Quantum Field Theory

2016 Exam

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1 Lagrangian density (Minkowski space)

$$\mathcal{L} = \frac{1}{2}\partial_\nu\phi(x)\partial^\nu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x) \qquad (*)$$

(a) the equation of motion (E.O.M)

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\left(m^2\phi(x) + \frac{\lambda}{3!}\phi^3(x)\right)$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} = \frac{\partial}{\partial(\partial_\nu\phi)}\left[\frac{1}{2}\eta^{\nu\rho}\partial_\nu\phi\partial_\rho\phi\right]$$

$$= \partial^\nu\phi$$

$$\therefore \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) = \partial_\nu\partial^\nu\phi = \partial^2\phi$$

$\therefore$ E.O.M $\cancel{\partial^2 + m^2 + \frac{\lambda}{3!}}$

$$(\partial^2 + m^2)\phi + \frac{\lambda}{3!}\phi^3 = 0$$

(b) Noether theorem :

For a variation of field

$$\phi(x) \to \phi'(x') = \phi(x) + \alpha\Delta\phi(x)$$

and variation of Lagrangian

$$\mathcal{L}' = \mathcal{L} + \alpha \partial_\mu J^\mu$$

the conserved quantity is $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu$

For spacetime translations:

$$x^\mu \to x'^\mu = x^\mu - a^\mu$$

$$\cancel{\phi(x) \to \phi'(x') = \phi(x)}$$

Active transformation:

Field

$$\phi \to \phi'(x) = \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

and Lagrangian

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + a^\nu \partial_\nu \phi = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L})$$

The quantity $J^\mu{}_\nu$ (with extra free index ν because of a^ν) is then

$$J^\mu{}_\nu = \mathcal{L} \delta^\mu{}_\nu, \quad \text{and} \quad \because \phi'(x) = \phi(x) + a^\nu \partial_\nu \phi(x)$$

$$\therefore \Delta\phi = \partial_\nu \phi(x)$$

$\therefore$ Noether current

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu{}_\nu$$

and variation of Lagrangian

$$\mathcal{L}' = \mathcal{L} + \alpha \partial_\mu J^\mu$$

the conserved quantity is $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu$

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From (*) $\quad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

$$\therefore \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$$

$$\therefore \quad T^\mu{}_\nu = \partial^\mu\phi\,\partial_\nu\phi - \left(\left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)\delta^\mu_\nu\right)$$

$$\cancel{= \partial_\rho\phi\,\partial^\rho\phi\,\delta^\rho_\alpha\,\delta_\rho}$$

$$\cancel{\partial^\mu\phi\,\partial_\nu\phi = \delta^\mu_\rho\,\partial^\rho\phi\,\delta_\nu^\kappa\,\partial_\kappa\phi}$$

$$\therefore T^\mu{}_\nu = \partial^\mu\phi\,\partial_\nu\phi - \delta^\mu_\nu\left[\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right]$$

(C)

$$\Omega^{\mu\nu} = T^{\mu\nu} + C(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^\rho\partial_\rho)\phi^2$$

$$\Omega^\mu_\mu = \eta_{\mu\nu}\Omega^{\mu\nu} = \eta_{\mu\nu}T^{\mu\nu} + C(\eta_{\mu\nu}\partial^\mu\partial^\nu - \underbrace{\eta_{\mu\nu}\eta^{\mu\nu}}_{=\delta^\mu_\mu = 4}\partial^\rho\partial_\rho)\phi^2$$

$$= T^\mu_\mu + C(\partial_\mu\partial^\mu - 4\partial_\rho\partial^\rho)\phi^2$$

$$= \cancel{T^\mu_\mu \pm 3} = T^\mu_\mu - 3C\,\partial_\mu\partial^\mu\phi^2$$

$$\underbrace{T^\mu_\mu}_{m=0} = \partial^\mu\phi\,\partial_\mu\phi - 4\left[\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{\lambda}{4!}\phi^4\right]$$

$$= -\partial_\mu\phi\,\partial^\mu\phi + \frac{\lambda}{3!}\phi^4$$

$$\rightarrow \quad \partial_\mu\partial^\mu\phi^2 = \partial_\mu(2\phi\,\partial^\mu\phi) = 2\phi\,\partial_\mu\partial^\mu\phi + 2\partial^\mu\phi\,\partial_\mu\phi$$

The Equation of motion $(m=0)$ is

$$\partial_\mu \partial^\mu \phi = -\frac{\lambda}{3!}\phi^3$$

$$\therefore \quad 2\phi \partial_\mu \partial^\mu \phi = -\frac{2\lambda}{3!}\phi^4$$

$$\therefore \Omega^\mu{}_\mu = \cancel{\partial\phi\partial} - \partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{3!}\phi^4$$

$$-3c\left[2\partial_\mu\phi\partial^\mu\phi - 2\frac{\lambda}{3!}\phi^4\right]$$

$$= \left[-\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{3!}\phi^4\right]\left[1+6c\right]$$

$$\Rightarrow \quad c = -\frac{1}{6}$$

(d) $m=0 \Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4!}\phi^4$.

$$S = \int d^4x \, \mathcal{L}$$

$$x \to x' = e^\alpha x \quad , \phi \to \phi' = e^{-\alpha}\phi \quad (\phi'(x') = e^{-\alpha}\phi(x))$$

$$\cancel{\partial_\mu = \partial_\mu \phi \partial} \qquad \partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = e^{-\alpha}\partial_\mu$$

$$\partial_\mu\phi\partial^\mu\phi \to \partial'_\mu\phi'\partial'^\mu\phi' = \eta^{\mu\nu}\partial'_\mu\phi'\partial'_\nu\phi'$$

$$= \cancel{\eta^{\mu\nu}\partial} \qquad = \eta^{\mu\nu}e^{-2\alpha}\partial_\mu\phi'\partial_\nu\phi'$$

$$= e^{-4\alpha}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$= e^{-4\alpha}\partial_\mu\phi\partial^\mu\phi$$

$$\therefore \mathcal{L} \to \mathcal{L}' = \cancel{e^{-4}\tfrac{1}{2}} \, e^{-4\alpha}\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - e^{-4\alpha}\frac{\lambda}{4!}\phi^4$$

$$= e^{-4\alpha}\mathcal{L}$$

$$dx^\mu \to dx'^\mu = e^\alpha dx^\mu$$

$$\therefore \quad d^4x' = e^{4\alpha}d^4x$$

$$\Rightarrow S \to S' = \int d^4x' \, \mathcal{L}'(x') = \int e^{4\alpha}d^4x \, e^{-4\alpha}\mathcal{L}$$

$$= \int d^4x \, \mathcal{L} = S$$

$\Rightarrow$ S is invariant

To use Noether Theorem, need to find infinitesimal transformation of ~~$\phi(x)$~~ ϕ and $\mathcal{L}$

$$\phi'(x) = \phi(x) + \alpha \Delta \phi(x)$$

$$\mathcal{L}'(x) = \mathcal{L}(x) + \alpha \partial_\mu J^\mu(x)$$

conserved quantity (Noether current) is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu$$

$$\because \quad \phi'(x') = e^{-\alpha}\phi(x), \quad x' = e^{\alpha} x$$

$$\therefore \phi'(e^{\alpha} x) = e^{-\alpha}\phi(x)$$

$$\therefore \phi'(x) = e^{-\alpha}\phi(e^{-\alpha} x)$$

$$\alpha \to 0 \implies \phi'(x) \simeq (1-\alpha)\,\phi(x - \alpha x)$$

$$\simeq (1-\alpha)(\phi(x) - \alpha x^\nu \partial_\nu \phi)$$

$$\simeq \phi(x) - \alpha\phi(x) - \alpha x^\nu \partial_\nu \phi(x)$$

$$= \phi(x) - \alpha\,\underbrace{(\phi(x) + x^\nu \partial_\nu \phi(x))}_{\Delta\phi \,=\, -(\phi(x) + x^\nu \partial_\nu \phi(x))}.$$

$$\therefore \Delta\phi = \phi(x) + x^\nu \partial_\nu \phi(x)$$

The Lagrangian $\mathcal{L}'(x) = \frac{1}{2}\partial_\mu \phi'(x)\,\partial^\mu \phi'(x) - \frac{\lambda}{4!}\phi'^4(x)$

$$\phi'^4(x) = e^{-4\alpha}\phi^4(e^{-\alpha}x)$$

$\because \partial_\mu' = e^{-\alpha}\partial_\mu$

$\therefore$ ~~$\mathcal{L}'(x) = \frac{1}{2}(e^{-\alpha}\partial_\mu)(e^{-\alpha}\partial\phi$~~

$$\mathcal{L}'(x) = \frac{1}{2}(e^{-\alpha}\partial_\mu)(e^{-\alpha}\phi(e^{-\alpha}x))(e^{-\alpha}\partial^\mu)(e^{-\alpha}\phi(e^{-\alpha}x))$$
$$- \frac{\lambda}{4!}e^{-4\alpha}\phi^4(e^{-\alpha}x)$$

$$= e^{-4\alpha}\left[\frac{1}{2}\partial_\mu\phi(e^{-\alpha}x)\partial^\mu\phi(e^{-\alpha}x) - \frac{\lambda}{4!}\phi^4(e^{-\alpha}x)\right]$$

$$= e^{-4\alpha}\mathcal{L}(e^{-\alpha}x)$$

$$\therefore \mathcal{L}'(x) = (1-4\alpha)(\mathcal{L}(\cancel{x}-\alpha x))$$

infinitesimal $= (1-4\alpha)(\mathcal{L}(x) - \alpha x^\rho\partial_\rho\mathcal{L})$

$$= \cancel{1-4\alpha}\ \mathcal{L}(x) - 4\alpha\mathcal{L}(x) - \alpha x^\rho\partial_\rho\mathcal{L}.$$

$\because 4 = \delta^\mu_\mu = \partial_\rho x^\rho$

$$\therefore \mathcal{L}'(x) = \mathcal{L}(x) - \alpha(\cancel{\partial_\rho}\mathcal{L}\partial_\rho x^\rho + x^\rho\partial_\rho\mathcal{L}).$$

$$= \mathcal{L}(x) - \alpha(\partial_\rho(x^\rho\mathcal{L})).$$

$$= \mathcal{L}(x) - \alpha\partial_\mu(\underbrace{x^\mu\mathcal{L}}_{J^\mu = -x^\mu\mathcal{L}})$$

$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$ as before

$$\therefore j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu = \boxed{-(\partial^\mu\phi)(\phi + x_\nu\partial^\nu\phi) + x^\mu\mathcal{L}}$$

$\cancel{\partial_\mu} \quad \partial_\mu j^\mu = \cancel{\partial_\mu} \cancel{\partial_\mu [-\partial_\mu \phi (\phi -}$

$$\partial_\mu[-\partial^\mu\phi(\phi + x_\nu\partial^\nu\phi) + x^\mu \mathcal{L}]$$

$$= -(\partial_\mu\partial^\mu\phi)(\phi + x_\nu\partial^\nu\phi)$$

$$- \partial^\mu\phi(\partial_\mu\phi + \partial_\mu(x_\nu\partial^\nu\phi))$$

$$+ \partial_\mu(x^\mu\mathcal{L})$$

$$= -(\partial_\mu\partial^\mu\phi)(\phi + x_\nu\partial^\nu\phi) - (\partial^\mu\phi)(\partial_\mu\phi + x_\nu\partial_\mu\partial^\nu\phi + \eta_{\mu\nu}\partial^\nu\phi)$$

$$+ \partial_\mu(x^\mu\mathcal{L})$$

$$= -(\partial_\mu\partial^\mu\phi)(\phi + x_\nu\partial^\nu\phi) - (\partial^\mu\phi)(\partial_\mu\phi + x_\nu\partial_\mu\partial^\nu\phi + \partial_\mu\phi)$$

$$+ \partial_\mu(x^\mu\mathcal{L})$$

$$= -(\partial_\mu\partial^\mu\phi)(\phi + x_\nu\partial^\mu\phi) - (\partial^\mu\phi)(2\partial_\mu\phi + x_\nu\partial_\mu\partial^\nu\phi)$$

$$+ \cancel{\partial_\mu(x^\mu}\underbrace{(\partial_\mu x^\mu)}_{4}\mathcal{L} + x^\mu\partial_\mu\mathcal{L}$$

$$= -(\partial_\mu\partial^\mu\phi)(\phi + x_\nu\partial^\nu\phi) - (\partial^\mu\phi)(\cancel{2\partial_\mu\phi} + x_\nu\partial_\mu\partial^\nu\phi)$$

$$+ 4(\cancel{\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi} - \tfrac{\lambda}{4!}\phi^4) + x^\mu\partial_\mu(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{\lambda}{4!}\phi^4)$$

$$= -(\partial_\mu\partial^\mu\phi)(\cancel{\phi} + x_\nu\partial^\nu\phi) - \partial^\mu\phi x_\nu\partial_\mu\partial^\nu\phi$$

$$- \cancel{\tfrac{\lambda}{3!}\phi^4} + \tfrac{1}{2}x^\nu\partial_\nu(\partial_\mu\phi\partial^\mu\phi) - \tfrac{\lambda}{4!}x^\nu\partial_\nu\phi^4$$

(E.O.M $\partial_\mu \partial^\mu \phi + \frac{\lambda}{3!}\phi^3 = 0$ is used)

$$= -(\partial_\mu \partial^\mu \phi)(x_\nu \partial^\nu \phi) - (\partial^\mu \phi) x_\nu \partial_\mu \partial^\nu \phi$$

$$+ \frac{1}{2} x^\nu \partial_\nu (\partial_\mu \phi \partial^\mu \phi) - \frac{\lambda}{4!} x^\nu \partial_\nu \phi^4$$

$$= -(\partial_\mu \partial^\mu \phi)(x^\nu \partial_\nu \phi) - (\partial^\mu \phi) x^\nu \partial_\mu \partial_\nu \phi$$

$$+ \frac{1}{2} x^\nu \partial_\nu \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4!} x^\nu \partial_\nu \phi^4$$

$$\cancel{= -(\partial_\mu \partial^\mu \phi)(x^\nu \partial_\nu \phi) - (\partial^\mu \phi) x^\nu (\partial_\mu \partial_\nu \phi)}$$

$$\cancel{+ \frac{1}{2} x^\nu \partial^\mu \phi (\partial_\mu \partial \phi)} \; \cancel{\frac{1}{4} x^\nu \partial_\nu (\phi (\frac{\lambda}{3!}\phi^3))}$$

$$\cancel{+ \frac{1}{2} x^\nu \partial_\nu (\partial_\mu \phi \partial^\mu \phi)} \quad \cancel{-\partial_\mu \partial^\mu \phi}$$

$$\cancel{= -(\partial_\mu \partial^\mu \phi)(x^\nu \partial_\nu \phi) - \frac{1}{2} x^\nu \partial^\mu \phi \, \partial_\mu \partial_\nu \phi}$$

$$\cancel{= -(\partial_\mu \partial^\mu \phi)(x^\nu \partial_\nu \phi) + \frac{1}{4} x^\nu \partial_\nu (\phi \partial_\mu \partial^\mu \phi)}$$

$$= -(\partial_\mu \partial^\mu \phi)(x^\nu \partial_\nu \phi) - \partial^\mu \phi \, x^\nu (\partial_\mu \partial_\nu \phi)$$

$$+ \frac{1}{2} x^\nu \partial_\nu (\partial_\mu \phi \partial^\mu \phi) - \frac{\lambda}{4!} x^\nu \partial_\nu \phi^4$$

$$= -(\partial_\mu \partial^\mu \phi)(x^\nu \partial_\nu \phi) - \frac{\lambda}{3!} x^\nu (\partial_\nu \phi) \phi^3$$

$$- (\partial^\mu \phi) x^\nu (\partial_\mu \partial_\nu \phi) + \frac{1}{2} x^\nu (\partial_\nu \partial_\mu \phi) \partial^\mu \phi$$

$$+ \frac{1}{2} x^\nu (\partial_\mu \phi)(\partial_\nu \partial^\mu \phi)$$

$$= -(x^\nu \partial_\nu \phi)\left[\underbrace{\partial_\mu \partial^\mu \phi + \frac{\lambda}{3!}\phi^3}_{=0 \text{ by E.o.m}}\right]$$

$$+ \underbrace{\left[\frac{1}{2}+\frac{1}{2}-1\right]}_{=0}\left[(\partial^\mu \phi) x^\nu (\partial_\mu \partial_\nu \phi)\right]$$

$$= 0$$

$\Rightarrow$ j^μ is conserved.

(2) Canonical quantisation:

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\pi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$[a_{\vec{p}}, a_{\vec{q}}^{\dagger}] = (2\pi)^3 \delta^{(3)}(\underline{p} - \underline{q}), \quad [a_{\vec{p}}, a_{\vec{q}}] = 0,$$

$$\cancel{[a_p^{\dagger}, a_q^{\dagger}} \quad [a_{\underline{p}}^{\dagger}, a_{\underline{q}}^{\dagger}] = 0$$

(a) $[\phi(\vec{x}), \pi(\vec{y})]$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{(-i)}{2} \sqrt{\frac{E_{\vec{q}}}{E_{\vec{p}}}} \left[a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}},\; a_{\vec{q}} e^{i\vec{q}\cdot\vec{y}} - a_{\vec{q}}^{\dagger} e^{-i\vec{q}\cdot\vec{y}} \right]$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \left(-\frac{i}{2}\right) \sqrt{\frac{E_{\vec{q}}}{E_{\vec{p}}}} \Big\{ \underbrace{[a_{\vec{p}}, a_{\vec{q}}]}_{0} e^{i(\vec{p}\cdot\vec{x} + \vec{q}\cdot\vec{y})}$$

$$+ \underbrace{[a_{\vec{p}}^{\dagger}, a_{\vec{q}}]}_{-(2\pi)^3 \delta^3(\vec{p}-\vec{q})} e^{i(-\vec{p}\cdot\vec{x} + \vec{q}\cdot\vec{y})} - \underbrace{[a_{\vec{p}}, a_{\vec{q}}^{\dagger}]}_{(2\pi)^3 \delta^3(\vec{p}-\vec{q})} e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})}$$

$$- \underbrace{[a_{\vec{p}}^{\dagger}, a_{\vec{q}}^{\dagger}]}_{0} e^{-i(\vec{p}\cdot\vec{x} + \vec{q}\cdot\vec{y})} \Big\}$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^{\cancel{6}\,3}} \cancel{(2\pi)^3} \sqrt{\frac{E_{\vec{q}}}{E_{\vec{p}}}} \left\{ e^{i(-\vec{p}\cdot\vec{x} + \vec{q}\cdot\vec{y})} + e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} \right\}$$

$$\times \delta^3(\vec{p} - \vec{q}) \times \left(+\frac{i}{2}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\cancel{E_p}}{\cancel{E_p}}} \left(\frac{i}{2}\right) \left[e^{i\vec{p}\cdot(\vec{y}-\vec{x})} + e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \right]$$

$$= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left(e^{i\vec{p}\cdot(\vec{y}-\vec{x})} + e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \right).$$

$$= \frac{i}{2} \delta^3(\underline{y}-\underline{x}) + \frac{i}{2} \delta^3(\underline{x}-\underline{y})$$

$$= \underline{\underline{i\delta^3(\vec{x}-\vec{y})}}$$

δ-function is even

(b) The vacuum state $|0\rangle$ is the state such that $a_{\vec{p}}|0\rangle = 0$ for all $\vec{p}$, it has $E=0$ and is the ground state.

~~$|\vec{p}\rangle$~~ $|\vec{p}\rangle \propto a^\dagger_{\vec{p}}|0\rangle$ has momentum $\vec{p}$ and energy $E_{\vec{p}}$ is the single particle state.

$|\vec{p}_1 \cdots \vec{p}_n\rangle \propto a^\dagger_{\vec{p}_1} \cdots a^\dagger_{\vec{p}_n}|0\rangle$ is the state with n particles and i^{th} particle has momentum $\vec{p}_i$ and energy $E_{\vec{p}_i}$.

$$|\vec{p}_1 \cdots \vec{p}_n\rangle = a^\dagger_{\vec{p}_1} \cdots a^\dagger_{\vec{p}_n}|0\rangle \underset{[a^\dagger_{\vec{p}_i},\, a^\dagger_{\vec{p}_j}]=0}{=} a^\dagger_{\vec{p}_{\sigma(1)}} \cdots a^\dagger_{\vec{p}_{\sigma(n)}}|0\rangle$$

$$= |\vec{p}_{\sigma(1)} \cdots \vec{p}_{\sigma(n)}\rangle$$

(c)

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}}\, a_{\vec{p}}^\dagger a_{\vec{p}}$$

$$\vec{P} = \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p}\, a_{\vec{p}}^\dagger a_{\vec{p}}$$

$\because a_{\vec{p}}|0\rangle = 0 \quad \therefore H|0\rangle = 0, \quad \vec{P}|0\rangle = 0$

$\Rightarrow$ $|0\rangle$ has 0 energy and momentum.

$|\vec{k}\rangle \propto a_{\vec{k}}^\dagger |0\rangle \qquad \because [a_{\vec{p}}^\dagger, a_{\vec{k}}^\dagger] = 0$

$$\therefore [H, a_{\vec{k}}^\dagger] = \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}}\, a_{\vec{p}}^\dagger \underbrace{[a_{\vec{p}}, a_{\vec{k}}^\dagger]}_{(2\pi)^3 \delta^3(\vec{p}-\vec{k})}$$

$$= \int \frac{d^3\vec{p}}{\cancel{(2\pi)^3}} E_{\vec{p}}\, a_{\vec{p}}^\dagger \cancel{(2\pi)^3}\, \delta^3(\vec{p}-\vec{k})$$

$$= E_{\vec{k}}\, a_{\vec{k}}^\dagger$$

$\because$ ~~$[H, a_{\vec{k}}]$~~

Similarly $[\vec{P}, a_{\vec{k}}^\dagger] = \cancel{\vec{P} a_{\vec{k}}^\dagger - \vec{P} a_{\vec{k}}^\dagger}\ \vec{k}\, a_{\vec{k}}^\dagger$

$$[H, a_{\vec{k}}^\dagger]|0\rangle = H a_{\vec{k}}^\dagger|0\rangle - a_{\vec{k}}^\dagger \underbrace{H|0\rangle}_{0} = H\cdot a_{\vec{k}}^\dagger|0\rangle$$

$$= E_{\vec{k}}\, a_{\vec{k}}^\dagger|0\rangle$$

$\because a_{\vec{k}}^\dagger|0\rangle \propto |\vec{k}\rangle \quad \therefore H|\vec{k}\rangle = E_{\vec{k}}|\vec{k}\rangle$

similarly $\cancel{H}\ \vec{P}|\vec{k}\rangle = \vec{k}\,|\vec{k}\rangle$

$\therefore |\vec{k}\rangle$ has energy $E_{\vec{k}}$, momentum $\vec{k}$.

(d)

If $|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a_{\vec{p}}^{\dagger} |0\rangle$ then

$$\langle \vec{p} | \vec{p}' \rangle = 2\sqrt{E_{\vec{p}} E_{\vec{p}'}} \langle 0 | a_{\vec{p}}\, a_{\vec{p}'}^{\dagger} | 0 \rangle$$

$$= 2\sqrt{E_{\vec{p}} E_{\vec{p}'}} \langle 0 | [a_{\vec{p}}, a_{\vec{p}'}^{\dagger}] + \underbrace{a_{\vec{p}'}^{\dagger} a_{\vec{p}} |0\rangle}_{0}$$

$$= 2\sqrt{E_{\vec{p}} E_{\vec{p}'}} \underbrace{\langle 0 | 0 \rangle}_{=1} (2\pi)^3 \delta^3 (\vec{p} - \vec{p}')$$

If $\vec{p} = \vec{p}'$, $E_{\vec{p}} = E_{\vec{p}'}$ $\quad = 2\sqrt{E_{\vec{p}} E_{\vec{p}'}} (2\pi)^3 \delta^3 (\vec{p} - \vec{p}')$

If $\vec{p} \neq \vec{p}'$, $\delta = 0$

$$= 2E_{\vec{p}} (2\pi)^3 \delta^3 (\vec{p} - \vec{p}')$$

Consider Lorentz boost in 3-direction.

$$p_3' = \gamma (p_3 + \beta E_{\vec{p}}), \quad E_{\vec{p}}' = \gamma (E + \beta p_3)$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \quad \beta = v.$$

use identity $\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$

$$\delta^{(3)}(\vec{p}' - \vec{q}') = \cancel{\frac{1}{dp}}\, \delta(p_1' - q_1')\, \delta(p_2' - q_2')\, \delta(p_3' - q_3')$$

$$= \delta(p_1 - q_1)\, \delta(p_2 - q_2) \frac{1}{dp_3'/dp_3} \delta(p_3 - q_3)$$

$$= \frac{dp_3}{dp_3'} \delta^{(3)} (\vec{p} - \vec{q})$$

$$\Rightarrow \delta^{(3)}(\vec{p}-\vec{q}) = \frac{dp_3'}{dp_3}\,\delta^{(3)}(\vec{p}\,'-\vec{q}\,')$$

$$= \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\,\gamma\left(1+\beta\frac{dE_{\vec{p}}}{dp_3}\right)$$

$$\left(\frac{dE_{\vec{p}}}{dp_3} = \frac{d}{dp_3}\Big(\underbrace{\sqrt{p_1^2+p_2^2+p_3^2+m^2}}_{E_{\vec{p}}}\Big) = \frac{p_3}{E_{\vec{p}}}\right).$$

$$= \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\,\frac{\gamma}{E_{\vec{p}}}(E_{\vec{p}}+\beta p_3).$$

$$= \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\,\frac{E_{\vec{p}\,'}}{E_{\vec{p}}} \Rightarrow E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}) = E_{\vec{p}\,'}\,\delta^{(3)}(\vec{p}\,'-\vec{q}\,').$$

For pure rotation ~~translation~~ the Jacobian of ~~a~~ ~~me~~ 3-D momentum subspace is 1

$$\therefore \delta^{(3)}(\vec{p}\,'-\vec{q}\,') = \delta^{(3)}(\vec{p}-\vec{q})$$

(using $dp_1'\,dp_2'\,dp_3' = dp_1\,dp_2\,dp_3$ basically)

~~$\Rightarrow$ For Lorentz transformation~~,

and $|\vec{p}|^2 = |\vec{p}\,'|^2 \Rightarrow E_{\vec{p}\,'} = E_{\vec{p}}$

$\Rightarrow$ Also $E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q}) = E_{\vec{p}\,'}\,\delta^{(3)}(\vec{p}\,'-\vec{q}\,')$

$\therefore$ For Lorentz transformation (general)

$E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q})$ is Lorentz invariant

$\Rightarrow \langle \vec{p}\,|\,\vec{p}\,'\rangle = 2E_{\vec{p}}(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')$ is Lorentz invariant.

(e) In Heisenberg picture

$$\cancel{\phi} \quad \phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^{\dagger} e^{ip\cdot x} \right)$$

Without loss of generality, assume $x^0 > y^0$

then $D_F(x-y) = \langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle$

$$\underbrace{=}_{x^0 > y^0} \langle 0 | \phi(x)\phi(y) | 0 \rangle$$

$$= \langle 0 | \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} \Big(a_{\vec{p}} a_{\vec{q}} e^{-i(p\cdot x + q\cdot y)}$$

$$+ a_{\vec{p}} a_{\vec{q}}^{\dagger} e^{-i(p\cdot x - q\cdot y)} + a_{\vec{p}}^{\dagger} a_{\vec{q}} e^{i(p\cdot x - q\cdot y)}$$

$$+ a_{\vec{p}}^{\dagger} a_{\vec{q}}^{\dagger} e^{+i(p\cdot x + q\cdot y)} \Big) | 0 \rangle$$

$$\underbrace{=}_{\substack{\text{orthogonality} \\ \& \ a_{\vec{p}}|0\rangle = 0}} \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} e^{-i(p\cdot x - q\cdot y)} \times \langle 0 | a_{\vec{p}} a_{\vec{q}}^{\dagger} | 0 \rangle$$

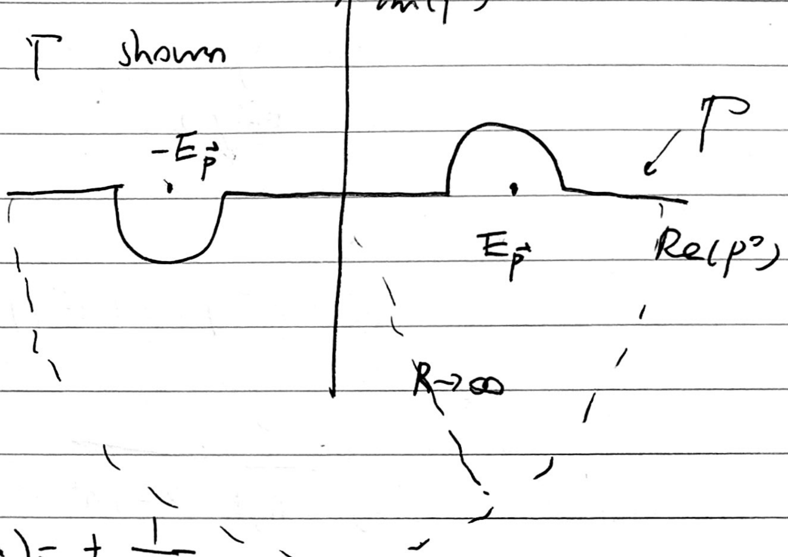
$$\Big(\langle 0 | a_{\vec{p}} a_{\vec{q}}^{\dagger} | 0 \rangle \underbrace{=}_{a_{\vec{p}}|0\rangle = 0} \langle 0 | [a_{\vec{p}}, a_{\vec{q}}^{\dagger}] | 0 \rangle \underbrace{=}_{\langle 0|0\rangle = 1} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \Big)$$

$$= \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^3} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} e^{-i(p\cdot x - q\cdot y)} \delta^{(3)}(\vec{p} - \vec{q}).$$

$$\underset{\substack{p^0 = E_{\vec{p}} \\ q^0 = E_{\vec{q}}}}{=} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

Consider $I = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip\cdot(x-y)}$ integrated

over contour Γ shown



$\because \frac{1}{p^2-m^2} = \frac{1}{(p^0)^2 - E_{\vec{p}}^2}$

$= \frac{1}{(p_0 - E_{\vec{p}})(p_0 + E_{\vec{p}})}$

$\therefore$ ~~res~~ $Res(\pm E_{\vec{p}}) = \pm \frac{1}{2E_{\vec{p}}}$

$\because x^0 > y^0$ ~~$\pm$~~ $e^{-ip^0(x^0-y^0)}$ vanishes as ~~p~~ $Im(p^0) \to -\infty$

$$\therefore I = \int \frac{d^3\vec{p}}{(2\pi)^4} \int dp^0 \frac{i}{p^2-m^2} e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^4} (-2\pi i)\left(\frac{1}{2E_{\vec{p}}}\right)(i) e^{-iE_{\vec{p}}(x^0-y^0) + i\vec{p}\cdot(\vec{x}-\vec{y})}$$

residue theorem

"$-$" ~~conge~~ comes from clockwise

integration

pole at $p^0 = E_{\vec{p}}$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \Big|_{p^0 = E_{\vec{p}}} = \Delta_F(x-y)$$

$$\Rightarrow \quad \Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

(assuming $y^0 > x^0$ everything works out similarly)

$$\therefore \quad (\partial^2 + m^2)\Delta_F(x-y)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} (\partial^2 + m^2) e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} ((-ip)\cdot(-ip) + m^2) e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4} (-i) \frac{\cancel{p^2 - m^2}}{\cancel{p^2 - m^2}} e^{-ip\cdot(x-y)}$$

$$= -i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} = \underline{-i\delta^{(4)}(x-y)}$$

$\Rightarrow$ $\Delta_F(x-y)$ is Green's function for Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0 \quad \square$$

③

$$S = \int d^Dx \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3 \right)$$

$$\langle \phi(y_1)\cdots\phi(y_N)\rangle = \frac{\int D\phi\, \phi(y_1)\cdots\phi(y_N)\, e^{iS}}{\int D\phi\, e^{iS}}$$

(a) $\because$ $[m]=1$ $[S]=0$ $[dx]=[x]=-1$

$\therefore$ $[d^Dx] = -D$

$\therefore$ $-D+2+2[\phi] = 0$ $\therefore$ $[\phi] = \underline{\underline{\frac{D-2}{2}}}$

$\downarrow$

from $d^Dx \cdot \frac{1}{2}m^2\phi^2$ term

is S

From $d^Dx\ \lambda\phi^3$ term :

$$[\lambda] + 3\,\frac{D-2}{2} - D = 0$$

$\therefore$ ~~λ~~ $\underline{\underline{[\lambda] = 3-\frac{D}{2}}}$

$[\lambda]=0$ for $D=D_c$ $\Rightarrow$ $\underline{\underline{D_c = 6}}$

in this case $[\phi] = \frac{D_c-2}{2} = \frac{6-2}{2} = \underline{\underline{2}}$

(b)

S_F = Free field $\quad S_I$ = interaction.

$$S = \underbrace{\int d^Dx \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m\phi^2 \right)}_{S_F} + \underbrace{\int d^Dx \left(-\frac{\lambda}{3!}\phi^3 \right)}_{S_I}.$$

$$e^{iS} = e^{i(S_F + S_I)} = e^{iS_F}e^{iS_I}$$

$$= e^{iS_F}\left(1 + \frac{(iS_I)}{1!} + \frac{(iS_I)^2}{2!} + \cdots \right)$$

$$= e^{iS_F}\sum_{n=0}^{\infty} \frac{i^n}{n!}\int d^Dx_1 \cdots d^Dx_n \mathcal{L}_I(x_1)\cdots\mathcal{L}_I(x_n).$$

where $\mathcal{L}_I = -\frac{\lambda}{3!}\phi^3$

$$\therefore e^{iS} = e^{iS_F}\sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!(3!)^n}\int d^Dx_1 \cdots d^Dx_n\, \phi^3(x_1)\cdots\phi^3(x_n)$$

Numerator

$$\int D\phi\, \phi(y_1)\cdots\phi(y_N)\, e^{iS}$$

$$= \int D\phi\, \phi(y_1)\cdots\phi(y_N)\, e^{iS_F}\sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{n!(3!)^n}\int d^Dx_1\cdots d^Dx_n\, \phi^3(x_1) \cdots \phi^3(x_n)$$

(absorb $\int d^Dx_1 \cdots d^Dx_n$ into ~~$D\phi$~~ $\int \prod dx_n$)

$$= \int \prod d^D x_n \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!(3!)^n} \int D\phi\, \phi(y_1)\cdots\phi(y_N)\, \phi^3(x_1)\cdots\phi^3(x_n)\, e^{iS_F}$$

$$\int \prod d^D x_n \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!(3!)^n} \underbrace{\langle \phi(y_1)\cdots\phi(y_N)\, \phi^3(x_1)\cdots \phi^3(x_n) \rangle_0}_{\text{Free theory correlator}}$$

1-point correlator $\langle \phi(y_1) \rangle =$

Numerator $\mathrm{Num}(\langle \phi(y_1) \rangle)$

$$\mathrm{Num}(\langle \phi(y_1)\rangle) = \int \prod d^D x_n \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!(3!)^n} \langle \phi(y_1)\, \phi^3(x_1)\cdots \phi^3(x_n) \rangle_0$$

$$= \langle \phi(y_1) \rangle_0 + \int d^D x_1 \langle \phi(y_1)\, \phi^3(x_1) \rangle \frac{(-i\lambda)}{3!}$$

$$\int d^D x_1 d^D x_2 \langle \phi(y_1)\, \phi^3(x_1)\, \phi^3(x_2) \rangle \frac{(-i\lambda)^2}{2(3!)^2} + O(\lambda^3)$$

By Wick's Theorem $\langle \phi(y_1) \rangle_0 = 0$

$\langle \phi(y_1)\, \phi^3(x_1)\, \phi^3(x_2) \rangle = 0$

Because they have odd number of terms of ϕ.

$$\langle \phi(y_1)\phi^3(x_1) \rangle = \Delta_F(y_1 - x_1)\, \Delta_F(x_1 - x_1)$$

$$+ \int d^D x_1\, \Delta_F(y_1 - x_1)\, \Delta_F(x_1 - x_1) + \int d^D x_1\, \Delta_F(y_1 - x_1)\, \Delta_F(x_1 - x_1)$$

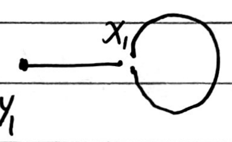
$$= \int d^D x_1\, 3 \Delta_F(y_1 - x_1)\, \Delta_F(x_1 - x_1)$$

$$\therefore \; \mathrm{Num}\left(\langle \phi(y_1) \rangle\right) = \frac{(-i\lambda)}{3!} \times 3 \int \Delta_F(y_1 - x_1) \times \Delta_F(x_1 - x_1) \times d^D x_1$$

$$= \int d^D x_1 \frac{(-i\lambda)}{2} \Delta_F(y_1 - x_1) \, \Delta_F(x_1 - x_1)$$

where Δ_F is the ~~Feynman~~ Feynman propagator

Diagram



(c) in $\langle \phi(y_1) \phi(y_2) \rangle$ all disconnected diagrams are cancelled by the denominator.

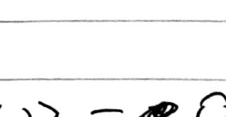
$\therefore \langle \phi(y_1) \phi(y_2) \rangle$ only contains connected diagrams [No $\Delta_F(y_1 - y_2)$ term]

$$\langle \phi(y_1)\phi(y_2)\rangle = \langle \phi(y_1)\phi(y_2)\rangle_0 + \int \langle \phi(y_1)\phi(y_2)\,\phi^3(x_1)\rangle_0 \frac{(-i\lambda)}{3!} d^D x_1$$

$$+ \int \langle \phi(y_1)\phi(y_2)\,\phi^3(x_1)\,\phi^3(x_2)\rangle_0 \frac{(-i\lambda)^2}{2!(3!)^2} d^D x_1 \, d^D x_2$$

Wick's Theorem : $\langle \phi(y_1)\phi(y_2)\,\phi^3(x_1)\rangle_0 = 0$

$\langle \phi(y_1)\phi(y_2)\rangle = \Delta_F(y_1 - y_2) \iff$



$$\langle \phi(y_1)\,\phi(y_2)\,\phi^3(x_1)\,\phi^3(x_2)\rangle_0 \, \frac{(-i\lambda)^2}{2!\,(3!)^2} \Bigg|_{\text{connected}}$$

$$= \frac{(-i\lambda)^2}{2!\,(3!)^2} \Bigg(\iint \Delta_F(y_1-x_1)\, \Delta_F^2(x_1-x_2)\, \Delta_F(x_2-y_2) \times (3\times 3\times 2\times 2)\; d^Dx_1\, d^Dx_2$$

$$+ 2 \int \Delta_F(y_1-x_1)\, \Delta_F(x_1-x_2)\, \Delta_F(x_1-y_2) \times \Delta_F(x_2-x_2) \times (3\times 2\times 3)\; d^Dx_1\, d^Dx_2 \Bigg).$$

$$= \frac{(-i\lambda)^2}{2} \int d^Dx_1\, d^Dx_2\, \Delta_F(y_1-x_1)\, \Delta_F^2(x_1-x_2)\, \Delta_F(x_2-y_2)$$

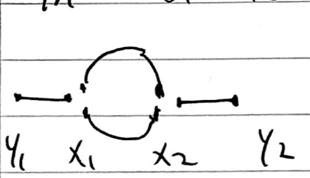
$$+ \frac{(-i\lambda)^2}{2} \int d^Dx_1\, d^Dx_2\, \Delta_F(y_1-x_1)\, \Delta_F(x_1-x_2)\, \Delta_F(x_1-y_2) \times \Delta_F(x_2-x_2)$$

(*)

In first term: 3 choices of ~~x~~ $y_1 - x_1$

3 " " ~~of~~ $x_2 - y_2$

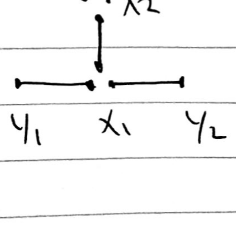
2×2 " " $x_1 - x_2$



In second term: 3 choices of $y_1 - x_1$

2 " " $x_1 - y_2$

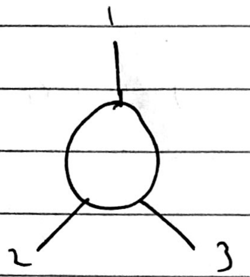
3 " " $x_1 - x_2$



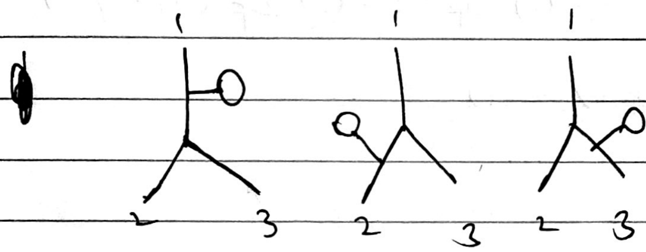
And an extra 2 ~~since~~ by exchanging x_1 and x_2.

(d) For $\langle \phi(y_1)\, \phi(y_2)\, \phi(y_3) \rangle$.

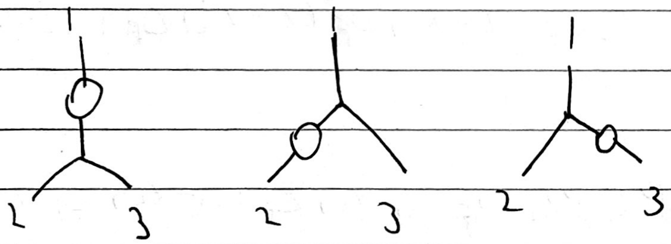
All connected diagrams | Symmetry Factor



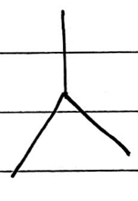
(λ^3) | 1



(λ^3) | 2



(λ^3) | 2



(λ) | 1

④

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda_0\phi^4$$

(a) Momentum Feynman Rule (Minkowski space) for ϕ^4 theory.

1: Draw all topologically distinct connected Feynman diagrams.

2 Assign momenta flowing along each line and impose momentum conservation at vertices.

3 To each vertex associate a factor ~~a~~. $-i\lambda$. ~~(Euclidean $-\lambda$)~~

4 To each line associate a factor

$$\frac{i}{p^2 - m^2 + i\varepsilon} \quad \left(\text{Euclidean } \frac{1}{p^2+m^2} \right)$$

5 integrating over remaining loop momenta

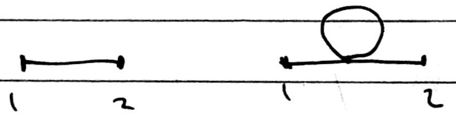
$$\cancel{\int \frac{d^D p}{(2\pi)^D}}$$

6 Multiplying by symmetry factor.

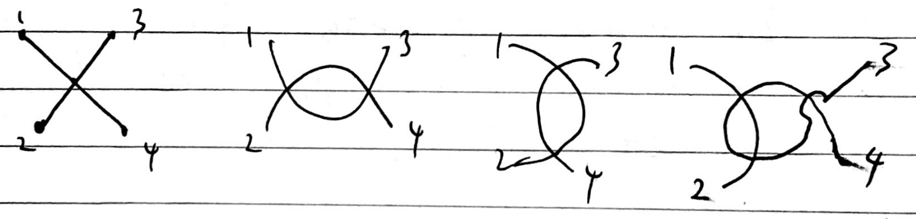
7. Include the momentum conserving delta function for all ~~momen~~ external momenta.

1-loop connected 1 particle irreducible diagrams.

2 point:



4 point:



(b) Schwinger parametrisation.

$$\frac{1}{A^x} = \frac{1}{\Gamma(x)}\int_0^\infty dt \cdot t^{x-1} e^{-At}.$$

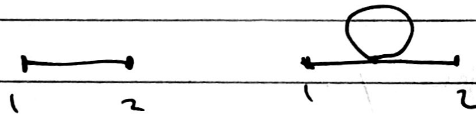
$$\therefore \frac{1}{(2\pi)^D}\int d^Dk \frac{1}{(k^2+m^2)^n} = \frac{1}{(2\pi)^D}\int d^Dk \frac{1}{\Gamma(n)}\int_0^\infty dt\, t^{n-1} e^{-(k^2+m^2)t}$$

$$= \frac{1}{(2\pi)^D}\frac{1}{(n-1)!}\int d^Dk\int_0^\infty dt\, t^{n-1}e^{-(k^2+m^2)t}$$

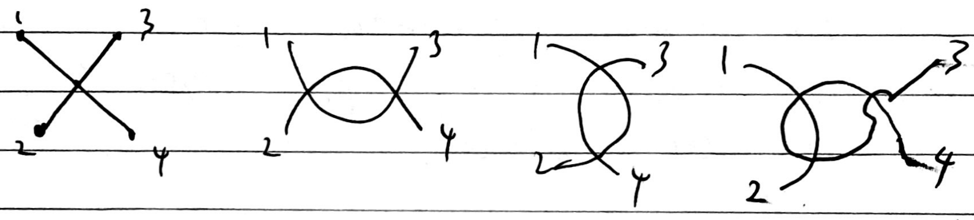
$$= \cancel{\frac{1}{(2\pi)^D}}\frac{1}{(n-1)!}\int_0^\infty dt\, t^{n-1}e^{-m^2t}\underbrace{\int\frac{d^Dk}{(2\pi)^D}e^{-k^2t}}_{(4\pi t)^{-\frac{D}{2}}}$$

1-loop connected 1 particle irreducible diagrams.

2 point :



4 point :



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$$= \frac{1}{(2\pi)^D} \frac{1}{(n-1)!} \int d^Dk \int_0^\infty dt\, t^{n-1} e^{-(k^2+m^2)t}$$

$$= \cancel{\frac{1}{(2\pi)^D}} \frac{1}{(n-1)!} \int_0^\infty dt\, t^{n-1} e^{-m^2 t} \underbrace{\int \frac{d^Dk}{(2\pi)^D} e^{-k^2 t}}_{(4\pi t)^{-\frac{D}{2}}}$$

$$= \cancel{\frac{1}{\Gamma(n)}} \frac{1}{(n-1)!} \int_0^\infty dt\, t^{n-1} e^{-m^2 t} (4\pi t)^{-D/2}$$

$$= \frac{1}{(n-1)!} \frac{1}{(4\pi)^{D/2}} \int_0^\infty dt\, t^{(n-\frac{D}{2})-1} e^{-m^2 t}$$

$s = m^2 t$

$t = \frac{s}{m^2}$

$dt = \frac{ds}{m^2}$

$$= \frac{1}{(n-1)!} \frac{1}{(4\pi)^{D/2}} \cancel{(m^2)^{-1}} ((m^2)^{-1})^{(n-\frac{D}{2})\cancel{-1}} \int_0^\infty ds\, s^{(n-\frac{D}{2})-1} e^{-s}$$

$$= \frac{1}{(n-1)!} \frac{1}{(4\pi)^{D/2}} (m^2)^{\frac{D}{2}-n} \underbrace{\int_0^\infty ds\, s^{(n-\frac{D}{2})-1} e^{-s}}_{\Gamma(n-\frac{D}{2})}$$

$$= \frac{1}{(n-1)!} \frac{1}{(4\pi)^{D/2}} (m^2)^{\frac{D}{2}-n} \Gamma(n-\frac{D}{2})$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+m^2)((p-k)^2+m^2)} \qquad D = 4-\varepsilon \quad (\varepsilon \to 0)$$

$$\left(\frac{1}{AB} = \int_0^1 dz \frac{1}{(zA+(1-z)B)^2} \right)$$

$$= \int \frac{d^D k}{(2\pi)^D} \int_0^1 dz \frac{1}{[(k^2+m^2)z + (1-z)((p-k)^2+m^2)]^2}$$

Denominator of integrand square-rooted:

$$= (k^2+m^2)\,z + (1-z)\left((p-k)^2+m^2\right)$$

$$= \cancel{k^2 z} + \cancel{m^2 z} + p^2 - 2pk + k^2 + m^2 - zp^2 + 2zpk - \cancel{zk^2} - \cancel{zm^2}$$

$$= k^2 - 2pk + 2zpk + p^2 - zp^2 + m^2$$

$$= k^2 - 2(1-z)pk + p^2 - zp^2 + m^2$$

$$= \left(k^2 - 2(1-z)pk + (1-z)^2p^2\right) - (1-z)^2p^2 + (1-z)p^2 + m^2$$

$$= (k-(1-z)p)^2 + (1-z)(\cancel{1}-\cancel{1}+z)\,p^2 + m^2$$

$$= (k-(1-z)p)^2 + (1-z)z\,p^2 + m^2$$

$$= \tilde{k}^2 + z(1-z)p^2 + m^2$$

where $\tilde{k} = k-(1-z)p$ and $d\tilde{k} = dk$

$$\therefore \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2+m^2)((p-k)^2+m^2)}$$

$$\tilde{m}^2 = m^2 + z(1-z)p$$

$$= \int_0^1 dz \int \frac{d^D\tilde{k}}{(2\pi)^D} \frac{1}{(\tilde{k}^2+\tilde{m}^2)^2}$$

$$= \int_0^1 dz \underbrace{\frac{1}{(2-1)!}}_{=1} \frac{1}{(4\pi)^{D/2}} \Gamma\left(2-\frac{D}{2}\right)\left(\tilde{m}^2\right)^{\frac{D}{2}-2}$$

$$= \frac{1}{(4\pi)^{D/2}} \Gamma\left(2-\frac{D}{2}\right) \int_0^1 dz \left(m^2 + z(1-z)p\right)^{\frac{D}{2}-2}$$

when $D = 4 - \varepsilon$ and $\varepsilon \to 0$

$\frac{D}{2} \to 2$

$$\Gamma(2-\tfrac{D}{2}) = \Gamma(2 - \tfrac{4-\varepsilon}{2})$$

$$= \Gamma(\tfrac{\varepsilon}{2}) \simeq \frac{2}{\varepsilon} + \gamma + O(\varepsilon) \sim \frac{2}{\varepsilon}$$

$$\int_0^1 dz\,(m^2 + z(1-z)p)^{\frac{D}{2}-2} \sim \int_0^1 dz\,(m^2 + pz(1-z))^{-\frac{\varepsilon}{2}}$$

$\because \frac{\varepsilon}{2} < 1 \quad \therefore$ converges $\sim O(1)$

$\therefore$ Highest order $O(\frac{1}{\varepsilon})$ has

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2+m^2)((p-k)^2+m^2)} \sim \frac{2}{\varepsilon} \frac{1}{(4\pi)^2}$$

$$= \frac{1}{8\pi^2 \varepsilon} \qquad \square$$

(C) Renormalisation Counterterms. (ϕ^4 theory)

$$\phi_0 = \underbrace{(1+\delta Z_\phi)^{1/2}}_{Z}\phi, \qquad Z m_0^2 = m^2 + \delta m^2$$

$$Z^2 \lambda_0 = \lambda + \delta\lambda$$

Lagrangian becomes: (Euclidean space)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z_\phi}{2}\partial_\mu\phi\,\partial^\mu\phi$$

$$+\frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4$$

~~(change "−" to "+")~~

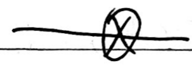
Euclidean space counter term Lagrangian

$$\mathcal{L}_{c.t} = +\frac{1}{2}\frac{\lambda_0 m^2}{16\pi^2\varepsilon}\phi^2 + \frac{1}{4!}\frac{3\lambda_0^2}{16\pi^2\varepsilon}\phi^4$$

From this we ~~reed~~ read off

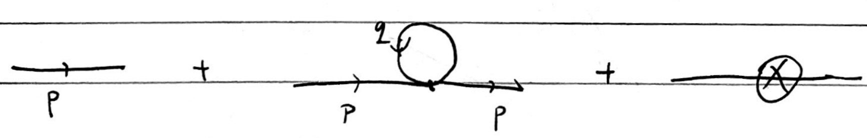
$$\delta m^2 = +\frac{\lambda_0 m^2}{16\pi^2\varepsilon}, \quad \delta\lambda = +\frac{3\lambda_0^2}{16\pi^2\varepsilon}$$

and $\delta Z_\phi = 0$

$\therefore$  $= -\delta m^2$

 $= -\delta\lambda$

two point function : (Euclidean)



$$= -\frac{\lambda_0}{2}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^2+m^2} - \delta m^2$$

symmetry factor

$$= -\frac{\lambda_0}{2}\frac{1}{(4\pi)^D}\Gamma\left(1-\frac{D}{2}\right)(m^2)^{\frac{D}{2}-1} - \delta m^2 \qquad 1-\frac{D}{2} = 1-\frac{4-\varepsilon}{2} = \frac{\varepsilon}{2}-1$$

$$= -\frac{\lambda_0}{2}\frac{1}{(4\pi)^D}\Gamma\left(\frac{\varepsilon}{2}-1\right)(m^2)^D - \delta m^2$$

$\Rightarrow$ divergent part $= +\frac{1}{(4\pi)^2}\frac{2}{\varepsilon}(m^2)^1 - \delta m^2$

cancel 1-loop divergence:

$$\delta m^2 = + \frac{2}{\varepsilon}\frac{1}{(4\pi)^2}\, m^2\, \frac{\lambda_0}{2}$$

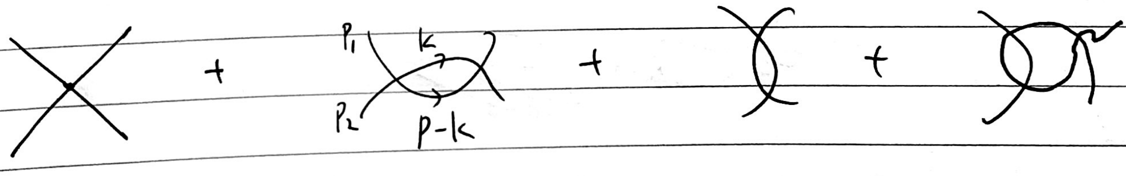
$$= + \frac{\lambda_0 m^2}{16\pi^2 \varepsilon} \qquad \text{consistent.}$$

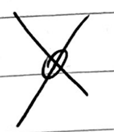
where I've used $\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum_{k=1}^{n} k^{-1} + O(x)\right]$

with $n=1$, $\cancel{x=\frac{2}{\varepsilon}}$ $x = \frac{\varepsilon}{2}$,

4-point function:

$p = p_1 + p_2$



+  $\Big|_{\text{amputated}}$

$$= 0 + 3 \times \frac{(-\lambda_0)^2}{2}\int \frac{d^D k}{(2\pi)^D}\frac{1}{(k^2+m^2)((p-k)^2+m^2)} - \delta\lambda$$

$$\Big| = \frac{3\lambda_0^2}{2} \times \frac{1}{8\pi^2\varepsilon} - \delta\lambda = \frac{3\lambda_0^2}{16\pi^2\varepsilon} - \delta\lambda$$

taking divergent part only

If divergent part $= 0$

$$\rightarrow \delta\lambda = \frac{3\lambda_0^2}{16\pi^2\varepsilon} \qquad \text{(consistent)}$$

This concludes the proof $\square$.

(d) $\because \delta Z_\phi = 0$ (loop part independent of p)

$\therefore Z = 1 + \delta Z_\phi = 1$

$Z^2 \lambda_0 = \lambda + \delta\lambda \quad \therefore \quad \lambda_0 = \lambda + \delta\lambda$

(Here we've redefined λ, λ_0 such that

$\delta\lambda = \frac{3\lambda^2}{16\pi^2\varepsilon}$, $g = \lambda \mu^{-\varepsilon}$

$g_0 = \lambda_0 \mu^{-\varepsilon}$

$\delta\lambda = \frac{3\lambda^2}{16\pi^2\varepsilon} = \frac{3}{16\pi^2\varepsilon} g^2 \mu^{2\varepsilon}$

$\because [\lambda_0] = 4-(4-\varepsilon) = \varepsilon$

keeping λ_0 and $\lambda + \delta\lambda$ with same mass dimension we put one factor of μ^ε into the finite terms (the log in the finite integral)

$\therefore \lambda_0 = \lambda + \frac{3}{16\pi^2\varepsilon} g^2 \mu^\varepsilon$

$\therefore g_0 = g + \frac{3}{16\pi^2\varepsilon} g^2 \rightarrow g = g_0 - \frac{3}{16\pi^2\varepsilon} g_0^2$

$\therefore$

$g = \lambda_0 \mu^{-\varepsilon} - \frac{3}{16\pi^2\varepsilon} \lambda_0^2 \mu^{-2\varepsilon}$

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}\bigg|_{\lambda_0} = -\varepsilon \lambda_0 \mu^{-\varepsilon} + 2\varepsilon \cdot \frac{3}{16\pi^2 \varepsilon} \lambda_0^2 \mu^{-2\varepsilon}$$

$$= -\varepsilon g_0 + \cancel{\varepsilon} \frac{3}{8\pi^2} g_0^2$$

$$g_0 \simeq g + \frac{3}{16\pi^2 \varepsilon} g^2 \qquad g_0^2 \simeq \cancel{\varepsilon} g^2$$

$$\therefore \beta(g) = -\varepsilon g - \frac{3 g^2}{16\pi^2 \cancel{\varepsilon}} + \frac{3}{8\pi^2} g^2$$

$$\cancel{= -\varepsilon g + 2 \frac{3}{8\pi^2 \varepsilon} g^2}$$

$$= -\varepsilon g + \frac{3}{16\pi^2} g^2 + O(g^3)$$