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A A A

Quantum Field theory.

TA : Johan Henriksson

Problem Set 3

wk. 7 Tues 3.30 - 5pm

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* For question (10) - (e), please see the last page, and ignore what I wrote previously.

⑧

$$\frac{1}{a_1 \cdots a_n} \stackrel{?}{=} (n-1)! \int_0^1 \prod_{i=1}^{n} dx_i \; \frac{\delta(\sum_{j=1}^{n} x_j - 1)}{(a_1 x_1 + \cdots + a_n x_n)}$$

Base case : $n=1$

$$RHS = 0! \int_0^1 dx_1 \frac{\delta(x_1 - 1)}{a_1 x_1}$$

$$= \frac{1}{a_1} \times \int_0^1 dx_1 \frac{\delta(x_1 - 1)}{x_1}$$

$\because$ $x_1 = 1$ is in the interval $[0, 1]$

$$\therefore RHS = \frac{1}{a_1} \int_0^1 dx_1 \frac{\delta(x_1 - 1)}{x_1} = \frac{1}{a_1} \int_{-\infty}^{\infty} dx_1 \frac{\delta(x_1 - 1)}{x_1}$$

$$= \frac{1}{a_1} \times \frac{1}{x_1}\Big|_{x_1 = 1} = \frac{1}{a_1} = LHS$$

$\therefore$ True for Base Case.

Inductive assumption :

Assume proposition is true for $n-1$

i.e.

$$\frac{1}{a_1 \cdots a_{n-1}} = (n-2)! \int_0^1 \prod_{i=1}^{n-1} dx_j \frac{\delta(\sum_{j=1}^{n-1} x_j - 1)}{(a_1 x_1 + \cdots + a_{n-1} x_{n-1})^{n-1}}$$

then consider the case for n:

$$RHS = (n-1)! \int_0^1 \prod_{i=1}^{n} dx_i \, \frac{\delta(\sum_{j=1}^{n} x_j - 1)}{(a_1x_1 + \cdots + a_nx_n)^n}$$

$$= \cancel{RHS\,(n-1)\,(n-2)! \int_0^1 \prod_{i=1}^{n-1} dx_i \int_0}$$

$$= (n-1)(n-2)! \int_0^1 \prod_{i=1}^{n-1} dx_i \int_0^1 dx_n \, \frac{\delta(x_n + \sum_{j=1}^{n-1} x_j - 1)}{(a_1x_1 + \cdots + a_nx_n)^n}$$

$$= \cancel{(n-1)(n-2)! \int_0^1 \prod_{i=1}^{n-2} dx_i \int_0 \frac{dx_{n-1}}{(a_1x_1 + \cdots + a_{n-1}x_{n-1} + a_n(1-\sum_{j=1}^{n-1} x_j))^n}}$$

$$= (n-1)(n-2)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \int \cdots \int_0^{1-\sum_{j=1}^{n-2} x_j} dx_{n-1}$$

$$\left[\frac{1}{(a_1x_1 + a_2x_2 + \cdots + a_n\underbrace{(1 - x_1 - x_2 - x_3 - \cdots)}_{1-\sum_{j=1}^{n-1} x_j})^n} \right].$$

The above equality holds because the $\delta(x_1 + x_2 + \cdots + x_n - 1)$ requires we have $x_1 + x_2 + \cdots + x_n = 1$ while integrating $x_1, x_2, \cdots, x_n$ from 0 to 1

$\because$ $x_n = 1 - x_1 - x_2 - \cdots - x_{n-1} \geq 0$

$\therefore$ $x_{n-1} \leq 1 - x_1 - x_2 - \cdots - x_{n-2}$

$\because \quad x_{n-1} \geq 0$

$\therefore \quad \cancel{x_2}\ x_{n-2} \leq 1 - x_1 - x_2 - \cdots - x_{n-3}$

And so on... we find that the upper limit of integration becomes

$$\int_0^{1-x_1-x_2-\cdots-x_{j-1}} dx_j \qquad \text{for the } dx_j \text{ integral.}$$

$$\therefore \text{RHS} = (n-1)(n-2)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1}$$

$$\times \left[\frac{1}{(a_1x_1 + \cdots + a_{n-1}x_{n-1} + a_n(1 - x_1 - \cdots - x_{n-1}))^n} \right]$$

$$= (n-1)(n-2)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1}$$

$$\times \left[\frac{1}{[(a_1-a_n)x_1 + (a_2-a_n)x_2 + \cdots + (a_{n-1}-a_n)x_{n-1} + a_n]^n} \right]$$

$$= \cancel{(n-1)}(n-2)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-3}} dx_{n-2}$$

$$\left[-\frac{1}{\cancel{(n-1)}[a_{n-1}-a_n]} \right] \times \left[\frac{1}{[(a_1-a_n)x_1 + \cdots + (a_{n-1}-a_n)x_{n-1} + a_n]^{n-1}} \right]_{x_{n-1}=0}^{x_{n-1}=K}$$

where $K = 1 - x_1 - \cdots - x_{n-2}$

$$= (n-2)!\,\frac{1}{a_{n-1}-a_n}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-3}} dx_{n-2}$$

$$\left\{\left[\frac{1}{(a_1-a_n)x_1+\cdots+(a_{n-2}-a_n)x_{n-2}+a_n}\right]^{n-1}\right.$$

$$\left.-\left[\frac{1}{\cancel{a_n}+(a_1-\cancel{a_n})x_1+\cdots+(a_{n-2}-\cancel{a_n})x_{n-2}+(a_{n-1}-\cancel{a_n})(1-x_1-\cdots-x_{n-2})}\right]^{n-1}\right\}$$

$$= (n-2)!\,\frac{1}{a_{n-1}-a_n}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-3}} dx_{n-2}$$

$$\left\{\left[\frac{1}{(a_1-a_n)x_1+\cdots+(a_{n-2}-a_n)x_{n-2}+a_n}\right]^{n-1}\right.$$

$$\left.-\left[\frac{1}{(a_1-a_{n-1})x_1+\cdots+(a_{n-2}-a_{n-1})x_{n-2}+a_{n-1}}\right]^{n-1}\right\}$$

$$= \frac{1}{a_{n-1}-a_n}\left\{(n-2)!\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-3}} dx_{n-2}\right.$$

$$\times\left[\frac{1}{(a_1-a_n)x_1+\cdots+(a_{n-2}-a_n)x_{n-2}+a_n}\right]^{n-1}$$

$$-(n-2)!\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-3}} dx_{n-2}$$

$$\left.\times\left[\frac{1}{(a_1-a_{n-1})x_1+\cdots+(a_{n-2}-a_{n-1})x_{n-2}+a_{n-1}}\right]^{n-1}\right\}.$$

$$= \frac{1}{a_{n-1}-a_n} \Big\{ (n-2)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-3}} dx_{n-2} \quad \times$$

$$\left[\frac{1}{[(a_1 x_1 + \cdots + a_{n-2}x_{n-2} + a_n(1-x_1-\cdots-x_{n-2})]^{n-1}} \right]$$

$$- (n-2)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-3}} dx_{n-2} \quad \times$$

$$\left[\frac{1}{[(a_1 x_1 + \cdots + a_{n-2}x_{n-2} + a_{n-1}(1-x_1-\cdots-x_{n-2})]^{n-1}} \right] \Big\}$$

$$= \frac{1}{a_{n-1}-a_n} \Big\{ (n-2)! \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_{n-1} \frac{\delta(x_1+x_2+\cdots+x_{n-1}-1)}{[a_1x_1 + a_2x_2 + \cdots + \cancel{a_{n-1}x_{n-1}}\, a_n x_{n-1}]^{n-1}}$$

$$- (n-2)! \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_{n-1} \frac{\delta(x_1+x_2+\cdots+x_{n-1}-1)}{[a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1}]^{n-1}}$$

By inductive assumption

$$= \frac{1}{a_{n-1}-a_n}\left[\frac{1}{a_1 \cdots a_{n-2}a_n} - \frac{1}{a_1 \cdots a_{n-2}a_{n-1}} \right]$$

$$= \frac{1}{a_1 \cdots a_{n-2}} \left[\frac{1}{a_{n-1}-a_n} \right] \left[\frac{1}{a_n} - \frac{1}{a_{n-1}} \right]$$

$$= \frac{1}{a_1 \cdots a_{n-2}} \left(\frac{1}{a_n a_{n-1}} \right) \left(\frac{\cancel{a_{n-1}-a_n}}{\cancel{a_{n-1}-a_n}} \right)$$

$$= \frac{1}{a_1 \cdots a_n} = \text{LHS} \Rightarrow$$ true for n.

✓ See class for a quicker proof.

proposition true for ~~~~ $n=1$ and if

$n-1$ is true then n is true.

~~~~ This concludes the proof $\square$
~~~~

(9)

(a) ~~$g_0 =$~~ ~~$\dot{=}\ \Sigma\lambda_0$~~ $D = 4-2\varepsilon$

$\therefore\ [\lambda_0] = 4 - D = 4-(4-2\varepsilon) = 2\varepsilon.$

let $[\mu] = 1$, ~~then~~ because μ is a mass scale.

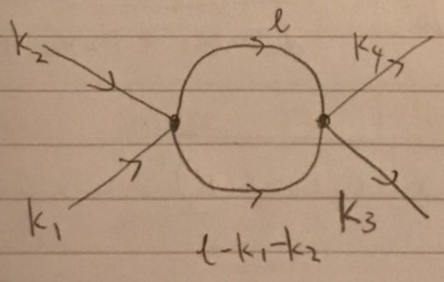
$\because\ g_0$ is dimensionless

$\therefore\ [g_0] = 0$.

$\therefore$ ~~The only poss~~ $\therefore$ we have

$$\boxed{g_0 = \mu^{-2\varepsilon}\lambda_0}$$

(b) $\lambda_0 = g_0\mu^{2\varepsilon}$ $\quad D = 4-2\varepsilon$



This diagram yields a contribution

2 vertices

$$(2\pi)^D\delta^{(D)}(k_1+k_2-k_3-k_4)\frac{1}{k_1^2+m^2}\frac{1}{k_2^2+m^2}\frac{1}{k_3^2+m^2}\frac{1}{k_2^2+m^2}$$

symmetry factor

$$\times\frac{(-\lambda_0)^2}{2}\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{\ell^2+m^2}\frac{1}{(\ell-k_1-k_2)^2+m^2}$$

in the Euclidean space by the Feynman rules in ~~a~~ momentum space.

Within the above expression, the loop contribution is

$$A = \frac{(-\lambda_0)^2}{2} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2+m^2} \frac{1}{(l-k_1-k_2)^2+m^2}$$

Substitute in ~~$\lambda_0 = 2g_0\mu^{-4\varepsilon}$~~, $D = 4-2\varepsilon$
~~$\lambda_0 = g_0\mu^{+2\varepsilon}$~~, $\lambda_0 = g_0\mu^{2\varepsilon}$
we have

$$A(\varepsilon) = \frac{g_0^2}{2}\mu^{4\varepsilon} \int \frac{d^{4-2\varepsilon} l}{(2\pi)^{4-2\varepsilon}} \frac{1}{l^2+m^2} \frac{1}{(l-k_1-k_2)^2+m^2}$$

(c) Feynman parametrisation for 2 variables:

$$\frac{1}{a_1 a_2} = \int_0^1 dx_1 \int_0^1 dx_2 \frac{\delta(x_1+x_2-1)}{(a_1x_1+a_2x_2)^2}$$

$$= \int_0^1 dx_1 \frac{1}{(a_1x_1+a_2(1-x_1))^2}$$

$$= \int_0^1 dx \frac{1}{(a_1x+a_2(1-x))^2}$$

In our case

$$a_1 = \frac{\cancel{1}}{l^2+m^2} \qquad a_2 = \frac{\cancel{1}}{(l-k_1-k_2)^2+m^2}$$

$$\therefore \quad A(\varepsilon) = \frac{g_0^2}{2}\mu^{4\varepsilon}\int \frac{d^{4-2\varepsilon} l}{(2\pi)^{4-2\varepsilon}} \frac{1}{(l^2+m^2)[(l-k_1-k_2)^2+m^2]}$$

$\Rightarrow$ ~~$A(\varepsilon)$~~

$$= \frac{g_0^2}{2}\mu^{4\varepsilon}\int \frac{d^{4-2\varepsilon} l}{(2\pi)^{4-2\varepsilon}} \int_0^1 dx \frac{1}{\left[(l^2+m^2)x + [(l-k_1-k_2)^2+m^2](1-x)\right]^2}$$

$$= \frac{g_0^2}{2}\mu^{4\varepsilon}\int_0^1 dx \int \frac{d^{4-2\varepsilon} l}{(2\pi)^{4-2\varepsilon}} \frac{1}{\left[\underbrace{(l^2+m^2)x + [l^2 - 2l(k_1+k_2) + (k_1+k_2)^2 + m^2](1-x)}\right]^2}$$

this denominator square rooted

is equal to

$$(l^2+m^2)(\underbrace{x+1-x}_{1}) + (k_1+k_2)^2 - 2l(k_1+k_2) - (k_1+k_2)^2 x$$
$$+ 2l(k_1+k_2)x$$

Now make the change of variable

~~$\tilde{l} = l + x(k_1+k_2)$, $dl = d\tilde{l}$ while integrating over $\tilde{l}$. $\Rightarrow l = \tilde{l} - x(k_1+k_2)$~~

$\Rightarrow$ ~~this equals to~~

$$\tilde{l} = l - (1-x)(k_1+k_2)$$

then $l = \tilde{l} + (1-x)(k_1+k_2)$

x is constant while integrating l, so

$dl = d\tilde{l}$

Denominator square rooted

$= l^2 + m^2 + (k_1+k_2)^2(1-x) - 2l(1-x)(k_1+k_2)$

$= (\tilde{l} + (1-x)(k_1+k_2))^2 + m^2 + (k_1+k_2)^2(1-x)$

$-2[\tilde{l} + (1-x)(k_1+k_2)](1-x)(k_1+k_2)$

$= \tilde{l}^2 + m^2 + \cancel{2\tilde{l}(1-x)(k_1+k_2)} + (1-x)^2(k_1+k_2)^2$

$+ (1-x)(k_1+k_2)^2 - \cancel{2\tilde{l}(1-x)(k_1+k_2)}$

$-2(1-x)^2(k_1+k_2)^2$

$= \tilde{l}^2 + m^2 + [(1-x) - (1-x)^2](k_1+k_2)^2$

$= \tilde{l}^2 + m^2 + (1-x)(\underbrace{1-(1-x)}_{x})(k_1+k_2)^2$

$= \tilde{l} + x(1-x)(k_1+k_2)^2 + m^2$

Hence

$$A(\varepsilon) = \frac{g_0^2}{2}\mu^{4\varepsilon}\int_0^1 dx \int \frac{d^{4-2\varepsilon}\tilde{l}}{(2\pi)^{4-2\varepsilon}} \frac{1}{[\tilde{l}^2 + x(1-x)(k_1+k_2)^2 + m^2]^2}$$

$\square$

(d) let $D = 4-2\varepsilon$, $q = m^2 + x(1-x)(k_1+k_2)^2$

$$A(\varepsilon) = \frac{g_0^2}{2} \mu^{4\varepsilon} \int_0^1 dx \int \frac{d^D\tilde{\ell}}{(2\pi)^D} \frac{1}{(\tilde{\ell}^2+q^2)^2}$$

Use the Schwinger parametrisation

$$\frac{1}{A^x} = \frac{1}{\Gamma(x)} \int_0^\infty dt\, t^{x-1} e^{-At}$$

$$\therefore \frac{1}{A^2} = \frac{1}{\Gamma(2)} \int_0^\infty dt\, t e^{-At}$$

$$\therefore \int \frac{d^D\tilde{\ell}}{(2\pi)^D} \frac{1}{(\tilde{\ell}^2+q^2)^2} = \int \frac{d^D\tilde{\ell}}{(2\pi)^D} \int_0^\infty dt \underbrace{\frac{1}{\Gamma(2)}}_{=1} t e^{-t(\tilde{\ell}^2+q^2)}$$

$$= \int_0^\infty dt \int \frac{d^D\tilde{\ell}}{(2\pi)^D} t e^{-t(\tilde{\ell}^2+q^2)}$$

$$\cancel{= \int_0^\infty dt \cdot t \int \frac{d^D\tilde{\ell}}{(2\pi)^D} e}$$

$$= \int_0^\infty dt \cdot t \cdot e^{-tq^2} \underbrace{\int \frac{d^D\tilde{\ell}}{(2\pi)^D} e^{-t\tilde{\ell}^2}}_{\frac{1}{(2\pi)^D}\left(\frac{\pi}{t}\right)^{D/2}}$$

$u = tq^2$

$dt = du \cdot q^{-2}$

$t^{1-D/2} = \left(\frac{u}{q^2}\right)^{1-D/2}$

$= u^{1-D/2} q^{\frac{D}{2}-1}$

$$\cancel{= \frac{1}{2^D} \int_0^\infty dt \cdot t \cdot e^{-tq^2}}$$

$$= \frac{1}{2^D} \frac{1}{\pi^{D/2}} \int_0^\infty dt\, t^{1-D/2} e^{-tq^2}$$

$$= \frac{1}{2^D} \frac{1}{\pi^{D/2}} \left[\int_0^\infty du\, u^{1-D/2} e^{-u}\right] q^{\frac{D}{2}-1} \cdot q^{-1}$$

$$= \frac{1}{2^D}\frac{1}{\pi^{D/2}}\Big[\underbrace{\int_0^\infty u^{(2-\frac{D}{2})-1} e^{-u}\, du}_{\Gamma(2-\frac{D}{2})}\Big]\, q^{\frac{D}{2}-2}$$

$$= \frac{1}{2^D}\frac{1}{\pi^{D/2}}\Gamma(2-\tfrac{D}{2})\, q^{\frac{D}{2}-2}$$

put $D = 4-2\varepsilon$, $2-\frac{D}{2} = \cancel{2-2}+\varepsilon \simeq \varepsilon$

$q = m^2 + x(1-x)(k_1+k_2)^2$

we have

$$A(\varepsilon) = \frac{g_0^2 \mu^{4\varepsilon}}{2^{5-2\varepsilon}\pi^{2-\varepsilon}}\int_0^1 dx\, \frac{\Gamma(\varepsilon)}{(m^2+x(1-x)(k_1+k_2)^2)^\varepsilon}$$

$$= \frac{g_0^2}{32\pi^2}\underbrace{\left(\frac{\mu^4}{(4\pi)^{-1}}\right)^\varepsilon}_{(4\pi\mu^4)^\varepsilon}\Gamma(\varepsilon)\int_0^1 \frac{dx}{(m^2+x(1-x)(k_1+k_2)^2)^\varepsilon}$$

we know that :

$$\Gamma(x) = \frac{1}{x} - \gamma_E + O(x)$$

$$\left(\frac{1}{\Delta}\right)^\varepsilon = 1 - \varepsilon\log(\Delta) + \ldots$$

$$\Delta^\varepsilon = 1 + \varepsilon\log(\Delta) + \ldots$$

$\therefore\ \cancel{\left(\frac{\mu^4}{4\pi}\right)^\varepsilon = 1 + \varepsilon\log\left(\frac{\mu^4}{4\pi}\right)}$

$\cancel{(4\pi\mu^4)^\varepsilon = 1 + \varepsilon\log(}$

$$(4\pi\mu^4)^\varepsilon = \mu^{2\varepsilon}(4\pi\mu^2)^\varepsilon$$

$$= \mu^{2\varepsilon}(1+\varepsilon\log(4\pi\mu^2)).$$

$$= \mu^{2\varepsilon}(1+\varepsilon\ln(4\pi\mu^2) + O(\varepsilon^2)).$$

$$\Gamma(\varepsilon) = \cancel{\tfrac{1}{\varepsilon}+\gamma_E}\ \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon)$$

$$\therefore (4\pi\mu^4)^\varepsilon\,\Gamma(\varepsilon)$$

$$\simeq \mu^{2\varepsilon}(1+\varepsilon\ln(4\pi\mu^2)+O(\varepsilon^2))\left(\frac{1}{\varepsilon}-\gamma_E+O(\varepsilon)\right)$$

$$\simeq \mu^{2\varepsilon}\left(\frac{1}{\varepsilon}-\gamma_E+\ln(4\pi\mu^2)+O(\varepsilon)\right).$$

$$\frac{1}{(m^2+(k_1+k_2)^2x(1-x))^\varepsilon} \simeq 1-\varepsilon\ln(m^2+(k_1+k_2)^2x(1-x)) + O(\varepsilon^2)$$

$$\therefore \int_0^1 \frac{(4\pi\mu^4)^\varepsilon\,\Gamma(\varepsilon)\,dx}{(m^2+(k_1+k_2)^2x(1-x))^\varepsilon} = \int_0^1 dx\, \mu^{2\varepsilon}\left(\frac{1}{\varepsilon}-\gamma_E+\ln(4\pi\mu^2)+O(\varepsilon)\right)$$

$$\times\left(1-\varepsilon\ln(m^2+(k_1+k_2)^2x(1-x)) + O(\varepsilon^2)\right)$$

$$= \mu^{2\varepsilon}\int_0^1 dx\left[\frac{1}{\varepsilon}-\gamma_E+\ln(4\pi\mu^2)-\ln(m^2+(k_1+k_2)^2x(1-x)) + O(\varepsilon)\right]$$

$$= \mu^{2\varepsilon}\left[\frac{1}{\varepsilon}-\gamma_E+\int_0^1 dx\,\ln\left(\frac{4\pi\mu^2}{m^2+(k_1+k_2)^2x(1-x)}\right) + O(\varepsilon)\right]$$

good!

multiply this by $\frac{g_0^2}{32\pi^2}$ and invert the ~~fn~~ log we get

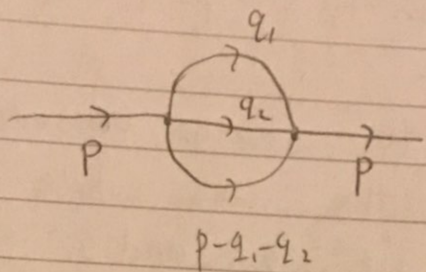
$$A(\varepsilon) = \mu^{2\varepsilon} \frac{g_0^3}{32\pi^2} \left[\frac{1}{\varepsilon} - \gamma_E - \int_0^1 dx \ln \left(\frac{m^2 + (k_1 + k_2)^2 x(1-x)}{4\pi\mu^2} \right) \right.$$

$$\left. + O(\varepsilon) \right]$$

$\square$

(10)

(a)



Consider only the loop contribution in Euclidean space. Use the Feynman rules in momentum space:
the symmetry factor for this diagram is $3! = 6$ and dimension $D=4$ (later $D=4-2\varepsilon$)

$$\therefore A = \frac{(-\lambda_0)^2}{6}\int\frac{d^Dq_1}{(2\pi)^D}\int\frac{d^Dq_2}{(2\pi)^D}\frac{1}{q_1^2+m^2}\frac{1}{q_2^2+m^2}\frac{1}{(p-q_1-q_2)^2+m^2}$$

(b) Use Feynmann parametrisation in 3 variables

$$\frac{1}{abc} = (2!)\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\frac{\delta(x+y+z-1)}{(ax+by+cz)^3}$$

$\therefore$ we have

$$\cancel{\frac{1}{q_1^2+m^2}\frac{1}{q_2^2+m^2}}\quad\frac{1}{q_1^2+m^2}\frac{1}{q_2^2+m^2}\frac{1}{(p-q_1-q_2)^2+m^2}$$

$$= 2\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\frac{\delta(x+y+z-1)}{[(q_1^2+m^2)x+(q_2^2+m^2)y+[(p-q_1-q_2)^2+m^2]z]^3}$$

Consider the denominator (inside the cube)

$$(q_1^2+m^2)x + (q_2^2+m^2)y + [(p-q_1-q_2)^2+m^2]z$$

~~wh~~ we want to change variables ~~p~~ q_1 and q_2 so that the ~~wh~~ whole thing only depends on p^2, q_1^2 and q_2^2

We also want linear change of variable since we want dq_1 and dq_2 to be the same as before. so

$$q_1 \to q_1 + (\text{independent of } q_1)$$
$$q_2 \to q_2 + (\text{independent of } q_2.)$$

the term $m^2(x+y+z)$ in the above equation will stay so we are not interested in it.

consider:

$$q_1^2x + q_2^2y + (p-q_1-q_2)^2z$$

$$= q_1^2x + q_2^2y + (p^2+q_1^2+q_2^2 - 2pq_1 - 2pq_2 + 2q_1q_2)z$$

$$= (x+z)q_1^2 + (y+z)q_2^2 - 2pzq_1 - 2pzq_2 + 2zq_1q_2 + p^2z$$

$$= (x+z)q_1^2 - 2pzq_1 + p^2z + (y+z)q_2^2 - 2z(p-q_1)q_2$$

$$= (y+z)\left[q_2^2 - \frac{2z(p-q_1)}{(y+z)}q_2\right] + \left[(x+z)q_1^2 - 2pzq_1 + p^2z\right]$$

$$= (y+z)\left[q_2^2 - \frac{2z(p-q_1)}{(y+z)}q_2 + \frac{z^2(p-q_1)^2}{(y+z)^2}\right] \ldots,$$

$$\cdots - \frac{z^2(p-q_1)^2}{(y+z)} + \left[(x+z)q_1^2 - 2pzq_1 + p^2 z\right]$$

$$= (y+z)\left[q_2 - \frac{z(p-q_1)}{y+z}\right]^2 - \frac{z^2(p-q_1)^2}{y+z} + \left[(x+z)q_1^2 - 2pzq_1 + p^2 z\right]$$

$$= (y+z)\left[q_2 - \frac{z(p-q_1)}{y+z}\right]^2 - \left(\frac{z^2}{y+z}\right)(p^2 + q_1^2 - 2pq_1) + \left((x+z)q_1^2 - 2pzq_1 + p^2 z\right)$$

$$= (y+z)\left[q_2 - \frac{z(p-q_1)}{y+z}\right]^2 + \left[(x+z) - \frac{z^2}{y+z}\right]q_1^2 - (2p)\left(z - \frac{z^2}{y+z}\right)q_1$$

$$+ \left(z - \frac{z^2}{y+z}\right)p^2$$

$$= (y+z)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{1}{y+z}\Big\{\left[(x+z)(y+z) - z^2\right]q_1^2$$

$$- 2p\left(z(y+z) - z^2\right)q_1 + \left(z(y+z) - z^2\right)p^2\Big\}$$

$$= (y+z)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{1}{y+z}\Big\{(xy+yz+xz)q_1^2$$

$$- 2pyzq_1 + yzp^2\Big\}$$

$$= (z+y)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{xy+yz+xz}{y+z}\Big\{q_1^2 - \frac{2pyz}{xy+yz+xz}q_1$$

$$+ \frac{yzp^2}{xy+yz+xz}\Big\}$$

$$= (z+y)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{xy+yz+xz}{y+z}\Big[\Big(q_1^2 - \frac{2pyz}{xy+yz+xz}q_1$$

$$+ \left(\frac{pyz}{xy+yz+xz}\right)^2\Big) + p^2\left(\frac{yz}{xy+yz+xz}\right)\left(1 - \frac{yz}{xy+yz+xz}\right)\Big]$$

$$= (z+y)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{xy+yz+xz}{y+z}\Big[\left(q_1 - \frac{pyz}{xy+yz+xz}\right)^2$$

$$+ p^2\left(\frac{yz}{xy+yz+xz}\right)\left(1 - \frac{yz}{xy+yz+xz}\right)\Big]$$

$$\cdots - \frac{z^2(p-q_1)^2}{(y+z)} + [(x+z)q_1^2 - 2pzq_1 + p^2 z]$$

$$= (y+z)\left[q_2 - \frac{z(p-q_1)}{y+z}\right]^2 - \frac{z^2(p-q_1)^2}{y+z} + [(x+z)q_1^2 - 2pzq_1 + p^2 z]$$

$$= (y+z)\left[q_2 - \frac{z(p-q_1)}{y+z}\right]^2 - \left(\frac{z^2}{y+z}\right)(p^2 + q_1^2 - 2pq_1) + \left((x+z)q_1^2 - 2pzq_1 + p^2 z\right)$$

$$= (y+z)\left[q_2 - \frac{z(p-q_1)}{y+z}\right]^2 + \left[(x+z) - \frac{z^2}{y+z}\right]q_1^2 - (2p)\left(z - \frac{z^2}{y+z}\right)q_1$$

$$+ \left(z - \frac{z^2}{y+z}\right)p^2$$

$$= (y+z)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{1}{y+z}\Big\{[(x+z)(y+z) - z^2]q_1^2$$

$$- 2p(z(y+z) - z^2)q_1 + (z(y+z) - z^2)p^2\Big\}$$

$$= (y+z)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{1}{y+z}\Big\{(xy+yz+xz)q_1^2$$

$$- 2pyz\,q_1 + yzp^2\Big\}$$

$$= (z+y)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{xy+yz+xz}{y+z}\Big\{q_1^2 - \frac{2pyz}{xy+yz+xz}q_1$$

$$+ \frac{yzp^2}{xy+yz+xz}\Big\}$$

$$= (z+y)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{xy+yz+xz}{y+z}\Bigg[\Big(q_1^2 - \frac{2pyz}{xy+yz+xz}q_1$$

$$+ \left(\frac{pyz}{xy+yz+xz}\right)^2\Big) + p^2\left(\frac{yz}{xy+yz+xz}\right)\left(1 - \frac{yz}{xy+yz+xz}\right)\Bigg]$$

$$= (z+y)\left(q_2 - \frac{z(p-q_1)}{y+z}\right)^2 + \frac{xy+yz+xz}{y+z}\left[\left(q_1 - \frac{pyz}{xy+yz+xz}\right)^2\right.$$

$$\left. + p^2\left(\frac{yz}{xy+yz+xz}\right)\left(1 - \frac{yz}{xy+yz+xz}\right)\right]$$

$$= (y+z)\left[q_2 - \frac{z(p-q_1)}{y+z}\right]^2 + \frac{xy+yz+xz}{y+z}\left(q_1 - \frac{pyz}{xy+yz+xz}\right)^2$$

$$+ \frac{yzp^2}{y+z}\left(1 - \frac{yz}{xy+yz+xz}\right).$$

to be more formal, can calculate the Jacobian of $\left|\frac{\partial(\tilde{q}_1, \tilde{q}_2)}{\partial(q_1, q_2)}\right|$ this gives $\begin{pmatrix} 1 & 0 \\ +\frac{z}{y+z} & 1 \end{pmatrix}$ and the determinate is $\underline{\underline{1}}$ so we can do this change of variable

Now consider the change of variable

$$\tilde{q}_2 = q_2 - \frac{z(p-q_1)}{y+z}, \quad \tilde{q}_1 = q_1 - \frac{pyz}{xy+yz+xz}$$

when integrating dq_2, $d\tilde{q}_2 = dq_2$

when integrating dq_1, $d\tilde{q}_1 = dq_1$

~~We can~~ the limits of integration extends to $\pm\infty$ so ~~does not~~ do not change.

$\therefore$ we can replace $\tilde{q}_2$ by q_2 and $\tilde{q}_1$ by q_1 to get

~~$(x+z)q_2^2 +$~~

$$(y+z)(q_2)^2 + \frac{xy+yz+xz}{y+z}(q_1)^2 + \frac{yzp^2}{y+z}\left(1 - \frac{yz}{xy+yz+xz}\right)$$

put the $m^2(x+y+z)$ back ~~we get~~

~~A =~~ and ~~remind~~ recall that

$$A = \frac{\lambda_0^2}{6} \times 2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \int \frac{d^D q_1}{(2\pi)^D} \int \frac{d^D q_2}{(2\pi)^D}$$

$$\frac{\delta(x+y+z-1)}{\left[(q_1^2+m^2)x + (q_2^2+m^2)y + [(p-q_1-q_2)^2 + m^2]z\right]^3}$$

we have

$$A(p^2) = \frac{\lambda_0^2}{3}\int_0^1 dx \int_0^1 dy \int_0^1 dz \int \frac{d^D q_1}{(2\pi)^D}\int \frac{d^D q_2}{(2\pi)^D}$$

$$\frac{\delta(x+y+z-1)}{\left[(y+z)q_2^2 + \frac{xy+yz+xz}{y+z}q_1^2 + \frac{yzp^2}{y+z}\left(1-\frac{yz}{xy+yz+xz}\right) + m^2(x+y+z)\right]^3}$$

compare to the question,

$C = \frac{\lambda_0^2}{3}$,

$$D = (y+z)q_2^2 + \frac{xy+yz+xz}{y+z}q_1^2 + \frac{yzp^2}{y+z}\left(1-\frac{yz}{xy+yz+xz}\right) + m^2(x+y+z)$$

(c) $\therefore$ $$A(p^2) = C\int_0^1 dx\int_0^1 dy\int_0^1 dz \int\frac{d^D q_1}{(2\pi)^D}\int\frac{d^D q_2}{(2\pi)^D}\frac{1}{D^3}$$

$$\therefore \frac{dA(p^2)}{dp^2} = C\int_0^1 dx\int_0^1 dy\int_0^1 dz\int\frac{d^D q_1}{(2\pi)^D}\int\frac{d^D q_2}{(2\pi)^D}(-3)\frac{1}{D^4}\frac{dD(p^2)}{dp^2}$$
$$\times\ \delta(x+y+z-1)$$

and from (b) it is clear that

$$\frac{dD(p^2)}{dp^2} = \frac{yz}{y+z}\left(1-\frac{yz}{xy+yz+xz}\right).$$ (we find $\frac{dD}{dp^2}$ independent of p^2).

consider $I = \int \frac{d^D q_1}{(2\pi)^D} \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{D^4}$

use Schwinger parametrisation:

$$\frac{1}{D^4} = \frac{1}{\Gamma(4)} \int_0^\infty dt\, t^3 e^{-tD}$$

$$\therefore \int \frac{d^D q_1}{(2\pi)^D} \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{\Gamma(4)} \int_0^\infty dt\, t^3 e^{-tD(q_1^2, q_2^2)}$$

where $D = (y+z) q_2^2 + \frac{xy+yz+xz}{y+z} q_1^2 + \frac{yz p^2}{y+z}\left(1 - \frac{yz}{xy+yz+xz}\right)$

$+ m^2(x+y+z)$

$$I = \frac{1}{\Gamma(4)} \int_0^\infty dt \cdot t^3 e^{-t\left(\frac{yz p^2}{y+z}\left(1-\frac{yz}{xy+yz+xz}\right) + m^2(x+y+z)\right)}$$

$$\times \int \frac{d^D q_1}{(2\pi)^D} \left[e^{-t\left(\frac{xy+yz+xz}{y+z}\right) q_1^2} \right.$$

$$\left. \times \int \frac{d^D q_2}{(2\pi)^D} \left[e^{-t(y+z) q_2^2} \right] \right.$$

Use momentum integral.

$$\int \frac{d^D k}{(2\pi)^D} e^{-\alpha k^2} = (4\pi\alpha)^{-\frac{D}{2}}$$

We have

$$J = \int \frac{d^D q_1}{(2\pi)^D} e^{-t\beta_1 q_1^2} \times \int \frac{d^D q_2}{(2\pi)^D} e^{-t\beta_2 q_2^2}$$

$$= (4\pi t\beta_1)^{-\frac{D}{2}} \times (4\pi t \beta_2)^{-\frac{D}{2}}$$

$$= (16\pi^2 t^2 \beta_1\beta_2)^{-\frac{D}{2}}$$

when $\beta_1 = \frac{xy+yz+xz}{y+z}$, $\beta_2 = y+z$

we have $\beta_1\beta_2 = xy+yz+xz$

$$\therefore \quad J = (16\pi^2 t^2 (xy+yz+xz))^{-\frac{D}{2}}$$

$$I = \frac{1}{\Gamma(4)} \int_0^\infty dt\, t^3 e^{-t\frac{yzp^2}{y+z}\left(1-\frac{yz}{xy+yz+xz}\right)} \times J$$

$$\cancel{= \frac{1}{3!}\int_0^\infty dt}$$

$$\cancel{I = \frac{1}{3!}(16\pi^2}$$

and $\frac{yzp^2}{y+z}\left(1-\frac{yz}{xy+yz+xz}\right)$

$$= \frac{yz}{y+z}\left(\frac{xy+xz}{xy+yz+xz}\right)p^2$$

$$= \frac{(yz) \times \cancel{(y+z)}}{\cancel{(y+z)}(xy+yz+xz)} p^2$$

$$= \frac{xyz\,p^2}{xy+yz+xz}$$

$$J = (4\pi)^{-D} t^{-D} (xy+yz+xz)^{-\frac{D}{2}}, \quad \Gamma(4) = 3! = 6$$

We have

$$J = \int \frac{d^D q_1}{(2\pi)^D} e^{-t\beta_1 q_1^2} \times \int \frac{d^D q_2}{(2\pi)^D} e^{-t\beta_2 q_2^2}$$

$$= (4\pi t\beta_1)^{-\frac{D}{2}} \times (4\pi t \beta_2)^{-\frac{D}{2}}$$

$$= (16\pi^2 t^2 \beta_1\beta_2)^{-\frac{D}{2}}$$

when $\beta_1 = \frac{xy+yz+xz}{y+z}$, $\beta_2 = y+z$

we have $\beta_1\beta_2 = xy+yz+xz$

$$\therefore\ J = (16\pi^2 t^2 (xy+yz+xz))^{-\frac{D}{2}}$$

$$I = \frac{1}{\Gamma(4)} \int_0^\infty dt\, t^3 e^{-t\frac{yzp^2}{y+z}\left(1-\frac{yz}{xy+yz+xz}\right)} \times J$$

$\cancel{= \frac{1}{3!}\int_0^\infty dt}$ and $\frac{yzp^2}{y+z}\left(1-\frac{yz}{xy+yz+xz}\right)$

$\cancel{I = \frac{1}{3!}(16\pi^2}$ $= \frac{yz}{y+z}\left(\frac{xy+xz}{xy+yz+xz}\right)p^2$

$$= \frac{(yz)\times \cancel{(y+z)}}{\cancel{(y+z)}(xy+yz+xz)}p^2$$

$$= \frac{xyz\,p^2}{xy+yz+xz}$$

$J = (4\pi)^{-D} t^{-D} (xy+yz+xz)^{-\frac{D}{2}}$, $\Gamma(4) = 3! = 6$

$I = \cancel{\frac{1}{6}\int}$

$\cancel{I = \frac{1}{6}}$ $I = \frac{1}{6}(4\pi)^{-D}(xy+yz+xz)^{-D/2}\int_0^\infty dt\, t^{3-D} e^{-ts}$

where $\cancel{\boxed{S = \frac{xyzp^2}{xy+yz+xz}}}$ $\boxed{S = \frac{xyzp^2}{xy+yz+xz} + m^2(x+y+z)}$.

let $u = ts$ $du = S\,dt$ $dt = \frac{du}{S}$. $t = \frac{u}{S}$.

$t^{3-D} = S^{D-3} u^{3-D}$

$\therefore \int_0^\infty dt\, t^{3-D} e^{-ts} = \cancel{S^{D-3}}\, S^{D-4}\int_0^\infty du\, u^{3-D} e^{-u}$

Also $\frac{dD}{dp^2} = \frac{yz}{y+z}\left(1 - \frac{yz}{xy+yz+xz}\right) = \frac{xyz}{xy+yz+xz}$

$\int_0^\infty du\, u^{3-D} e^{-u} = \int_0^\infty du\, u^{(4-D)-1} e^{-u}$.

$= \Gamma(4-D)$

$\therefore I = \frac{1}{6}(4\pi)^{-D}(\cancel{xy}+yz+xz)^{-D/2} S^{D-4}\,\Gamma(4-D)$.

Also $c = \frac{\lambda_0^2}{3}$ $(-3)c = -\lambda_0^2$

$\frac{dA}{dp^2} = (-3)c\int_0^1 dx\int_0^1 dy\int_0^1 dz\, \delta(x+y+z-1) \times I \times \frac{dD}{dp^2}$

$= -\lambda_0^2\int_0^1 dx\int_0^1 dy\int_0^1 dz\, \delta(x+y+z-1) \times I \cdot \frac{dD}{dp^2}$.

$$= -\lambda_0^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \delta(x+y+z-1) \quad \times$$

$$\frac{1}{6}(4\pi)^{-D} \, (xy+yz+xz)^{-D/2} \, S^{D-4} \, \Gamma(4-D). \quad \times$$

$$\left(\frac{xyz}{xy+yz+xz} \right).$$

Now $\quad D = 4-2\varepsilon \qquad 4-D = 2\varepsilon, \; D-4 = -2\varepsilon$

$$-D = -4+2\varepsilon \qquad -\frac{D}{2} = -2+\varepsilon$$

$$\therefore \frac{dA}{dp^2} = -\frac{\lambda_0^2}{6} \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \delta(x+y+z-1) \frac{xyz}{xy+yz+xz} \times$$

$$(4\pi)^{-4} (4\pi)^{2\varepsilon} (xy+yz+xz)^{-2} (xy+yz+xz)^{+\varepsilon}$$

$$S^{-2\varepsilon} \, \Gamma(2\varepsilon)$$

$$= -\frac{\lambda_0^2}{6(4\pi)^4} \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \delta(x+y+z-1) \frac{xyz}{(xy+yz+xz)^3} \times$$

$$\Big[\left(1 + 2\varepsilon \log(4\pi) + O(\varepsilon^2)\right) \times \left(1 + \varepsilon \log(xy+yz+xz) + O(\varepsilon^2)\right)$$

$$\times \left(1 - 2\varepsilon \log(S) + O(\varepsilon^2)\right) \times \left(\frac{1}{2\varepsilon} - \gamma_E + O(\varepsilon)\right) \Big].$$

$$= \frac{-\lambda_0^2}{6(4\pi)^4} \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \delta(x+y+z-1) \frac{xyz}{(xy+yz+xz)^3}$$

$$\times \left[\frac{1}{2\varepsilon} - \gamma_E + \log(4\pi) + \frac{1}{2} \log(xy+yz+xz) \right.$$

$$\left. - \log\left(m^2 \left[\frac{xyz}{xy+yz+xz} \frac{p^2}{m^2} + x+y+z \right] \right) \right]$$

$$= -\frac{\lambda_0^2}{24(4\pi)^4} \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \delta(x+y+z-1) \frac{xyz}{(xy+yz+xz)^3}$$

$$\times \left[\frac{2}{\varepsilon} - 4\gamma_E + 4\log(4\pi) - 4\log(m^2) \right.$$

$$\left. + 2\log(xy+yz+xz) - 4\log\left[\frac{xyz}{xy+yz+xz} \frac{p^2}{m^2} + x+y+z \right] \right]$$

$$= \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \delta(x+y+z-1) \underbrace{\left[-\frac{\lambda_0^2}{24(4\pi)^4} \frac{xyz}{(xy+yz+xz)^3} \right]}_{F(x,y,z)} \times$$

$$\left[\frac{2}{\varepsilon} - 4\gamma_E + \underbrace{4\log(4\pi) - 4\log(m^2)}_{\text{const.}} \right.$$

$$\left. + \underbrace{\log\left[\frac{(xy+yz+xz)^2}{\left(\frac{xyz}{xy+yz+xz} \frac{p^2}{m^2} + (x+y+z) \right)^4} \right]}_{G\left(x,y,z,\frac{p^2}{m^2}\right)} \right] \quad \square$$

good

(d) when $p^2 = m^2$ in Minkowski space,

$p^2 = -m^2 \Rightarrow \frac{p^2}{m^2} = -1$ in Euclidean space

after wick rotation.

$$\therefore G(x, y, z, \frac{p^2}{m^2} = -1) = \frac{\cancel{xy}\,(xy + yz + xz)^2}{\left(\frac{-xyz}{xy+yz+xz} + x+y+z\right)^4}$$

$$= \frac{(xy+yz+xz)^6}{((x+y+z)(xy+yz+xz) - xyz)^4}$$

Because of the presence of $\delta(x+y+z-1)$

$\therefore$ effectively we can substitute 1 for $x+y+z$

$$\therefore G = \left[\frac{(xy+yz+xz)^3}{(xy+yz+xz-xyz)^2}\right]^2 \Rightarrow$$

$$\therefore \frac{dA}{dp^2}\bigg|_{p^2=m^2} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left\{\left[\delta(x+y+z-1)\left(\frac{-\lambda_0^2}{6144\pi^4}\right)\right.\right.$$

$$\left.\left.\left(\frac{xyz}{(xy+yz+xz)^3}\right)\left[\frac{2}{\varepsilon} - 4\gamma_E + 4\log(4\pi) - 4\log(m^2)\right]\right]\right.$$

$$+ \left[\delta(x+y+z-1)\left(-\frac{\lambda_0^2}{6144\pi^4}\right)\left[\frac{xyz}{(xy+yz+xz)^3} \times\right.\right.$$

$$\left.\left.\left.2\log\left(\frac{(xy+yz+xz)^3}{(xy+yz+xz-xyz)^2}\right)\right]\right\}\right\}$$

$$\therefore \left.\frac{dA}{dp^2}\right|_{p^2=m^2} = \text{first}\{\quad\} + \text{second}\{\quad\}$$

$$= -\frac{\lambda_0^2}{6144\pi^2}\left[\left(\frac{1}{2}\right)\left(\frac{2}{\varepsilon} - 4\gamma_E + 4\log(4\pi) - 4\log(m^2)\right).\right.$$

$$\left.\left(-\frac{3}{4}\right)(2)\right]$$

$$= -\frac{\lambda_0^2}{6144\pi^2}\left(\frac{1}{\varepsilon} - 2\gamma_E + 2\log(4\pi) + 2\log\left(\frac{1}{m^2}\right)\right.$$

$$\left. - \frac{3}{2} + O(\varepsilon)\right).$$

Now, like question ⑨, we introduce

g_0 such that $[g_0] = 0$ is dimensionless. ~~renormalised~~
coupling constant

$$[\lambda_0] = 4 - D = 4 - (4 - 2\varepsilon) = 2\varepsilon \quad , \text{ with } [\mu] = 1 = [m]$$

$$\Rightarrow \quad g_0 = \mu^{-2\varepsilon}\lambda_0$$

$$\therefore \lambda_0^2 = g_0^2 \mu^{4\varepsilon}$$

for $\varepsilon \to 0 \quad D \to 4$

$$\mu^{4\varepsilon} \simeq 1 + 4\varepsilon\log\mu = 1 + 2\varepsilon\log\mu^2$$

$$\therefore \left.\frac{dA}{dp^2}\right|_{p^2=m^2} = -\frac{g_0^2}{6144\pi^4}\left(1 + 2\varepsilon\log\mu^2\right) \times$$

$$\left[\frac{1}{\varepsilon} - 2\gamma_E + 2\log(4\pi) + 2\log\left(\frac{1}{m^2}\right) - \frac{3}{2} + O(\varepsilon)\right)$$

$$= \frac{-g_0^2}{6144\pi^4}\left(\frac{1}{\varepsilon} + 2\log\mu^2 - 2\log\left(\frac{1}{m^2}\right) + 2\log(4\pi)\right.$$

$$\left. -2\gamma_E - \frac{3}{2} + O(\varepsilon)\right)$$

$$= -\frac{g_0^2}{6144\pi^4}\left(\frac{1}{\varepsilon} + 2\log\left(\frac{\mu^2}{m^2}\right) + 2\log(4\pi) - 2\gamma_E - \frac{3}{2}\right.$$

$$\left. + O(\varepsilon)\right) \quad \square$$

(e)

$$Z_\phi^{-1} = \frac{1}{1 + \left.\frac{dA(p^2)}{dp^2}\right|_{p^2=m^2}}$$

$$\therefore Z_\phi = 1 + \left.\frac{dA}{dp^2}\right|_{p^2=m^2}$$

$$\gamma(g_0) = \mu\frac{\partial}{\partial\mu}\log(Z_\phi) = \frac{\mu}{Z_\phi}\frac{\partial}{\partial\mu}Z_\phi$$

$$= \mu Z_\phi^{-1}\frac{\partial}{\partial\mu}Z_\phi$$

$$\frac{\partial}{\partial\mu}Z_\phi = \frac{\partial}{\partial\mu}\left(1 + \left.\frac{dA}{dp^2}\right|_{p^2=m^2}\right) = \frac{\partial}{\partial\mu}\left(\left.\frac{dA}{dp^2}\right|_{p^2=m^2}\right)$$

$$= \frac{\partial}{\partial \mu}\left[-\frac{g_0^2}{6144\pi^4}\left(\frac{1}{\varepsilon} + 2\log\frac{\mu^2}{m^2} + 2\log(4\pi) - 2\gamma_E - \frac{3}{2} + O(\varepsilon)\right)\right]$$

$$= \frac{\partial}{\partial \mu}\left(-\frac{g_0^2}{6144\pi^4} \times 2\left(\log\mu^2 - \log m^2\right)\right)$$

$$= \frac{\partial}{\partial \mu}\left(-\frac{g_0^2}{6144\pi^4} \times 4\log\mu\right)$$

$$= \frac{-4g_0^2}{6144\pi^4}\frac{\partial}{\partial \mu}\log\mu = -\frac{4g_0^2}{6144\pi^4\mu}$$

$$\gamma(g_0) = \mu Z_\phi^{-1}\frac{\partial}{\partial\mu}Z_\phi$$

$$= \cancel{\mu}\left(-\frac{4g_0^2}{6144\pi^4\cancel{\mu}}\right) Z_\phi^{-1}$$

$$= -\frac{4g_0^2}{6144\pi^4}\frac{1}{1 - \frac{g_0^2}{6144\pi^4}\left(\frac{1}{\varepsilon} + 2\log\left(\frac{4\pi\mu^2}{m^2}\right) - 2\gamma_E + \frac{3}{2} + O(\varepsilon)\right)}$$

$$= \frac{4\left(\frac{g_0^2}{6144\pi^4}\right)}{\left(\frac{g_0^2}{6144\pi^4}\right)\frac{1}{\varepsilon}\left[1 + \varepsilon\frac{6144\pi^4}{g_0^2}2\log\left(\frac{4\pi\mu^2}{m^2}\right) - \varepsilon\frac{6144\pi^4}{g_0^2}2\gamma_E - \varepsilon\frac{6144\pi^4}{g_0^2} + \varepsilon\frac{6144\pi^4}{g_0^2}\cdot\frac{3}{2} + O(\varepsilon^2)\right]}$$

$$= \frac{4\varepsilon}{1 + O(\varepsilon)}$$

$$= 4\varepsilon\left[1+O(\varepsilon)\right]^{-1}$$

$$= 4\varepsilon\left(1-O(\varepsilon)\right) = 4\varepsilon\left(1+O(\varepsilon)\right)$$

$$\simeq 4\varepsilon + O(\varepsilon^2)$$

is the ~~anatda~~ anomalous dimension

(e)

$$\gamma(g_0) = \mu\frac{\partial}{\partial\mu}\log Z_\phi = \mu Z_\phi^{-1}\frac{\partial}{\partial\mu}Z_\phi \quad \text{with}$$

$$Z_\phi = \frac{1}{1+\frac{dA}{dp^2}\Big|_{p^2=m^2}} = \left(1-\frac{g_0^2}{6144\pi^4}\left(\frac{1}{\varepsilon}+2\log\left(\frac{4\pi\mu^2}{m^2}\right)-2\gamma_E-\frac{3}{2}+O(\varepsilon)\right)\right)^{-1}$$

$$\mu\frac{\partial}{\partial\mu}Z_\phi = -\underbrace{\left(1-\frac{g_0^2}{6144\pi^4}\left(\frac{1}{\varepsilon}+2\log\left(\frac{4\pi\mu^2}{m^2}\right)-2\gamma_E+\frac{3}{2}\right)\right)^{-2}}_{Z_\phi^2}\left(\frac{-g_0}{6144\pi^4}\right)\frac{4}{\mu}\cancel{\alpha\mu} = +\frac{4g_0^2}{6144\pi^4}Z_\phi^2$$

$$Z_\phi^{-1} = 1-\frac{g_0^2}{6144}\left(\frac{1}{\varepsilon}+2\log\left(\frac{4\pi\mu^2}{m^2}\right)-2\gamma_E+\frac{3}{2}\right)$$

$$\gamma(g_0) = Z_\phi^{-1}\left(\mu\frac{\partial}{\partial\mu}Z_\phi\right)$$

$$= Z_\phi^{-1}\left(\frac{4g_0}{6144\pi^4}\right)Z_\phi^2 = \frac{4g_0}{6144\pi^4}Z_\phi$$

$$= \frac{4g_0^2}{6144\pi^4}\,\frac{1}{1-}$$

(e) $\gamma(g_0) = \mu \frac{\partial}{\partial \mu} \log Z_\phi = Z_\phi^{-1} \left(\mu \frac{\partial}{\partial \mu} Z_\phi \right)$

$$Z_\phi = \frac{1}{1 + \frac{dA}{dp^2}\Big|_{p^2 = m^2}} = \left(1 - \frac{g_0^2}{6144\pi^2}\left(\frac{1}{\varepsilon} + 2\log\left(\frac{4\pi\mu^2}{m^2}\right) - 2\gamma_E - \frac{3}{2} + O(\varepsilon)\right)\right)^{-1}$$

$$\mu \frac{\partial}{\partial \mu} Z_\phi = -\left(1 - \frac{g_0^2}{6144\pi^2}\left(\frac{1}{\varepsilon} + 2\log\left(\frac{4\pi\mu^2}{m^2}\right) - \gamma_E - \frac{3}{2} + O(\varepsilon)\right)\right)^{-2}$$

$$\times \left(-\frac{g_0}{6144\pi^4}\right) \frac{4}{\cancel{\mu}} \cdot \cancel{\mu} = Z_\phi^2 \frac{4 g_0^2}{6144\pi^2}$$

$$\therefore \gamma(g_0) = \cancel{Z_\phi^{-1}} \cancel{Z_\phi^2} \frac{4 g_0^2}{6144\pi^2}$$

$$= \frac{4 g_0^2}{6144\pi^2} \Bigg(\underbrace{\frac{1}{1 + O(g_0^2)}}_{Z_\phi} \Bigg)$$

$$\simeq \frac{4 g_0^2}{6144\pi^2} + O(g_0^4)$$

$$\simeq \frac{1}{6}\left(\frac{g_0}{16\pi^2}\right)^2 + O(g_0^4)$$

Same as the one in lecture notes.

good!