Quantum Field Theory

2017 Exam.

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(1) $\mathcal{L} = (\partial_\nu \phi^*)(\partial^\nu \phi) - f(\phi^* \phi)$

(Minkowski)

(a) Euler Lagrange equation

ϕ: $\quad \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{df}{d|\phi|^2} \frac{\partial |\phi|^2}{\partial \phi} = -f' \phi^*$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} = \partial^\nu \phi^* \qquad f' = \frac{df}{d|\phi|^2}.$$

$$\therefore \partial_\nu \partial^\nu \phi^* = -f' \phi^*$$

$$\therefore \underline{(\partial_\nu \partial^\nu + f') \phi^* = 0}$$

Similarly for ϕ^*:

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -f' \phi$$

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} \right) = \partial_\nu \partial^\nu \phi \quad \Rightarrow \underline{(\partial_\nu \partial^\nu + f') \phi = 0}$$

(b) $\phi \to \phi' = e^{i\alpha} \phi \quad , \phi^* \to \phi'^* = e^{-i\alpha} \phi^*$

$$\mathcal{L} \to \mathcal{L}' = \partial_\nu \phi'^* \partial^\nu \phi' - f(\phi'^* \phi')$$

$$= e^{i\alpha} e^{-i\alpha} \partial_\nu \phi^* \partial^\nu \phi - f(e^{i\alpha} e^{-i\alpha} \phi^* \phi)$$

$$= \partial_\nu \phi^* \partial^\nu \phi - f(\phi^* \phi) = \mathcal{L}$$

$\therefore$ Lagrangian invariant. , $\delta \mathcal{L} = 0$

$\Rightarrow \tilde{J}^\mu = 0.$

~~#~~

Noether current $J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta\phi^*$

Infinitesimal:

$$\phi' = e^{i\alpha}\phi = (1+i\alpha)\phi = \phi + \alpha \underbrace{(i\phi)}_{\delta\phi.}$$

$$\phi^{*\prime} = e^{-i\alpha}\phi^* = (1-i\alpha)\phi^* = \phi^* + \alpha\underbrace{(-i\phi^*)}_{\delta\phi^*}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^* \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi$$

$$J^\mu = (\partial^\mu \phi^*)\delta\phi + (\partial^\mu\phi)\delta\phi^*$$

$$= i(\partial^\mu\phi^*)\phi - i\phi^*(\partial^\mu\phi)$$

(c) $\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(b_{\vec{p}} e^{-ip\cdot x} + c_{\vec{p}}^\dagger e^{ip\cdot x}\right)\Big|_{p_0 = \omega_{\vec{p}}}$

$$[b_{\vec{p}}, b_{\vec{q}}^\dagger] = [c_{\vec{p}}, c_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

$$\phi^*(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(b_{\vec{p}}^\dagger e^{ip\cdot x} + c_{\vec{p}} e^{-ip\cdot x}\right)\Big|_{p_0 = \omega_{\vec{p}}}$$

$$J^0 = i(\partial^0\phi^*)\phi - i\phi^*(\partial^0\phi) = i\dot{\phi}^*\phi - i\phi^*\dot{\phi}$$

$$\therefore \quad Q = \int d^3\vec{x}\, J^0 = \int d^3\vec{x} \left[i\dot{\phi}^*\phi - i\phi^*\dot{\phi} \right]$$

$$= \int d^3\vec{x} \Big[i\Big(\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(+i\omega_{\vec{p}}\, b_{\vec{p}}^\dagger e^{+ip\cdot x} - i\omega_{\vec{p}}\, c_{\vec{p}}\, e^{-ip\cdot x} \right) \Big) (\phi)$$

$$- i(\phi^*) \Big(\int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{q}}}} \left(-i\omega_{\vec{q}}\, b_{\vec{q}}\, e^{-iq\cdot x} + i\omega_{\vec{q}}\, c_{\vec{q}}^\dagger e^{iq\cdot x} \right) \Big).$$

$$= i\int d^3\vec{x} \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} \Big[\Big(i\sqrt{\frac{\omega_{\vec{p}}}{2}} \Big) \Big(\frac{1}{\sqrt{2\omega_{\vec{q}}}} \Big) \left(b_{\vec{p}}^\dagger e^{ip\cdot x} - c_{\vec{p}}\, e^{-ip\cdot x} \right) \left(b_{\vec{q}}\, e^{-iq\cdot x} \right.$$

$$\left. + c_{\vec{q}}^\dagger e^{iq\cdot x} \right) - \Big(\frac{-i}{2} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \Big) \left(b_{\vec{p}}^\dagger e^{ip\cdot x} + c_{\vec{p}}\, e^{-ip\cdot x} \right)$$

$$\left(b_{\vec{q}}\, e^{-iq\cdot x} - c_{\vec{q}}^\dagger e^{iq\cdot x} \right) \Big]$$

$$= -\frac{1}{2} \int \frac{d^3\vec{x}\, d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} \Big[\Big(\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}} \Big) \Big(b_{\vec{p}}^\dagger b_{\vec{q}}\, e^{ix(p-q)}$$

$$+ \cancel{b_{\vec{p}}^\dagger c_{\vec{q}}^\dagger e^{ix(p+q)}} - \cancel{c_{\vec{p}}\, b_{\vec{q}}\, e^{-ix(p+q)}}$$

$$- c_{\vec{p}}\, c_{\vec{q}}^\dagger e^{-ix(p-q)} \Big) + \Big(\sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \Big) \Big(b_{\vec{p}}^\dagger b_{\vec{q}}\, e^{ix(p-q)}$$

$$- \cancel{b_{\vec{p}}^\dagger c_{\vec{q}}^\dagger e^{ix(p+q)}} + \cancel{c_{\vec{p}}\, b_{\vec{q}}\, e^{-ix(p+q)}}$$

$$- c_{\vec{p}}\, c_{\vec{q}}^\dagger e^{-ix(p-q)} \Big)$$

$$\left(\omega_{\vec{p}} = \omega_{-\vec{p}}, \quad \omega_{\vec{q}} = \omega_{-\vec{q}} \quad \because |\vec{p}|^2 = |-\vec{p}|^2, \ |\vec{q}|^2 = |-\vec{q}|^2 \right).$$

$$\cancel{\left(\int \frac{d^3 x}{(2\pi)^3} e^{\pm ix(p-q)} = e^{it(\omega_{\vec p}-\omega_{\vec q})}\right.}$$

$$\left(\int \frac{d^3x}{(2\pi)^3} e^{\pm ix(p-q)} = e^{it(\omega_{\vec p}-\omega_{\vec q})} \delta^{(3)}(\vec p - \vec q) \right)$$

$$Q = -\frac{1}{2} \int \frac{d^3\vec p\, d^3\vec q}{(2\pi)^3} \delta^{(3)}(\vec p - \vec q) \left(\sqrt{\frac{\omega_{\vec p}}{\omega_{\vec q}}} + \sqrt{\frac{\omega_{\vec q}}{\omega_{\vec p}}} \right) \left(b_{\vec p}^\dagger b_{\vec q} - c_{\vec p} c_{\vec q}^\dagger \right)$$

$$= \int \frac{d^3\vec p}{(2\pi)^3} \left(c_{\vec p} c_{\vec p}^\dagger - b_{\vec p}^\dagger b_{\vec p} \right)$$

$$= \int \frac{d^3\vec p}{(2\pi)^3} \left(c_{\vec p}^\dagger c_{\vec p} - b_{\vec p}^\dagger b_{\vec p} \right).$$

Normal ordering

= Number of ~~anti~~ particles

− Number of ~~particles~~ antiparticles

(d) $$[Q, \phi] = \int \frac{d^3\vec p\, d^3\vec q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_{\vec q}}} \Big[c_{\vec p}^\dagger c_{\vec p} - b_{\vec p}^\dagger b_{\vec p},$$

$$b_{\vec q} e^{-iq\cdot x} + c_{\vec q}^\dagger e^{iq\cdot x} \Big]$$

$$= \int \frac{d^3\vec p\, d^3\vec q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_{\vec q}}} \left([c_{\vec p}^\dagger c_{\vec p}, c_{\vec q}^\dagger e^{iq x}] - [b_{\vec p}^\dagger b_{\vec p}, b_{\vec q} e^{-iq\cdot x}] \right)$$

$$= \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} \frac{1}{\sqrt{2\omega_{\vec{q}}}} \Big(c_{\vec{p}}^{\dagger} \overbrace{[c_{\vec{p}}, c_{\vec{q}}^{\dagger}]}^{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})} e^{iq\cdot x}$$

$$- \underbrace{[b_{\vec{p}}^{\dagger}, b_{\vec{q}}]}_{-(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})} e^{-iq\cdot x} b_{\vec{p}} \Big)$$

$$= \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^{\cancel{6}\,3}} \frac{1}{\sqrt{2\omega_{\vec{q}}}} \cancel{(2\pi)^3} \delta^{(3)}(\vec{p}-\vec{q}) \left(c_{\vec{p}}^{\dagger} e^{iq\cdot x} + b_{\vec{p}} e^{-iq\cdot x} \right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(b_{\vec{p}} e^{-ip\cdot x} + c_{\vec{p}}^{\dagger} e^{ip\cdot x} \right)$$

$$= \phi$$

$\therefore [Q, \phi] = \phi$

$\therefore$ ~~$[Q,\phi]$~~ if $\underline{Q|q\rangle = q|q\rangle}$

$[Q, \phi]|q\rangle = \phi|q\rangle$

$\Rightarrow \phi|q\rangle = Q\phi|q\rangle - \phi Q|q\rangle = Q\phi|q\rangle - q\phi|q\rangle.$

$\therefore Q\phi|q\rangle = (q+1)\phi|q\rangle$

$\therefore$ $\phi|q\rangle$ is the eigenstate of Q with eigenvalue $\underline{\underline{q+1}}$.

$\therefore$ ϕ add a particle to a state of q number of particles

(q = # particle − # antiparticle).

②

$$S = \int d^dx \left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\kappa}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 \right]$$

(Minkowski)

(a) $[x] = -1 \quad [m] = 1 \quad [\partial_\nu] = 1 \quad [dx] = -1$

$\therefore [d^dx] = -d \qquad \because [S] = 0$

$\therefore \quad -d + 2 + 2[\phi] = 0$

$\therefore [\phi] = \dfrac{d-2}{2}$

$[\kappa] + 3[\phi] - d = 0 \qquad \therefore [\kappa] = d - 3\dfrac{d-2}{2} = 3 - \dfrac{d}{2}$

$[\lambda] + 4[\phi] - d = 0 \qquad \therefore [\lambda] = d - 4\dfrac{d-2}{2} = 4 - d$

critical dimension :

$[\kappa] = 0 \Rightarrow D_c(\kappa) = 6,$

$[\lambda] = 0 \Rightarrow D_c(\lambda) = 4,$

$\therefore$ No critical dimension such that both $[\kappa]$ and $[\lambda]$ are 0.

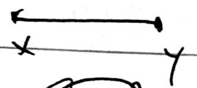
(b) Two point function $\langle \phi(x)\phi(y) \rangle$

connected diagrams :

$$\langle \phi(x)\phi(y)\rangle = \frac{\int D\phi\, \phi(x)\phi(y)\, e^{iS}}{\int D\phi\, e^{iS}}$$

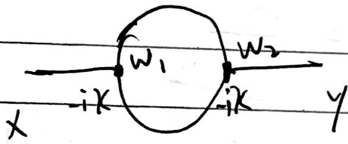
$$= \int D\phi\, \phi(x)\phi(y)\, e^{iS} \Big|_{\text{connected}}$$

1-loop connected diagrams :



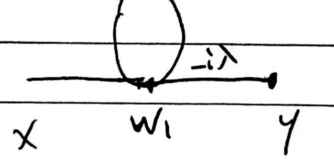
$\longrightarrow$ $\Delta_F(x-y)$

Wick Theorem

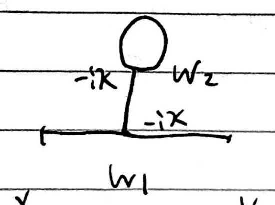


$$\frac{(-i\lambda)^2}{2}\int d^Dw_1\, d^Dw_2\, \Delta_F(x-w_1)\, \Delta_F^2(w_1-w_2)\, \Delta_F(w_2-y)$$

symmetry factor

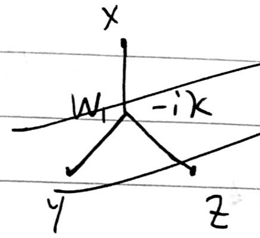


$$\frac{(-i\lambda)}{2}\int d^Dw_1\, \Delta_F(x-w_1) \times \Delta_F(w_1-w_1) \times \Delta_F(w_1-y)$$

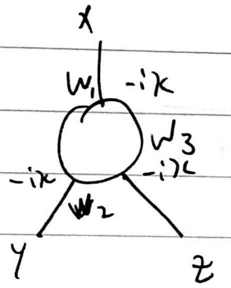


$$\frac{(-i\lambda)^2}{2}\int d^Dw_1\, d^Dw_2\, \Delta_F(x-w_1) \times \Delta_F(w_1-w_2) \times \Delta_F(w_2-w_2) \times \Delta_F(w_1-y)$$

(c) $\langle \phi(x)\, \phi(y)\, \phi(z) \rangle$



$$\cancel{(-i\lambda)\int d^D w_1\, \Delta_F(x-w_1)\, \Delta_F(y-w_1)\, \Delta_F(z-w_1)}$$



$$(-i\lambda)^3 \int d^D w_1\, d^D w_2\, d^D w_3\, \Delta_F(x-w_1)\, \Delta_F(w_1-w_2)$$

$$\times \Delta_F(y-w_2) \times \Delta_F(w_2-w_3) \times \Delta_F(z-w_3)$$

$$\times \Delta_F(w_1-w_3)$$

$$\Delta_F(x-y) = \int \frac{d^Dp}{(2\pi)^D} \frac{i}{p^2-m^2+i\varepsilon} \times e^{-ip(x-y)}$$

$$= (-i\lambda)^3 \int d^D w_1\, d^D w_2\, d^D w_3 \int \frac{d^D p_1}{(2\pi)^D} \frac{i}{p_1^2-m^2+i\varepsilon} e^{-ip_1(x-w_1)}$$

$$\int \frac{d^D p_2}{(2\pi)^D} \frac{i}{p_2^2-m^2+i\varepsilon} e^{-ip_2(w_1-w_2)} \int \frac{d^D p_3}{(2\pi)^D} \frac{i}{p_3^2-m^2+i\varepsilon} e^{-ip_3(y-w_2)}$$

$$\int \frac{d^D p_4}{(2\pi)^D} \frac{i}{p_4^2-m^2+i\varepsilon} e^{-ip_4(w_2-w_3)} \int \frac{d^D p_5}{(2\pi)^D} \frac{i}{p_5^2-m^2+i\varepsilon} e^{-ip_5(z-w_3)}$$

$$\int \frac{d^D p_6}{(2\pi)^D} \frac{i}{p_6^2-m^2+i\varepsilon} e^{-ip_6(w_1-w_3)}$$

$$= (-i\lambda)^3 \int d^D w_1\, e^{ip_1 w_1} e^{-ip_2 w_1} e^{-ip_6 w_1} \int d^D w_2\, e^{ip_2 w_2} e^{ip_3 w_2} e^{-ip_4 w_2}$$

$$\int d^D w_3\, e^{ip_4 w_3} e^{ip_5 w_3} e^{-ip_6 w_3}$$

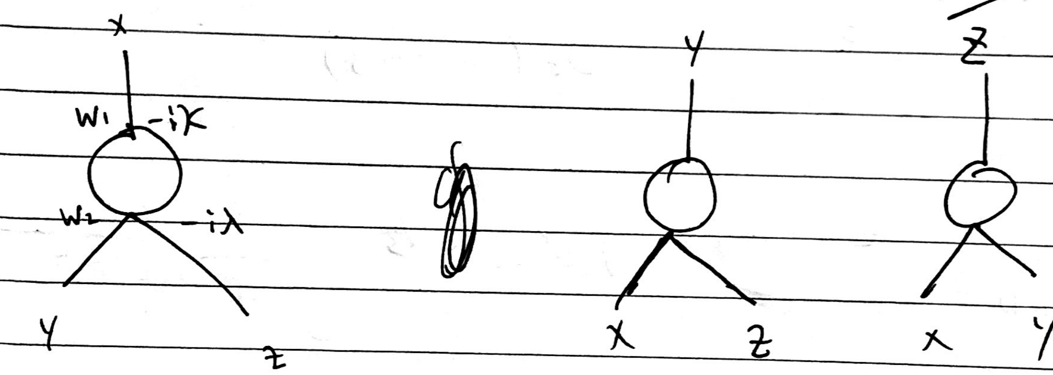
$$\int \frac{d^D p_1 \cdots d^D p_6}{(2\pi)^{6D}} e^{-ip_1 x} e^{-ip_3 y} e^{-ip_5 z} \frac{i}{p_1^2-m^2+i\varepsilon} \cdots \frac{i}{p_6^2-m^2+i\varepsilon}$$

$$= \cancel{(-i\lambda)^3 (2\pi)^{3D} \delta^D(p_1-p_2-p_6)\, \delta^D(p_2+p_3}$$

$$= (-i\lambda)^3 (2\pi)^{3D} \int \frac{d^Dp_1 \cdots d^Dp_6}{(2\pi)^{6D}} \delta^D(p_1-p_2-p_6)\, \delta^D(p_2+p_3-p_4)$$

$$\delta^D(p_4+p_5-p_6)\, e^{-ip_1x} e^{-ip_3y} e^{-ip_5z} \times$$

$$\frac{i}{p_1^2-m^2+i\varepsilon} \frac{i}{p_2^2-m^2+i\varepsilon} \cdots \frac{i}{p_6^2-m^2+i\varepsilon}$$



3 diagrams in ~~to~~ total

↓

$$\frac{3(-i\lambda)(-i\lambda)}{2} \int d^Dw_1\, d^Dw_2\, \Delta_F(x-w_1)\, \Delta_F^2(w_1-w_2)\, \Delta_F(w_2-y)$$

$$\Delta_F(w_2-z)$$

✓ symmetry factor

$$= \frac{3(-i\lambda)(-i\lambda)}{2} \int d^Dw_1\, d^Dw_2 \int \frac{d^Dp_1}{(2\pi)^D} \frac{i}{p_1^2-m^2+i\varepsilon} e^{-ip_1(x-w_1)}$$

$$\int \frac{d^Dp_2}{(2\pi)^D} \frac{i}{p_2^2-m^2+i\varepsilon} e^{-ip_2(w_1-w_2)} \int \frac{d^Dp_3}{(2\pi)^D} \frac{i}{p_3^2-m^2+i\varepsilon} e^{-ip_3(w_1-w_2)}$$

$$\int \frac{d^Dp_4}{(2\pi)^D} \frac{i}{p_4^2-m^2+i\varepsilon} e^{-ip_4(w_2-y)} \int \frac{d^Dp_5}{(2\pi)^D} \frac{i}{p_5^2-m^2+i\varepsilon} e^{-ip_5(w_2-z)}$$

$$= \cancel{-3}\ \frac{3(-i\kappa)(-i\lambda)}{2}\int d^D w_1\, e^{ip_1w_1}\, e^{-ip_2w_1}\, e^{-ip_3w_1}$$

$$\int d^D w_2\, e^{ip_2w_2}\, e^{ip_3w_2}\, e^{-ip_4w_2}\, e^{-ip_5w_2}$$

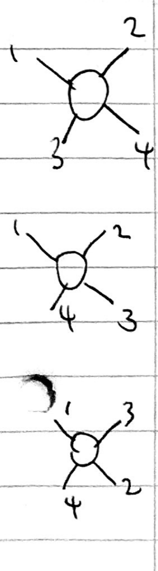
$$\int \frac{d^Dp_1\cdots d^Dp_5}{(2\pi)^{5D}}\ \frac{i}{p_1^2-m^2+i\epsilon}\cdots\frac{i}{p_5^2-m^2+i\epsilon}\ e^{-ip_1x}\, e^{ip_4y}\, e^{ip_5z}$$

$$= \frac{3(-i\kappa)(-i\lambda)}{2}\,(2\pi)^{2D}\int\frac{d^Dp_1\cdots d^Dp_5}{(2\pi)^{5D}}\ \delta^{(D)}(p_1-p_2-p_3)\ \times$$

$$\delta^{(D)}(p_2+p_3-p_4-p_5)\ e^{-ip_1x}\, e^{ip_4y}\, e^{ip_5z}\ \times$$

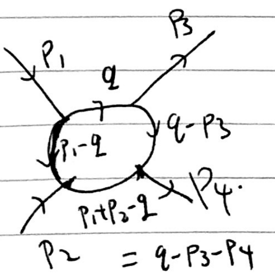
$$\frac{i}{p_1^2-m^2+i\epsilon}\ \frac{i}{p_2^2-m^2+i\epsilon}\cdots\frac{i}{p_5^2-m^2+i\epsilon}$$

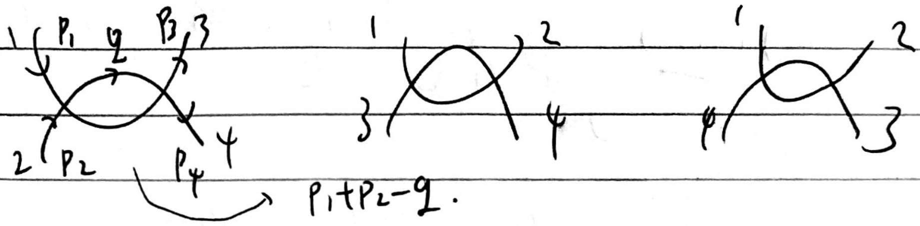
(d) ~~out~~ one loop contributions to $\Gamma^{(4)}(p_1,p_2,p_3,p_4)$ are the "loop part" of all one particle irreducible connected 1-loop diagrams.



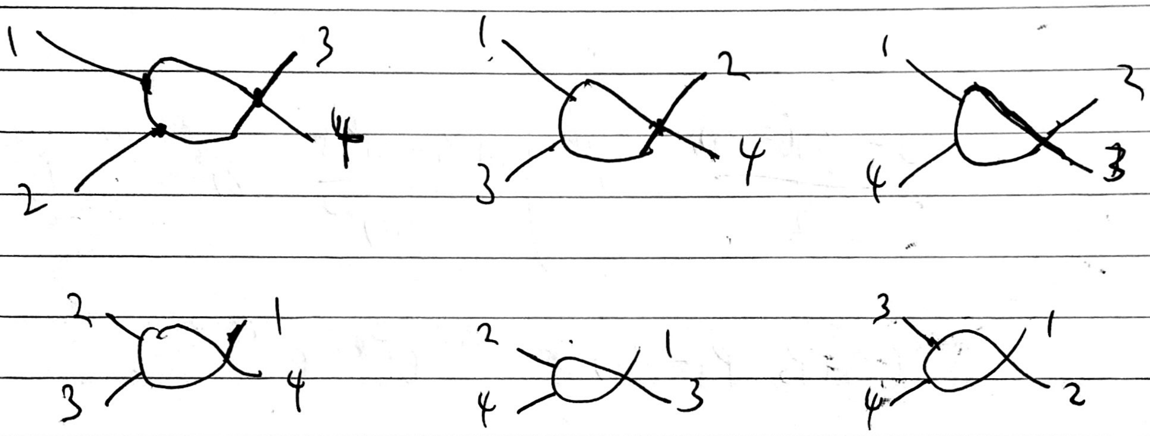
$$\therefore\ 3\times\frac{(-i\kappa)^4}{1}\int\frac{d^Dq}{(2\pi)^D}\ \frac{i}{q^2-m^2+i\epsilon}\ \frac{i}{(p_1-q)^2-m^2+i\epsilon}\ \frac{i}{(q-p_3)^2-m^2+i\epsilon}$$

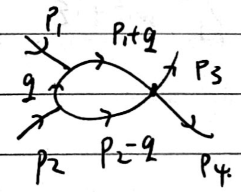
$$\frac{i}{(p_1+p_2-q)^2-m^2+i\epsilon}$$





$$\frac{3(-i\lambda)^2}{2}\int\frac{d^Dq}{(2\pi)^D}\frac{i}{q^2-m^2+i\varepsilon}\frac{i}{(p_1+p_2-q)^2-m^2+i\varepsilon}$$





$$\frac{6(-i\kappa)(-i\lambda)}{1}\int\frac{d^Dq}{(2\pi)^D}\frac{i}{(p_1+q)^2-m^2+i\varepsilon}\frac{i}{(p_2-q)^2-m^2+i\varepsilon}\frac{i}{q^2-m^2+i\varepsilon}$$

(3)

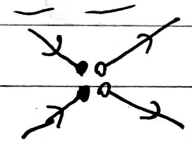
$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m_0^2 \phi^* \phi - \frac{\lambda_0}{4}(\phi^* \phi)^2$$

(~~Ab~~ Minkowski).

(a)

Feynman Rules (momentum space) :

• ϕ ∘ ϕ^*

-  $-i\lambda_0$ vertex.

-  $\frac{i}{p^2 - m^2 + i\varepsilon}$ propagator

$\because \langle \varphi(x) \varphi(y) \rangle_0 = 0$ and $\langle \phi(x) \phi^*(y) \rangle = \Delta_F(x-y)$

$\therefore$ ~~only~~ propagator can only connect ϕ and ϕ^* (not $\phi - \phi$ or $\phi^* - \phi^*$).

- integrate loop momenta
- divide by symmetry factors.

(b) $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ $\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$.

$\therefore$ $\phi^* \phi = \frac{1}{2}(\phi_1 + i\phi_2)(\phi_1 - i\phi_2)$

$$= \frac{1}{2}(\phi_1^2 + \phi_2^2)$$

$$\partial_\mu \phi^* \partial^\mu \phi = \frac{1}{2}[\partial_\mu(\phi_1 - i\phi_2)][\partial^\mu(\phi_1 + i\phi_2)]$$

$$= \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2}\partial_\mu \phi^2 \partial^\mu \phi^2$$

$$\therefore \mathcal{L} = \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2}m_0^2 \phi_1^2$$

$$+ \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2}m_0^2 \phi_2^2$$

$$- \frac{\lambda_0}{4^2}(\phi_1^2 + \phi_2^2)^2$$

$$= \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2}m_0^2 \phi_1^2 - \frac{(3\lambda_0/2)}{4!}\phi_1^4$$

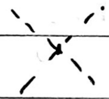
$$+ \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2}m_0^2 \phi_2^2 - \frac{(3\lambda_0/2)}{4!}\phi_2^4$$

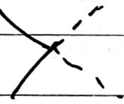
$$- \frac{2(3\lambda_0/2)}{4!}\phi_1^2 \phi_2^2$$

(c)

vertices :

ϕ_1^4  $-i\,\frac{4!}{4!} \times \frac{3\lambda_0}{2} = \cancel{3\lambda_0}\; -i\,\frac{3\lambda_0}{2}.$

ϕ_2^4  $-i\,\frac{4!}{4!}\,\frac{3\lambda_0}{2} = -i\,\frac{3\lambda_0}{2}$

$\phi_1^2\phi_2^2$  $-i\,\frac{2!\cdot 2!\cdot 2}{4!}\,\frac{3\lambda_0}{2} = -i\,\frac{\lambda_0}{2}.$

propagators.

 $\frac{i}{p^2-m_0^2+i\varepsilon}$

 $\frac{i}{p^2-m_0^2+i\varepsilon}$

} same m_0

integrate over loop momenta.

divide by symmetry factors.

(d) $\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m_0^2\phi^*\phi - \frac{\lambda_0}{4}(\phi^*\phi)^2.$

(No field renormalisation ∵ )

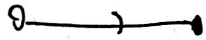
$\cancel{\lambda_0}$ $\lambda = g\mu^\varepsilon$ $\lambda_0 = \lambda(1+\delta g)$

$m_0^2 = m^2 + \delta m^2$

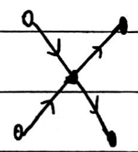
i. $\mathcal{L} = (\partial_\nu \phi^*)(\partial^\nu \phi) - m^2 \phi^* \phi - \delta_{m^2} \phi^* \phi$

$$- \frac{\lambda}{4}(\phi^* \phi)^2 \, , \; - \frac{\delta_g \lambda}{4} (\phi^* \phi)^2$$

Feynman rules including counterterms:

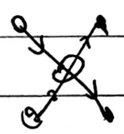


$\frac{i}{p^2 - m^2 + i\varepsilon}$



$-i\lambda$

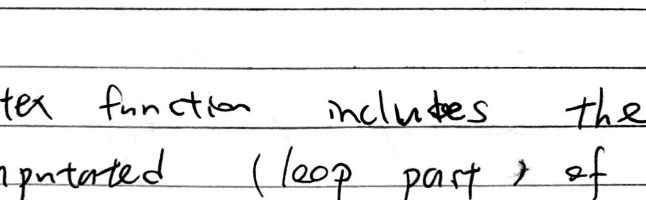
$-i\delta m^2$



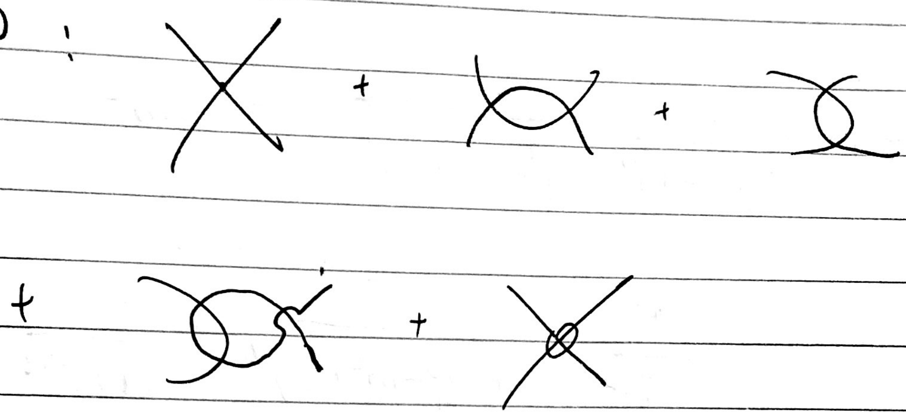
$-i\delta_g \lambda$

Γ - the vertex function includes the ~~amputed~~ amputated (loop part) of the ~~following diag~~ 1-particle irreducible connected diagrams.

i. $\Gamma^{(2)}$:



$\Gamma^{(4)}$:



Divergent part of $\Gamma^{(2)}$ has 1-vertex $\sim$ ~~x~~ $O(\lambda)$

$\therefore \delta_m \sim O(\lambda) \sim \underline{\underline{O(g)}}$

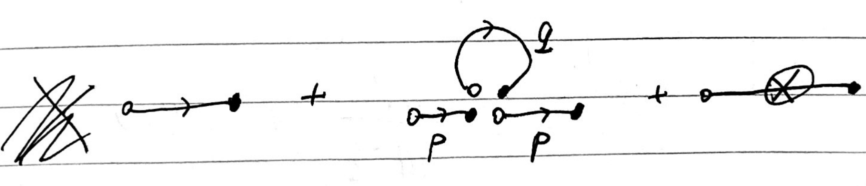
Divergent part of $\Gamma^{(4)}$ has 2-vertices $\sim O(\lambda^2)$

$\therefore \lambda \delta g \sim O(\lambda^2)$ ~~$\sim O(g)$~~

$\therefore \delta g \sim O(\lambda) \sim \underline{\underline{O(g)}}$

Both first order.

(e) $\Gamma^{(2)} =$



amputated.

$$= 0 + (-i\lambda)\int \frac{d^D q}{(2\pi)^D} \frac{i}{q^2 - m^2 + i\epsilon'} - i\delta m^2$$

$$= \lambda \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 - m^2 + i\varepsilon'} - i\delta m^2$$

use $\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + 2qp - m^2 + i\varepsilon')^\alpha} = \frac{(-1)^\alpha i}{(4\pi)^{D/2}} \frac{\Gamma(\alpha - \frac{D}{2})}{\Gamma(\alpha)} (p^2 + m^2)^{D/2 - \alpha}$.

and $\alpha = 1$ $\quad p = 0$ $\quad D = 4 - \varepsilon$

gives

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 - m^2 - i\varepsilon'} = \frac{(-1) i}{(4\pi)^{2 - \varepsilon/2}} \frac{\Gamma(\frac{\varepsilon}{2} - 1)}{\Gamma(1)} (m^2)^{1 - \frac{\varepsilon}{2}}$$

$$= \frac{(-1) i}{16\pi^2} m^2 \, \Gamma(\tfrac{\varepsilon}{2} - 1)$$

leading order

$O(\frac{1}{\varepsilon})$

$$= \frac{-i m^2}{16\pi^2} \frac{(-1)}{1!} \left(\frac{2}{\varepsilon} + \underbrace{1 - \gamma}_{O(1)} \right)$$

$$= \frac{i m^2}{8\pi^2 \varepsilon} + O(1)$$

Minimal subtraction :

$$-i\delta m^2 + \left(\frac{i m^2}{8\pi^2 \varepsilon} + O(1) \right) \lambda = 0$$

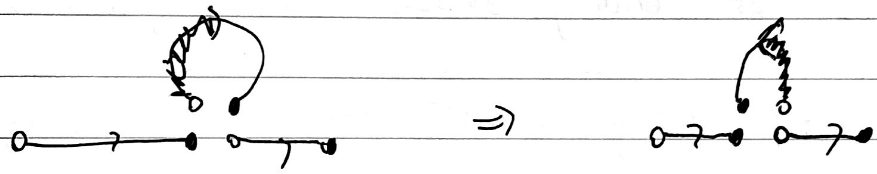
$\lambda = g\mu^\varepsilon$ $\qquad [\delta m^2] = [m^2]$

absorb the factor $\underbrace{\mu^\varepsilon \simeq 1+\varepsilon\ln\mu}$ ($\varepsilon\ln\mu$ in) into the $O(1)$ term

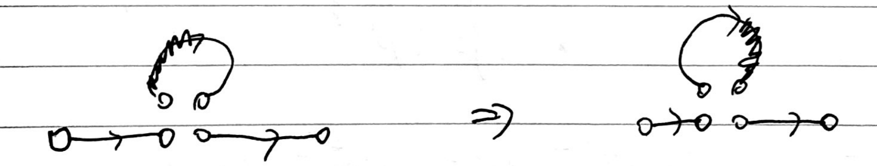
$$-i\,\delta m^2 + \frac{i m^2 g}{8\pi^2\varepsilon} = 0$$

$$\therefore\ \delta m^2 = \frac{g m^2}{8\pi^2\varepsilon}$$

(Note, symmetry factor $= 1$ because we are dealing with a complex field



$S=1$



$S=2$.

exchanging the end point of the self connection reverse the ~~the~~ flow of momentum in the case of complex ~~fiel~~ field, but not for the case of real field).

more "brute force":

$\phi(x)\phi^*(y)\ \phi^*\phi\ \phi^*\phi$ $\qquad$ $\phi(x)\phi^*(y)\ \phi^*\phi\,\phi^*\phi$

$\phi(x)\phi^*(y)\ \phi^*\phi\,\phi^*\phi$ $\qquad$ $\phi(x)\phi(y)\ \phi^*\phi\ \phi^*\phi$

4 ways to contract

$\frac{4}{4} = \underline{\underline{1}}$

the denominator of $\frac{\lambda}{4}$

(f)

$\because \lambda_0 = g\mu^{\varepsilon}(1+\delta g)$ and $\delta g \sim O(\lambda) \sim O(g)$

$= \cancel{\lambda(1+\delta g)}$

$\therefore \lambda_0 = g\mu^{\varepsilon}(1+O(g))$

$= g\mu^{\varepsilon} + O(g^2) = \lambda + O(\lambda^2)$

$\cancel{\Rightarrow \lambda = \lambda_0 + O(\lambda_0^2)} \Rightarrow \lambda = \lambda_0 + O(\lambda_0^2)$

$\therefore$ At leading order of g, $\lambda_0 = g\mu^{\varepsilon}$

At leading order of λ_0, $\lambda = \lambda_0$

$\therefore g = \lambda_0 \mu^{-\varepsilon}$

$\cancel{m^2 = m_0^2 - \delta m^2 = m_0^2 - \delta m\; m_0^2 - \frac{gm^2}{8\pi\varepsilon}}$

$m_0^2 = m^2 + \delta m^2 = \left(1 + \frac{g}{8\pi^2\varepsilon}\right)m^2$

$\therefore m^2 = \left(1 + \frac{g}{8\pi^2\varepsilon}\right)^{-1} m_0^2$

$= \left(1 - \frac{g}{8\pi^2\varepsilon}\right)m_0^2 = \left(1 - \frac{\lambda_0\mu^{-\varepsilon}}{8\pi^2\varepsilon}\right)m_0^2.$

to leading order in g

$\gamma_m(g) = \frac{1}{2}\mu \frac{\partial \log m^2}{\partial \mu}\Big|_{\lambda_0, m_0}$

$$= \frac{1}{2m^2} \mu \frac{\partial m^2}{\partial \mu}\bigg|_{m_0, \lambda_0}$$

$$= \frac{1}{2m^2} \left(\frac{-(\cancel{-}\varepsilon)}{\cancel{\varepsilon}} \frac{\lambda_0 \mu^{-\varepsilon} m_0^2}{8\pi^2} \right)$$

$$= \frac{1}{2} \frac{1}{(1+\frac{g}{8\pi^2\varepsilon})^{-1} \cancel{m_0^2}} \frac{(\lambda_0 \mu^{-\varepsilon}) \cancel{m_0^2}}{8\pi^2}$$

$$= \frac{1}{16\pi^2} \left(1 + \frac{g}{8\pi^2 \varepsilon}\right) g + O(g^2) + O(g^3)$$

$$= \frac{g}{16\pi^2} + O(g^2)$$

④ (a)

$$\Gamma^{(n)}(\{P_k\}, g(\lambda_0,\mu), \mu) = Z_\phi^{-n/2}(\lambda_0,\mu)\,\Gamma_0^{(n)}(\{P_k\}, \lambda_0)$$

$$\therefore \Gamma_0^{(n)}(P_k, \lambda_0) = Z_\phi^{-n/2}(\lambda_0,\mu)\,\cancel{\times}\,\Gamma^{(n)}(P_k, g(\lambda_0,\mu), \mu)$$

the bare vertex function ~~sho~~ $\Gamma_0^{(n)}(P_k, \lambda_0)$ should not dependent on the renormalisation mass scale μ (it only depends on P_k, λ_0 and m_0 and $\because m=0 \therefore m_0 = m_0(\lambda_0)$ $\therefore$ ~~Γ_0~~ ~~Γ_0~~ $\Gamma_0^{(n)}$ only ~~is~~ depends on P_k and λ_0)

$$\therefore \mu \frac{d}{d\mu}\Gamma_0^{(n)}(P_k, \lambda_0) = 0$$

$$\therefore \mu \frac{d Z_\phi^{-n/2}}{d\mu}\Gamma^{(n)} + \mu Z_\phi^{-n/2}\frac{d\Gamma^{(n)}}{d\mu} = 0$$

$$\therefore 0 = \cancel{\mu\partial} - \mu\frac{n}{2} Z_\phi^{-\frac{n}{2}-1}\left.\frac{\partial Z_\phi}{\partial\mu}\right|_{\lambda_0}\Gamma^{(n)} + \mu Z_\phi^{-n/2}\frac{d\Gamma^{(n)}}{d\mu}$$

$$\Rightarrow 0 = \Gamma^{(n)}\left(-\frac{n}{2}\mu\frac{1}{Z_\phi}\left.\frac{\partial Z_\phi}{\partial\mu}\right|_{\lambda_0}\right) + \mu\frac{d\Gamma^{(n)}}{d\mu}$$

$$= -\frac{n}{2}\left(\mu\left.\frac{\partial \log Z_\phi}{\partial\mu}\right|_{\lambda_0}\right)\Gamma^{(n)} + \mu\left(\frac{\partial}{\partial\mu}\Gamma^{(n)} + \left.\frac{\partial}{\partial g}\Gamma^{(n)}\frac{\partial g}{\partial\mu}\right|_{\lambda_0}\right)$$

$$= \left(\mu\frac{\partial}{\partial\mu} + \underbrace{\left(\mu\left.\frac{\partial g}{\partial\mu}\right|_{\lambda_0}\right)}_{\beta(g)}\frac{\partial}{\partial g} - \frac{n}{2}\underbrace{\left(\mu\left.\frac{\partial\log Z_\phi}{\partial\mu}\right|_{\lambda_0}\right)}_{\gamma(g)}\right)\Gamma^{(n)}$$

$$= \left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\gamma(g)\right)\Gamma^{(n)}(\{P_k\}, g, \mu)$$

(b) Now $\hat{\Gamma}^{(n)}(\{p_k\}, g, \mu) = \Gamma^{(n)}(\{p_k\}, g_0, \mu)$

$$= \Gamma_0^{(n)}(\{p_k\}, \lambda_0) \quad , \quad g = g_0 \Rightarrow \frac{\partial}{\partial g} = \frac{\partial}{\partial g_0}$$

and $g_0 = \lambda_0 \mu^{-\varepsilon}$

$$\therefore \beta_0(g_0) = \mu \frac{\partial}{\partial \mu} g_0 \Big|_{\lambda_0} = \mu \frac{\partial}{\partial \mu}\left(\lambda_0 \mu^{-\varepsilon}\right) = -\varepsilon \underbrace{\lambda_0 \mu^{-\varepsilon}}_{g_0}$$

$$= -\varepsilon g_0$$

$$\therefore \left(\mu \frac{\partial}{\partial \mu} + \beta_0(g_0)\frac{\partial}{\partial g_0} - \frac{n}{2}\gamma_0(g_0)\right)\hat{\Gamma}^{(n)}(\{p_k\}, g_0, \mu) = 0 \, ,$$

and $\lambda_0 = g_0 \mu^{+\varepsilon}$ and $\because \lambda_0 = \text{const}$

$$\therefore 0 = d\lambda_0 = \mu^{\varepsilon} dg_0 + \varepsilon \mu^{\varepsilon} \frac{d\mu}{\mu} g_0$$

$$\therefore dg_0 = + \frac{\varepsilon \, d\mu}{\mu} \quad \therefore \frac{dg_0}{\varepsilon} = + \frac{d\mu}{\mu} g_0$$

$$\therefore \frac{-\varepsilon g_0}{dg_0} = \frac{\mu}{d\mu} \Rightarrow -\varepsilon g_0 \frac{\partial}{\partial g_0} = \mu \frac{\partial}{\partial \mu}$$

$$\therefore \mu \frac{\partial}{\partial \mu} \mp \varepsilon g_0 \frac{\partial}{\partial g_0} .$$

$$g_0 = \lambda_0 \mu^{-\varepsilon} \quad \ln(g_0) = \ln \lambda_0 - \varepsilon \ln \mu$$

$$\ln \mu = -\frac{1}{\varepsilon}\ln(g_0) + \frac{1}{\varepsilon}\ln(\lambda_0)$$

$$\hat{\Gamma}^{(n)}(\{P_k\}, g_0, \mu) = \Gamma_0^{(n)}(\{P_k\}, \lambda_0)$$

$$\therefore \left(\mu\frac{\partial}{\partial\mu} + \beta_0(g_0)\frac{\partial}{\partial g_0} - \frac{n}{2}\gamma_0(g_0)\right)\Gamma_0^{(n)}(\{P_k\}, \lambda_0) = 0.$$

$\because$ $\Gamma_0^{(n)}(\{P_k\}, \lambda_0)$ independant of μ, g_0

$\therefore$ $-\frac{n}{2}\gamma_0(g_0)\Gamma_0^{(n)} = 0 \Rightarrow \gamma_0(g_0) = 0$

<u>OR</u> $\because$ $\Gamma^{(n)}(\{P_k\}, g, \mu) = Z_\phi^{n/2}(\lambda_0, \mu)\,\Gamma_0^{(n)}(\{P_k\}, \lambda_0)$

and $\Gamma^{(n)}(\{P_k\}, g_0, \mu) = \Gamma_0^{(n)}(\{P_k\}, \lambda_0)$

$\Rightarrow Z_\phi = 1$

$$\Rightarrow \gamma_0(g_0) = \mu\frac{\partial \log Z_\phi}{\partial\mu}\Big|_{\lambda_0} = 0.$$

(c) $g(\lambda_0, \mu) = g(g_0)$, $Z_\phi(\lambda_0, \mu) = Z_\phi(g_0)$

$$\beta(g(g_0)) = \mu\frac{\partial g(g_0)}{\partial\mu} = \mu\frac{\partial g}{\partial g_0}\frac{\partial g_0}{\partial\mu} = \underbrace{\mu\frac{\partial g_0}{\partial\mu}}_{\beta_0(g_0)}\frac{\partial g}{\partial g_0}$$

$$= \beta_0(g_0)\frac{\partial g}{\partial g_0}$$

$$\gamma(g(g_0)) = \mu \frac{\partial \log Z_\phi(g_0)}{\partial \mu}\Big|_{\lambda_0}$$

$$= \frac{\mu}{Z_\phi}\frac{\partial Z_\phi(g_0)}{\partial \mu} = \frac{\mu}{Z_\phi}\frac{\partial Z_\phi}{\partial g_0}\frac{\partial g_0}{\partial \mu}$$

$$= \frac{\mu \frac{\partial g_0}{\partial \mu}}{Z_\phi}\frac{\partial Z_\phi}{\partial g_0} = \frac{\beta(g_0)}{Z_\phi}\frac{\partial Z_\phi}{\partial g_0}$$

$$= \underbrace{\gamma_0(g_0)}_{=0} + \frac{\beta_0(g_0)}{Z_\phi(g_0)}\frac{\partial Z_\phi}{\partial g_0}$$

OR

(b) $\left(\mu\frac{\partial}{\partial\mu} + \beta_0(g_0)\frac{\partial}{\partial g_0} - \frac{n}{2}\gamma_0(g_0)\right) T_0^{(n)}(\{p_k\}, \lambda_0) = 0.$

$\because$ ~~$T_0^{(n)}$ independent of μ.~~ $-\varepsilon g_0$

$\therefore$ ~~and~~ $\because$ $\left(\mu\frac{\partial}{\partial\mu} + \beta_0(g_0)\frac{\partial}{\partial g_0}\right) T_0^{(n)}$

$$= \cancel{\frac{\partial}{\partial}}\left(\frac{\partial}{\partial\log\mu} - \varepsilon\frac{\partial}{\partial\log g_0}\right) T_0^{(n)}$$

and $\because$ $g_0 = \lambda_0\mu^{-\varepsilon}$ $\therefore$ $\log g_0 = \log\lambda_0 - \varepsilon\log\mu.$

$$\therefore \frac{\partial}{\partial \log g_0} = \frac{\partial}{\partial \log \lambda_0} \quad \text{for} \quad \cancel{\frac{\partial}{\partial \mu}=0}$$

constant μ

and $\dfrac{\partial}{\partial \log \mu} = \varepsilon \dfrac{\partial}{\partial \log \lambda_0}$ for constant g_0

$$\therefore \left(\frac{\partial}{\partial \log \mu} - \varepsilon \frac{\partial}{\partial \log g_0}\right) \Gamma_0^{(n)} = \left(\varepsilon \frac{\partial}{\partial \log \lambda_0} - \varepsilon \frac{\partial}{\partial \log \lambda_0}\right) \Gamma_0^{(n)}$$

$$= 0$$

$$\Rightarrow \quad -\frac{n}{2} \gamma_0(g_0) \Gamma_0^{(n)} = 0 \quad \Rightarrow \gamma_0(g_0) = 0$$

(c) $$0 = \left(\mu \frac{\partial}{\partial \mu} + \beta_0(g_0) \frac{\partial}{\partial g_0} - \frac{n}{2} \gamma_0(g_0)\right) \Gamma^{(n)}(\{p_k\}, g_0, \mu)$$

$$= \left(\mu \frac{\partial}{\partial \mu} + \beta_0(g_0) \frac{\partial}{\partial g_0} - \frac{n}{2} \gamma_0(g_0)\right) (Z_\phi(\lambda_0, \mu)^{-n/2} \times$$

$$\Gamma^{(n)}(\{p_k\}, g(\lambda_0, \mu), \mu)).$$

$$= \left(\mu \frac{\partial}{\partial \mu} + \beta_0(g_0) \frac{\partial}{\partial g_0} - \frac{n}{2} \gamma_0(g_0)\right) \left(Z_\phi^{-n/2}(g_0)\, \Gamma^{(n)}(p_k, g(g_0), \mu)\right)$$

$$= Z_\phi^{-n/2} \mu \frac{\partial}{\partial \mu} \Gamma^{n} + \frac{\beta_0(g_0)}{Z_\phi} Z_\phi^{-n/2} \frac{\partial Z_\phi}{\partial g_0} \left(-\frac{n}{2}\right) \Gamma^{(n)}$$

$$+ Z_\phi^{-n/2} \beta_0(g_0) \frac{\partial g}{\partial g_0} \frac{\partial}{\partial g} \Gamma^{(n)} - \frac{n}{2} \gamma_0(g_0) Z_\phi^{-n/2} \Gamma^{(n)}$$

$\Rightarrow$ divided by $Z_\phi^{-n/2}$

$$0 = \left[\mu\frac{\partial}{\partial\mu} + \underbrace{\left(\beta_0(g_0)\frac{\partial g}{\partial g_0}\right)}_{\beta(g)}\frac{\partial}{\partial g} - \frac{n}{2}\underbrace{\left(\gamma_0(g_0) + \frac{\beta_0(g_0)}{Z_\phi}\frac{\partial Z_\phi}{\partial g_0}\right)}_{\gamma(g)}\right]$$

$\Gamma^{(n)}$

$$\Rightarrow \quad \beta(g(g_0)) = \beta_0(g_0)\frac{\partial g}{\partial g_0}$$

$$\gamma(g(g_0)) = \gamma_0(g_0) + \frac{\beta_0(g_0)}{Z_\phi(g_0)}\frac{\partial Z_\phi}{\partial g_0}.$$

(d) $g(g_0) = g_0 + g_0^2\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon + \cdots\right) \cdots$

$Z_\phi(g_0) = 1 + g_0\left(\frac{z_1}{\varepsilon} + z_2 + z_3\varepsilon + \cdots\right)$

$\beta(g) = \beta(g(g_0)) = \cancel{1 + 2g_0\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon\right)}$

~~g~~ invert: $g_0 = g - g^2\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon\right) + \cdots$

$g_0^2 = g^2 + \cdots$

$\therefore$ ~~β~~

$\therefore \beta(g) = \beta(g(g_0)) = \beta_0(g_0)\frac{\partial g}{\partial g_0}$

$= -\varepsilon g_0\frac{\partial g}{\partial g_0} = -\varepsilon g_0\left(1 + 2g_0\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon\right)\right).$

$= -\varepsilon g_0 \mp 2\varepsilon g_0^2\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon\right)$

$= -\varepsilon g_0 \mp 2g_0^2\left(a_1 + a_2\varepsilon + a_3\varepsilon^2\right)$

$\Rightarrow$ divided by $Z_\phi^{-n/2}$

$$0 = \left[\nu \frac{\partial}{\partial \nu} + \underbrace{\left(\beta_0(g_0) \frac{\partial g}{\partial g_0}\right)}_{\beta(g)} \frac{\partial}{\partial g} - \frac{n}{2} \underbrace{\left(\gamma_0(g_0) + \frac{\beta_0(g_0)}{Z_\phi} \frac{\partial Z_\phi}{\partial g_0}\right)}_{\gamma(g)}\right]$$

$\Gamma^{(n)}$

$$\Rightarrow \beta(g(g_0)) = \beta_0(g_0) \frac{\partial g}{\partial g_0}$$

$$\gamma(g(g_0)) = \gamma_0(g_0) + \frac{\beta_0(g_0)}{Z_\phi(g_0)} \frac{\partial Z_\phi}{\partial g_0}.$$

(d) $g(g_0) = g_0 + g_0^2 \left(\frac{a_1}{\varepsilon} + a_2 + a_3 \varepsilon + \cdots \right) \cdots$

$Z_\phi(g_0) = 1 + g_0 \left(\frac{z_1}{\varepsilon} + z_2 + z_3 \varepsilon + \cdots \right)$

$\beta(g) = \beta(g(g_0)) = \cancel{1 + 2g_0 \left(\frac{a_1}{\varepsilon} + a_2 + a_3 \varepsilon \right)}$

invert : $g_0 = g - g^2 \left(\frac{a_1}{\varepsilon} + a_2 + a_3 \varepsilon \right) + \cdots$

$g_0^2 = g^2 + \cdots$

$\therefore$ ~~β~~

$$\therefore \beta(g) = \beta(g(g_0)) = \beta_0(g_0) \frac{\partial g}{\partial g_0}$$

$$= -\varepsilon g_0 \frac{\partial g}{\partial g_0} = -\varepsilon g_0 \left(1 + 2g_0 \left(\frac{a_1}{\varepsilon} + a_2 + a_3 \varepsilon \right) \right).$$

$$= -\varepsilon g_0 - 2\varepsilon g_0^2 \left(\frac{a_1}{\varepsilon} + a_2 + a_3 \varepsilon \right)$$

$$= -\varepsilon g_0 - 2g_0^2 \left(a_1 + a_2 \varepsilon + a_3 \varepsilon^2 \right)$$

$$= -\varepsilon\left(g - g^2\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon\right)\right) - 2g^2(a_1 + a_2\varepsilon + a_3\varepsilon^2)$$

$$= -\varepsilon g + g^2(a_1 + a_2\varepsilon + a_3\varepsilon^2) - 2g^2(a_1 + a_2\varepsilon + a_3\varepsilon^2)$$

$$= -\varepsilon g - g^2(a_1 + a_2\varepsilon + a_3\varepsilon^2) + O(g^3).$$

$g^2 a_1$ = finite

as regulater is removed. $\varepsilon \to 0$.

$$\gamma(g) = \frac{\beta_0(g_0)}{Z_\phi}\frac{\partial Z_\phi}{\partial g_0} = -\varepsilon \frac{g_0}{Z_\phi}\frac{\partial Z_\phi}{\partial g_0}$$

$$= -\varepsilon g_0\left(1 + g_0\left(\frac{Z_1}{\varepsilon} + Z_2 + Z_3\varepsilon\right)\right)^{-1}\left(\frac{Z_1}{\varepsilon} + Z_2 + Z_3\varepsilon\right)$$

$$= -\varepsilon g_0\left(1 - g_0\left(\frac{Z_1}{\varepsilon} + Z_2 + Z_3\varepsilon\right) + \cdots\right)\left(\frac{Z_1}{\varepsilon} + Z_2 + Z_3\varepsilon\right)$$

$$= \left[-\varepsilon\left(g - g^2\left(\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon\right)\right) + \varepsilon g^2\left(\frac{Z_1}{\varepsilon} + Z_2 + Z_3\varepsilon\right)\right]$$

$$\times\left[\frac{Z_1}{\varepsilon} + Z_2 + Z_3\varepsilon\right]$$

$$\simeq -g(Z_1 + Z_2\varepsilon) + O(g^2) = \text{finite}.$$

as $\varepsilon \to 0$.

($\beta(g)$ to 2nd order $\because$ g in $\frac{\partial g}{\partial g}$ is to 2nd order

$\gamma(g)$ to 1st order $\because$ Z_ϕ in $\frac{\partial Z_\phi}{\partial g_0}$ is to 1st order)

(e)

$$g(g_0) = g_0 + g_0^2 \frac{a_1}{\varepsilon} + g_0^3 \frac{b_1}{\varepsilon^2}$$

$$\beta(g) = \beta_0(g_0) \frac{\partial g}{\partial g_0} = -\varepsilon g_0 \left(1 + \frac{2a_1}{\varepsilon} g_0 + \frac{3b_1}{\varepsilon^2} g_0^2\right)$$

$$= -\varepsilon g_0 - 2a_1 g_0^2 - \frac{1}{\varepsilon} 3b_1 g_0^3$$

Now we invert the perturbation.

$$g = g_0 + g_0^2 \frac{a_1}{\varepsilon} + g_0^3 \frac{b_1}{\varepsilon^2}$$

let $g_0 = g + \frac{m_1}{\varepsilon} g^2 + \frac{m_2}{\varepsilon^2} g^3 + O(g^4)$

$$\therefore\ g = g + \frac{m_1}{\varepsilon} g^2 + \frac{m_2}{\varepsilon^2} g^3$$

$$+ \frac{a_1}{\varepsilon}\left(g + \frac{m_1}{\varepsilon} g^2 + \frac{m_2}{\varepsilon^2} g^3\right)^2$$

$$+ \frac{b_1}{\varepsilon^2}\left(g + \frac{m_1}{\varepsilon} g^2 + \frac{m_2}{\varepsilon^2} g^3\right)^3$$

$$= g + \frac{m_1}{\varepsilon} g^2 + \frac{m_2}{\varepsilon^2} g^3 + \frac{a_1}{\varepsilon} g^2 + \frac{2a_1 m_1}{\varepsilon^2} g^3$$

$$+ \frac{b_1 g^3}{\varepsilon^2}$$

$\Rightarrow\ m_1 + a_1 = 0 \ \rightarrow\ m_1 = -a_1$

$m_2 + 2a_1 m_1 + b_1 = 0 \ \rightarrow\ m_2 - 2a_1^2 + b_1 = 0$

$m_2 = 2a_1^2 - b_1$

$$\therefore\ g_0 = g - \frac{a_1}{\varepsilon} g^2 + \frac{2a_1^2 - b_1}{\varepsilon^2} g^3 + O(g^4)$$

$$\beta(g) = -\varepsilon g_0 - 2a_1 g_0^2 - \frac{1}{\varepsilon} 3 b_1 g_0^3$$

$$= -\varepsilon \left(g - \frac{a_1}{\varepsilon} g^2 + \frac{2a_1^2 - b_1}{\varepsilon^2} g^3\right)$$

$$-2a_1 \left(g - \frac{a_1}{\varepsilon} g^2 + \frac{2a_1^2 - b_1}{\varepsilon^2} g^3\right)^2$$

$$-\frac{1}{\varepsilon} 3b_1 \left(g - \frac{a_1}{\varepsilon} g^2 + \frac{2a_1^2 - b_1}{\varepsilon^2} g^3\right)^3$$

$$= -\varepsilon g + a_1 g^2 - \frac{2a_1^2 - b_1}{\varepsilon^2} g^3 - 2a_1 g^2 + \frac{4a_1^2}{\varepsilon} g^3$$

$$- \frac{3b_1 g^3}{\varepsilon} + O(g^4)$$

$$= -\varepsilon g + a_1 g^2 - 2a_1 g^2 + \frac{1}{\varepsilon}(-2a_1^2 + b_1 + 4a_1^2 - 3b_1) g^3$$

$$+ O(g^4)$$

$$= \underbrace{-\varepsilon g}_{\text{finite}} \underbrace{- \cancel{2} a_1 g^2}_{\text{finite}} + \frac{1}{\varepsilon}(2a_1^2 - 2b_1) g^3 + O(g^4)$$

(to ~~order~~ 3rd order $\because$ g in $\frac{\partial g}{\partial g_0}$ is to 3rd order

For $\beta(g)$ to be finite up to 3rd order as $\varepsilon \to 0$, need $2a_1^2 - 2b_1 = 0$ $\to$ $\underline{\underline{b_1 = a_1^2}}$

$\Rightarrow$ two loop coefficient ~~dependen~~ depends on the lower order ones.

*Note: ~~ta~~ $\cancel{\alpha\frac{1}{\varepsilon^2}}$ $O(\frac{1}{\varepsilon}, g^2)$ term in $\gamma(g)$ need to be cancelled by the ~~$O(g_0^2)$~~ ~~ε~~ ~~te~~ $O(g_0^2, \frac{1}{\varepsilon^2})$ term (2-loop term) in Z_ϕ, which is not ~~ta~~ included in this question.