \frac{4}{3(32\pi^2)\varepsilon} g_x^2$$

try series expansion

$$\begin{cases} g_\lambda = g_{\lambda 0} + a g_{\lambda 0}^2 + b g_{\lambda 0} g_{x0} + c g_{x0}^2 + \dots \\ g_x = g_{x0} + \alpha g_{x0}^2 + \beta g_{\lambda 0} g_{x0} + \gamma g_{\lambda 0}^2 + \dots \end{cases}$$

$$\rightarrow g_{\lambda 0} = g_{\lambda 0} + a g_{\lambda 0}^2 + b g_{\lambda 0} g_{x0} + c g_{x0}^2$$

$$+ \frac{3}{32\pi^2\varepsilon}\left(g_{\lambda 0}^2\right) + \frac{1}{3(32\pi^2)\varepsilon}\left(g_{x0}^2\right)$$

$\Rightarrow$

$\Rightarrow \quad b=0$

$$a = -\frac{3}{32\pi^2\varepsilon} \qquad c = -\frac{1}{3(32\pi^2)\varepsilon}$$

$\cancel{g_{\lambda 0}}$

$$\therefore g_\lambda = g_{\lambda_0} - \frac{3 g_{\lambda_0}^2}{32\pi^2\varepsilon} - \frac{1}{3(32\pi^2)\varepsilon} g_{\kappa_0}^2 .$$

$$\rightarrow \quad g_{\kappa_0} = g_{\kappa_0} + \alpha g_{\kappa_0}^2 + \beta g_{\lambda_0} g_{\kappa_0} + \gamma g_{\lambda_0}^2$$
$$+ \frac{2}{32\pi^2\varepsilon} g_{\lambda_0} g_{\kappa_0} + \frac{4}{3(32\pi^2)\varepsilon} g_{\kappa_0}^2$$

$$\therefore \quad \alpha = -\frac{4}{3(32\pi^2)\varepsilon} \qquad \beta = -\frac{2}{32\pi^2\varepsilon} \qquad \gamma = 0$$

$\therefore \cancel{g_{\kappa_0} = Zg}$

$$g_\kappa = g_{\kappa_0} - \frac{2}{32\pi^2\varepsilon} g_{\lambda_0} g_{\kappa_0} - \frac{4}{3(32\pi^2)\varepsilon} g_{\kappa_0}^2$$

$$\therefore g_\lambda = \lambda_0 \mu^{-2\varepsilon} - \frac{3}{32\pi^2\varepsilon} \lambda_0^2 \mu^{-4\varepsilon} - \frac{1}{3(32\pi^2)\varepsilon} \kappa_0^2 \mu^{-4\varepsilon}$$

$$g_\kappa = \kappa_0 \mu^{-2\varepsilon} - \frac{2}{32\pi^2\varepsilon} \lambda_0 \kappa_0 \mu^{-4\varepsilon} - \frac{4}{3(32\pi^2)\varepsilon} \kappa_0^2 \mu^{-4\varepsilon}$$

$$\boxed{\beta_\lambda = \mu \frac{\partial g_\lambda}{\partial \mu} = -2\varepsilon \lambda_0 \mu^{-2\varepsilon} + \frac{12\lambda_0^2}{32\pi^2} \mu^{-4\varepsilon} + \frac{4\kappa_0^2}{3(32\pi^2)} \mu^{-4\varepsilon}}$$

$$\boxed{\beta_\kappa = \mu \frac{\partial g_\kappa}{\partial \mu} = -2\varepsilon \kappa_0 \mu^{-2\varepsilon} + \frac{8\lambda_0\kappa_0}{32\pi^2} \mu^{-4\varepsilon} + \frac{16\kappa_0^2}{3(32\pi^2)} \mu^{-4\varepsilon}}$$

a) when $D = 4 \rightarrow \varepsilon = 0$

$$\beta_\lambda = \frac{1}{32\pi^2}\left(12\lambda_0^2 + \frac{4}{3}\kappa_0^2\right) = \frac{1}{16\pi^2}\left(6\lambda_0^2 + \frac{2}{3}\kappa_0^2\right).$$

$$= \underline{\frac{1}{8\pi^2}\left(3\lambda_0^2 + \frac{1}{3}\kappa_0^2\right)}$$

$$\underline{\beta_\kappa = \frac{1}{8\pi^2}\left(2\lambda_0\kappa_0 + \frac{4}{3}\kappa_0^2\right).}$$

~~b) with ε,~~

$$\cancel{\beta_\lambda = \frac{1}{8\pi^2}\left(3\lambda_0^2 + \frac{1}{3}\kappa_0^2\right) - 2\varepsilon\lambda_0}$$

$$\cancel{\beta_\kappa = \frac{1}{8\pi^2}\left(2\lambda_0\kappa_0 + \frac{4}{3}\kappa_0^2\right) - 2\varepsilon\kappa_0}$$

b) $$\mu\frac{\partial}{\partial\mu}(v) = \mu\frac{\partial}{\partial\mu}\left(\frac{\kappa}{\lambda}\right) = \mu\frac{\partial}{\partial\mu}\left(\frac{g_\kappa\mu^{2\varepsilon}}{g_\lambda\mu^{2\varepsilon}}\right)$$

$$= \mu\frac{\partial}{\partial\mu}\left(\frac{g_\kappa}{g_\lambda}\right) = \frac{1}{g_\lambda}\underbrace{\left(\mu\frac{\partial}{\partial\mu}g_\kappa\right)}_{=\beta_\kappa}$$

$$-\frac{g_\kappa}{g_\lambda^2}\underbrace{\left(\mu\frac{\partial}{\partial\mu}g_\lambda\right)}_{\beta_\lambda}.$$

$$= \frac{\beta_\kappa}{g_\lambda} - \frac{g_\kappa}{g_\lambda^2}\beta_\lambda = \frac{\beta_\kappa}{g_\lambda} - \frac{v\beta_\lambda}{g_\lambda}.$$

$\lambda = g_\lambda$, $\kappa = g_\kappa$ for $\varepsilon = 0$.

$$\beta_\kappa = \frac{1}{8\pi^2}\left(2\lambda\kappa + \frac{4}{3}\kappa^2\right) = \frac{1}{8\pi^2}\left(2g_\lambda g_\kappa + \frac{4}{3}g_\kappa^2\right)$$

$$\beta_\lambda = \frac{1}{8\pi^2}\left(3\lambda^2 + \frac{1}{3}\kappa^2\right) = \frac{1}{8\pi^2}\left(3g_\lambda^2 + \frac{1}{3}g_\kappa^2\right).$$

$$\cancel{\mu\frac{\partial}{\partial\mu}(\nu) = \frac{1}{g_\lambda 8\pi^2}\left(2\lambda\kappa + \frac{4}{3}\kappa^2 - \nu\left(3\lambda^2 + \frac{1}{3}\kappa^2\right)\right)}$$

$$\cancel{= \frac{\kappa^2}{g_\lambda 8\pi^2}\left(\ \right)^2}$$

$$\mu\frac{\partial}{\partial\mu}(\nu) = \frac{1}{8\pi^2}\left(2g_\kappa + \frac{4}{3}g_\kappa\nu - 3\nu \underbrace{g_\lambda}_{g_\kappa} - \frac{1}{3}\nu^2 g_\kappa\right)$$

$$= \frac{g_\kappa}{8\pi^2}\left(2 + \frac{4}{3}\nu - 3 - \frac{1}{3}\nu^2\right).$$

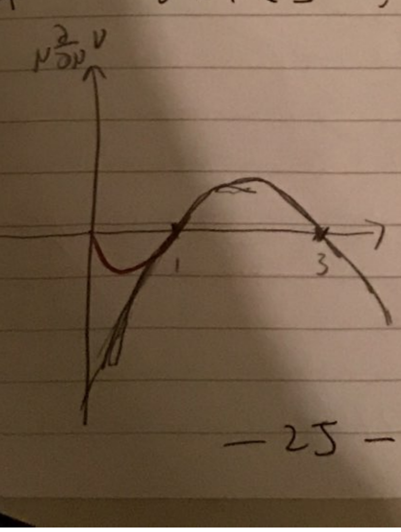
$$= \frac{-g_\kappa}{24\pi^2}\left(\nu^2 - 4\nu + 3\right)$$

$$= -\frac{g_\kappa}{24\pi^2}(\nu-3)(\nu-1)$$

$= \nu g_\lambda$, So also $\nu = 0$ is a zero.

If $1 < \nu < 3$, $\frac{\partial\nu}{\partial\mu} > 0$

If $\nu < 1 < 3$, $\frac{\partial\nu}{\partial\mu} < 0$



at large distance, μ becomes small

$\Rightarrow$ $1 < \nu < 3 \to \nu$ $\mu \to$ small

$\nu < 1 < 3 \to \nu \to$ large.

$\Rightarrow$ ν approaches $\underline{1}$ ✓

c) For $\varepsilon \neq 0$.

$$\cancel{\beta_\lambda = \frac{1}{8\pi^2}(3\lambda_0^2 + \frac{1}{3}\kappa_0^2}$$

$$\beta_\lambda = \frac{1}{8\pi^2}(3\lambda^2 + \frac{\kappa^2}{3}) - 2\varepsilon\lambda.$$

$$\beta_\kappa = \frac{1}{8\pi^2}(2\lambda\kappa + \frac{4}{3}\kappa^2) - 2\varepsilon\kappa.$$

there are 4 fixed points:

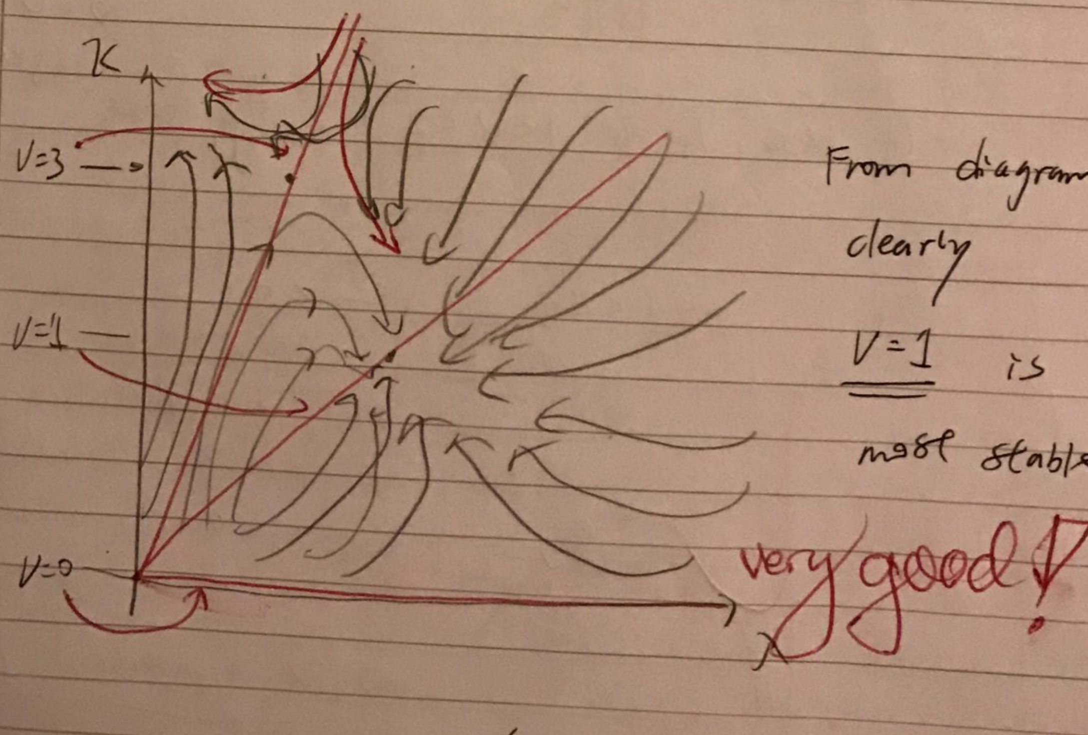
$\lambda = \kappa = 0$ (trivial).

$\lambda = \frac{16\pi^2}{3}\varepsilon, \ \kappa = 0 \Rightarrow V = 0$

$\lambda = \frac{24\pi^2}{5}\varepsilon = \kappa \Rightarrow V = 1$

$\lambda = \frac{8\pi^2}{3}\varepsilon, \ \kappa = 8\pi^2\varepsilon \Rightarrow V = 3$

$\Rightarrow V = 0, 1, 3 \rightarrow$ non-trivial fixed points.



From diagram clearly $\underline{\underline{V=1}}$ is most stable.

13

Callan - Symanzik equation for generic N-point vertex function.

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-\frac{N}{2}\gamma(g)\right)\Gamma^{(N)}(p_k,g,\mu)=0$$

$$\beta(g)=\mu\frac{\partial}{\partial\mu}(g(\mu))\Big|_{\lambda_0}\qquad \gamma(g)=\mu\frac{\partial\log(Z_\phi)}{\partial\mu}\Big|_{\lambda_0}$$

a) For bare vertex function.

$$\Gamma_0^{(N)}=\Gamma^{(N)}(p_k,g_0=\lambda_0\mu^{-\varepsilon},\mu).=\Gamma_0^{(N)}(p_k,\lambda_0)$$

~~$0=\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-\frac{N}{2}\gamma(g)\right)\Gamma^{(N)}(p_k,g_0=\lambda_0\mu^{-\varepsilon},\mu).$~~

$$\beta_0(g_0)=\mu\frac{\partial}{\partial\mu}g_0\Big|_{\lambda_0}=\mu\frac{\partial}{\partial\mu}(\lambda_0\mu^{-\varepsilon})$$

$$=-\varepsilon\lambda_0\mu^{-\varepsilon}=-\varepsilon g_0 \quad ✓$$

Also $\because\ \Gamma^{(N)}(p_k,g_0,\mu)=\Gamma_0^{(N)}(p_k,\lambda_0)$

and $\Gamma^{(N)}(p_k,\underbrace{g(\lambda_0,\mu)}_{g_0},\mu)=Z_\phi(\lambda_0,\mu)^{N/2}\Gamma_0^{(N)}(p_k,\lambda_0)$

$\Rightarrow$ ~~$Z_\phi=$~~ $Z_\phi(\lambda_0,\mu)^{N/2}=1\Rightarrow Z_\phi=1$

$\therefore\ \gamma_0(g_0)=\mu\frac{\partial}{\partial\mu}(\log Z_\phi)=0$ ✓

b).

$$\beta(g(g_0)) = \cancel{\beta_0(g_0)}\ \mu\frac{\partial}{\partial\mu}(g(g_0))\Big|_{\lambda_0}$$

$$= \cancel{\mu\frac{\partial}{\partial g_0}}\ \mu\frac{\partial g}{\partial g_0}\frac{\partial g_0}{\partial\mu} = \frac{\partial g}{\partial g_0}\underbrace{\mu\frac{\partial g_0}{\partial\mu}}_{\beta_0(g_0)} =$$

$$= \beta_0(g_0)\frac{\partial g}{\partial g_0}$$

$$\gamma(g(g_0)) = \mu\frac{\partial\log Z_\phi}{\partial\mu}\Big|_{\lambda_0.}$$

$$= \cancel{\phi}\ \frac{1}{Z_\phi}\mu\frac{\partial Z_\phi}{\partial\mu} = \frac{1}{Z_\phi}\mu\frac{\partial Z_\phi(g_0)}{\partial\mu} =$$

$$= \frac{1}{Z_\phi}\mu\frac{\partial Z_\phi}{\partial g_0}\frac{\partial g_0}{\partial\mu} = \frac{1}{Z_\phi}\frac{\partial Z_\phi}{\partial g_0}\underbrace{\mu\frac{\partial g_0}{\partial\mu}}_{\beta_0(g_0)} =$$

$$= \frac{\beta_0}{Z_\phi}\frac{\partial Z_\phi}{\partial g_0} = 0 + \frac{\beta_0}{Z_\phi}\frac{\partial Z_\phi}{\partial g_0}$$

$$= \gamma_0(g_0) + \frac{\beta_0(g_0)}{Z_\phi(g_0)}\frac{\partial Z_\phi}{\partial g_0}$$

$$\underbrace{=}_{\gamma_0(g_0)=0}$$

c) $$g(g_0) = g_0 + g_0^2\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon + \dots\right)$$

$$Z_\phi(g_0) = 1 + g_0\left(\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon + \dots\right)$$

$$\beta(g(g_0)) = \beta_0(g_0)\frac{\partial g}{\partial g_0}$$

$$= (-\varepsilon g_0)\frac{\partial}{\partial g_0}\left(g_0 + g_0^2\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon + \cdots\right)\right)$$

$$= (-\varepsilon g_0)\left(1 + 2g_0\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon + \cdots\right)\right)$$

$$= -\varepsilon g_0 - 2g_0^2\left(a_1 + a_2\varepsilon + a_3\varepsilon^2 + \cdots\right)$$

~~$Z_\phi(g_0,\varepsilon)$~~

$$\gamma(g) = \frac{\beta_0(g_0)}{Z_\phi(g_0)}\frac{\partial Z_\phi}{\partial g_0}$$

$$= (-\varepsilon g_0)\left(1 + g_0\left(\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon\right)\right)^{-1} \times \left[\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon\right]$$

$$\cancel{= (-\varepsilon g_0)\left(\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon\right)\left[1 - g_0\left(\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon\right) + \cdots\right]}$$

$$= (-\varepsilon g_0)\frac{\left[\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon\right]}{\left(\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon + \cdots\right)\left[g_0 + \frac{1}{\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon + \cdots}\right]}$$

$$= \frac{-\varepsilon g_0}{g_0 + \frac{1}{\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon + \cdots}}$$

Finish expansion in small g_0 before you let $\varepsilon \to 0$.

clearly β and γ are finite as $\varepsilon \to 0$

d) $\beta(g) = \beta_0(g_0)\frac{\partial g}{\partial g_0} = (-\varepsilon g_0)\left(1 + \frac{2a_1}{\varepsilon}g_0 + \frac{3b_1 g_0^2}{\varepsilon^2} + \dots\right)$

$$= -\varepsilon g_0 - 2a_1 g_0^2 - \frac{1}{\varepsilon}3b_1 g_0^3$$

Now we invert the perturbation.

$$g = g_0 + g_0^2\frac{a_1}{\varepsilon} + g_0^3\frac{b_1}{\varepsilon^2}$$

let $\quad g_0 = g + \frac{m_1}{\varepsilon}g^2 + \frac{m_2}{\varepsilon^2}g^3 + O(g^4)$

$$\therefore g = \left(g + \frac{m_1}{\varepsilon}g^2 + \frac{m_2}{\varepsilon^2}g^3\right)$$

$$+ \frac{a_1}{\varepsilon}\left(g + \frac{m_1}{\varepsilon}g^2 + \frac{m_2}{\varepsilon^2}g^3\right)^2$$

$$+ \frac{b_1}{\varepsilon^2}\left(g + \frac{m_1}{\varepsilon}g^2 + \frac{m_2}{\varepsilon^2}g^3\right)^3$$

$$= g + \frac{m_1}{\varepsilon}g^2 + \frac{m_2}{\varepsilon^2}g^3 + \frac{a_1}{\varepsilon}g^2 + \frac{2a_1 m_1}{\varepsilon^2}g^3$$

$$+ \frac{b_1 g^3}{\varepsilon^2}$$

$\Rightarrow \quad m_1 + a_1 = 0 \quad \rightarrow \quad m_1 = -a_1$

$m_2 + 2a_1 m_1 + b_1 = 0 \rightarrow \quad m_2 - 2a_1^2 + b_1 = 0$

$\qquad m_2 = 2a_1^2 - b_1$

$$g_0 = g - \frac{a_1}{\varepsilon}g^2 + \frac{2a_1^2 - b_1}{\varepsilon^2}g^3$$

$$\beta(g) = -\varepsilon g_0 - 2a_1 g_0^2 - \frac{1}{\varepsilon} 3b_1 g_0^3$$

$$= -\varepsilon\left(g - \frac{a_1}{\varepsilon} g^2 + \frac{2a_1^2 - b_1}{\varepsilon^2} g^3\right)$$

$$- 2a_1 \left(g - \frac{a_1}{\varepsilon} g^2 + \frac{2a_1^2 - b_1}{\varepsilon^2} g^3\right)^2$$

$$- \frac{1}{\varepsilon} 3b_1 \left(g - \frac{a_1}{\varepsilon} g^2 + \frac{2a_1^2 - b_1}{\varepsilon^2} g^3\right)^3$$

$$= -\varepsilon g + a_1 g^2 - \frac{2a_1^2 - b_1}{\varepsilon^2} g^3$$

$$- 2a_1 g^2 + \frac{4a_1^2}{\varepsilon} g^3 - \frac{3b_1 g^3}{\varepsilon} + \cdots$$

$$= -\varepsilon g + a_1 g^2 - 2a_1 g^2 + \frac{1}{\varepsilon}(-2a_1^2 + b_1 + 4a_1^2 - 3b_1) g^3$$

$$= \underset{\substack{\downarrow \\ \text{finite}}}{-\varepsilon g} - \underset{\substack{\downarrow \\ \text{finite}}}{a_1 g^2} + \frac{1}{\varepsilon}(2a_1^2 - 2b_1) g^3$$ good

for $\beta(g)$ to be finite at $\varepsilon \to 0$.

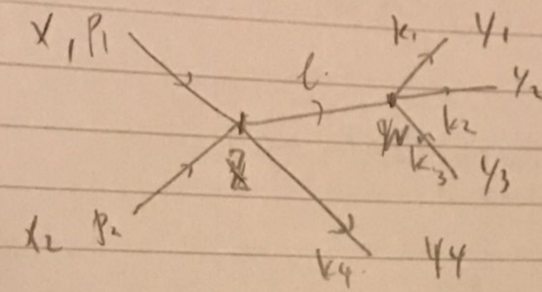
need $2a_1^2 = 2b_1 \Rightarrow b_1 = a_1^2$ ✓

$\Rightarrow$ b_1 depends on a_1

$\Rightarrow$ two loop coefficient depends on the lower order ones.

(14)

a)



the external propagators gives.

$$\phi(x)\, a^{\dagger}_{\vec{p}_1} |0\rangle = e^{ip_1 \cdot x} |0\rangle.$$

~~For~~ position space Feynman rule, we represent ~~extect~~ external propagators as $e^{ip\cdot x}$ and internal ones as $\Delta_F(x-y)$, the Feynman propagator.

the contribution

$$= \frac{(-i\lambda)^2}{2} \int d^4z \int d^4w \, e^{-i(p_1+p_2-k_4)\cdot z} \, \Delta_F(z-w)$$

$$\times \, e^{iw\cdot(k_1+k_2+k_3)}$$

$$= -\lambda^2 \int d^4z \int d^4w \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2-m^2+i\varepsilon} e^{-il(z-w)}$$

$$\times e^{iw(k_1+k_2+k_3)} e^{-iz(p_1+p_2-k_4)}.$$

$$= -i\lambda^2 \int d^4z\, e^{-iz(p_1+p_2-k_4+l)}.$$

$$\times \int d^4w\, e^{iw(k_1+k_2+k_3+l)}.$$

$$\times \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2-m^2+i\varepsilon}$$

$$= -i\lambda^2 \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2-m^2+i\varepsilon} (2\pi)^4 \delta^{(4)}(p_1+p_2-k_4+l).$$

$$\times (2\pi)^4 \delta^{(4)}(k_1+k_2+k_3+l)$$

$$\cancel{= -i\lambda^2 \int \frac{d^4l}{2\pi}}$$

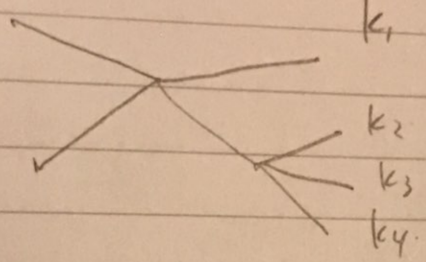
$$= -i\lambda^2 (2\pi)^4 \frac{1}{(p_1+p_2-k_4)^2-m^2+i\varepsilon} \delta^{(4)}(k_1+k_2+k_3+k_4-p_1-p_2)$$

replace $\underline{l = k_4 - p_1 - p_2}$.

□.

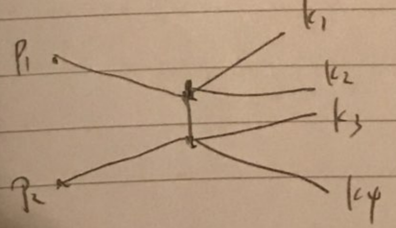
The δ-function represents momentum conservation.

c) Similar !



$$-i\lambda^2 (2\pi)^4 \frac{1}{(p_1+p_2-k_4)^2-m^2+i\varepsilon} \delta^{(4)}(k_1+k_2+k_3+k_4-p_1-p_2)$$

Different !



$$-i\lambda^2 (2\pi)^4 \frac{1}{(p_1-k_1-k_2)^2-m^2+i\varepsilon} \delta^{(4)}(k_1+k_2+k_3+k_4-p_1-p_2).$$