Quantum Field Theory

Problem Set 2

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4	5	6	7
A	A	A	A

TA: Mr. Johan Henriksson.

Tues 3:30 - 5:20.

④

(a) $\phi(x) = \phi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_{\vec{p}}^{\dagger} e^{ip\cdot x} + a_{\vec{p}} e^{-ip\cdot x})$

can be decomposed into

$$\phi(x) = \phi^{+}(x) + \phi^{-}(x) \quad \text{where}$$

$$\phi^{+}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} a_{\vec{p}} e^{-ip\cdot x} \quad \text{and}$$

$$\phi^{-}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} a_{\vec{p}}^{\dagger} e^{ip\cdot x}$$

$\because a_{\vec{p}}|0\rangle = 0 \quad , \quad \langle 0| a_{\vec{p}}^{\dagger} = 0$

$\therefore \quad \phi^{+}|0\rangle = 0$ and $\langle 0|\phi^{-} = 0$

consider $x_1^0 > x_2^0$: (without loss of generality)

$$T\{\phi(x_1)\phi(x_2)\} \underset{x_1^0 > x_2^0}{=} \phi^{+}(x_1)\phi^{+}(x_2) + \phi^{+}(x_1)\phi^{-}(x_2)$$
$$+ \phi^{-}(x_1)\phi^{+}(x_2) + \phi^{-}(x_1)\phi^{-}(x_2)$$

All these 4 terms ~~expec~~ except $\phi^{+}(x_1)\phi^{-}(x_2)$ are already normal ordered ~~since~~ since in $\phi^{-}(x_1)\phi^{+}(x_2)$ $a_p^{\dagger}$s are on the left of a_p's and in $\phi^{\pm}(x_1)\phi^{\pm}(x_2)$ there is no mix of $\dagger$ and non-$\dagger$'s.

The commutator $[\phi^+(x_1), \phi^-(x_2)] = \phi^+(x_1)\phi^-(x_2) - \phi^-(x_2 \ldots \phi^+(x_1)$

$$\therefore T\{\phi(x_1)\phi(x_2)\} = \Big[\phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_2)\phi^-(x_1) + \phi^-(x_1)\phi^-(x_2)\Big]$$

$$+ [\phi^+(x_1), \phi^-(x_2)]$$

$\phi^-(x_2)\phi^+(x_1)$ is the normal ordering of $\phi^+(x_1)\phi^-(x_2)$

$$\therefore \phi^-(x_2)\phi^+(x_1) = :\phi^+(x_1)\phi^-(x_2):$$

and terms in [red bracket] $= :\phi(x_1)\phi(x_2):$

This can be checked explicitly

~~$:\phi^+(x_1)\phi^-(x_2): = \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} e^{-i(p-q)\cdot x} :a_{\vec{p}} a_{\vec{q}}^\dagger: \frac{1}{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}$~~

~~$= \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} e^{-i(p-q)\cdot x} a_q^\dagger a_p \frac{1}{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}$~~

~~(swap $\vec{p}, \vec{q}$)~~ ~~$= \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} e^{+i(p-q)\cdot x} a_{\vec{p}}^\dagger a_{\vec{q}} \frac{1}{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}^{-1}}}$~~

~~$= \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} e^{-i(p_0-q_0)x^0} e^{+i(\vec{p}-\vec{q})} a_{\vec{q}}^\dagger a_{\vec{p}} \frac{1}{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}$~~

~~swap $\vec{p}, \vec{q}$~~

~~$= \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6} \frac{1}{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}} e^{-i(p_0-q_0)x_0} e^{-i(\vec{p}-\vec{q})} (a_{\vec{p}}^\dagger a_{\vec{q}})$~~

~~$= (-1)^k \int_{-\vec{p}=+\infty=-\vec{q}}^{-\vec{p}=-\vec{q}=-\infty} \frac{d^3(-\vec{p})}{(2\pi)^6} d^3(-\vec{q}) \frac{1}{\sqrt{\omega_p\omega_q}} e^{-i(p_0-q_0)x_0} e^{i((-\vec{p})-(-\vec{q}))} a_{-\vec{p}}^\dagger a_{-\vec{q}}$~~

$$:\phi^+(x_1)\,\phi^-(x_2): \; = \; :\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{\omega_{\vec p}\times 2}}\,a_p\,e^{-ip\cdot x_1}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{\omega_{\vec q}\times 2}}\,a_q^\dagger\,e^{+iq\cdot x_2}$$

$$= \int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{\omega_{\vec p}\,\omega_{\vec q}}}\,:(a_{\vec p}\,a_{\vec q}^\dagger):\,e^{-i(p\cdot x_1)+iq\cdot x_2}$$

$$= \int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{\omega_{\vec p}\,\omega_{\vec q}}}\,a_q^\dagger a_p\,e^{-ip\cdot x_1+iq\cdot x_2}$$

$$= \int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{\omega_{\vec q}\times 2}}\,a_q^\dagger\,e^{iq\cdot x_2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec p}}}\,a_p\,e^{-ip\cdot x_1} = \phi^-(x_2)\,\phi^+(x_1) \quad \square$$

$$\therefore \quad T\{\phi(x_1)\phi(x_2)\} = :\phi(x_1)\phi(x_2): + [\phi^+(x_1), \phi^-(x_2)]$$

Define contraction of $\phi(x_1)\phi(x_2)$ to be

$$\overbrace{\phi(x_1)\,\phi(x_2)} = \begin{cases} [\phi^+(x_1), \phi^-(x_2)] & \text{for } x_1^0 > x_2^0 \\ [\phi^+(x_2), \phi^-(x_1)] & \text{for } x_2^0 > x_1^0 \end{cases}$$

For $x_1^0 > x_2^0$ ~~(without loss of generality)~~, the Feynman Propagator

you should here consider both cases (which you do).

$$\Delta_F(x_1 - x_2) = \langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle \quad \text{(for real field } \phi)$$

For $x_1^0 > x_2^0$

$$D_F(x_1 - x_2) = \langle 0|\phi(x_1)\phi(x_2)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip\cdot(x_1 - x_2)}$$

$$\overline{\phi(x_1)\,\phi(x_2)} = [\phi^+(x_1), \phi^-(x_2)]$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{\omega_{\vec p}\,\omega_{\vec q}}}\, e^{-i(p\cdot x_1 - q\cdot x_2)} \underbrace{[a_{\vec p}, a_{\vec q}^\dagger]}_{(2\pi)^3 \delta^{(3)}(\vec p - \vec q)}$$

$$= \int \frac{d^3\vec p\, d^3\vec q}{(2\pi)^3} \frac{1}{2\sqrt{\omega_{\vec p}\,\omega_{\vec q}}}\, e^{-i(\omega_{\vec p} x_1^0 - \omega_{\vec q} x_2^0)}\, e^{i(\vec p\cdot \vec x_1 - \vec q \cdot \vec x_2)}\, \delta^{(3)}(\vec p - \vec q)$$

$$= \int \frac{d^3\vec p}{(2\pi)^3} \frac{1}{2\omega_{\vec p}}\, e^{-i(\omega_{\vec p} x_1^0 - \omega_{\vec p} x_2^0)}\, e^{i\vec p\cdot(\vec x_1 - \vec x_2)}$$

$$= \int \frac{d^3\vec p}{(2\pi)^3} \frac{1}{2\omega_{\vec p}}\, e^{-ip\cdot(x_1 - x_2)} = \Delta_F(x_1 - x_2)$$

$$\therefore\ T\{\phi(x_1)\phi(x_2) = :\phi(x_1)\phi(x_2): + \Delta_F(x_1 - x_2)$$

$\Delta_F(x_1 - x_2)$ is just a function and is of course normal ordered

$$\therefore\ T\{\phi(x_1)\phi(x_2)\} = :\phi(x_1)\phi(x_2) + \Delta_F(x_1 - x_2):$$

□

(b) Proof by induction

Wick's theorem :

$$T\{\phi(x_1)\cdots\phi(x_m)\} = :\phi(x_1)\cdots\phi(x_m) + \text{all possible contractions}:$$

Assume statement is true for m-1 fields.

~~New consid~~ ~~$\phi(x_2), \cdots, \phi(x_m)$~~ and ~~con~~ without loss of generality consider $x_1^0 > x_2^0 > \cdots > x_m^0$, then for m fields.

$$T\{\phi(x_1)\cdots\phi(x_m)\} = \phi(x_1)\cdots\phi(x_m)$$

inductive assumption. →

$$= \phi(x_1) : \phi(x_2)\cdots\phi(x_m) + \text{all contractions not involving } \phi(x_1) :$$

$$= (\phi^+(x_1)+\phi^-(x_1)) : \phi(x_2)\cdots\phi(x_m) + \text{all contractions not involving } \phi(x_1) : \quad (*)$$

- we can safely move $\phi^-(x_1)$ into : : and put it on the left of everything else, since $\phi^-(x_1)$ only has $a_p^\dagger$'s and doesn't need to be swapped with any term originally in : :

For $\phi^+(x_1)$:

∵ $\phi^+(x_1)$ has only a_p normal ordered on the right ∴ this $= :\phi^+(x_1)\phi(x_2)\cdots\phi(x_m):$

$$\phi^+(x_1) :\phi(x_2)\cdots\phi(m): = :\phi(x_2)\cdots\phi(x_m):\phi^+(x_1)$$

$$+ [\phi_1^+(x_1), :\phi(x_2)\cdots\phi(x_m):]$$

~~$= :\phi(x_2)\cdots\phi(x_m):\phi^+(x_1) +$~~

$$= :\phi^+(x_1)\phi(x_2)\cdots\phi(x_m): + [\phi^+(x_1), \phi(x_2)\cdots\phi(x_m)]$$

$$= :\phi^+(x_1)\phi(x_2)\cdots\phi(x_m): + [\phi^+(x_1),\phi(x_2)]\phi(x_3)\cdots\phi(x_m)$$

$$+ \phi(x_2)[\phi^+(x_1),\phi(x_3)]\phi(x_4)\cdots\phi(x_m) + \cdots + \phi(x_2)\cdots\phi(x_{m-1})[\phi^+(x_1),\phi(x_m)]$$

$$= :\phi^+\underbrace{(x_1)\phi(x_2)\cdots\phi}_{\textcircled{1}}(x_m): + :[\phi^+(x_1),\phi(x_2)]\phi(x_3)\cdots\phi(x_m):$$

$$+ :\phi(x_2)\cdots\phi(x_{m-1})[\phi^+(x_1),\phi(x_m)]:$$

Terms with commutator can be put into $: \, :$ as they are because $[\phi^+(x_1), \phi(x_i)]$ ~~is~~ $= \Delta_F(x$ is a function, not operator, so doesn't affect normal ordering. yes

→ term ① combine with term with $\phi^-(x_1)$ form

$:\phi(x_1)\cdots\phi(x_m):$

→ terms with $:\phi(x_2)\cdots[\phi^+(x_1), \phi(x_i)]\cdots\phi(x_m):$ gives

$:\overline{\phi(x_1)\cdots\phi(x_i)}\cdots\phi(x_m):$ ⇒ gives all contractions between $\phi(x_1)$ and $\phi(x_i)$ $i \in [2, m]$, $i \in \mathbb{Z}$

→ Similarly, a term in $(*)$ with involving one contraction will produce all possible terms involving both ~~both~~ that contraction and a contraction of $\phi(x_1)$ with one of the other fields.

Doing this with all terms of $(*)$, we get all possible contraction of all fields.

The base case $m=2$ is proven before in (a)

⇒ induction complete, Wick's theorem ✓ □

(c) ∵ $a_{\vec{p}}|0\rangle = 0$ ∴ all ~~vacuum~~ vacuum expectation value of normally ordered ϕ operators are 0 since the right-most $a_{\vec{p}}$ will give the 0.

Hence we may only have fully contracted terms as non-zero terms in $T\{\phi(x_1)\cdots\phi(x_n)\}$

* $n=4$, the vacuum expectation value :

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle$$

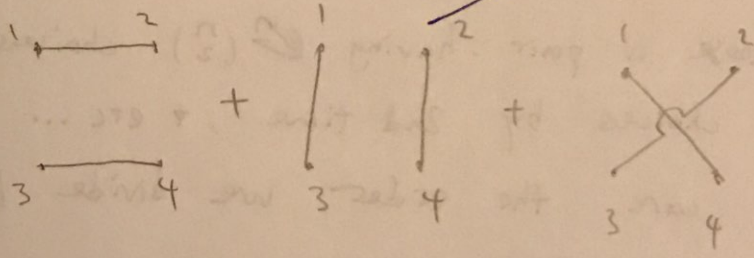
$$= \langle 0|\wick{\c1\phi(x_1)\c1\phi(x_2)\c2\phi(x_3)\c2\phi(x_4)}|0\rangle + \langle 0|\wick{\c1\phi(x_1)\c2\phi(x_2)\c1\phi(x_3)\c2\phi(x_4)}|0\rangle$$

$$+ \langle 0|\wick{\c1\phi(x_1)\c2\phi(x_2)\c2\phi(x_3)\c1\phi(x_4)}|0\rangle$$

$$= \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4)$$

$$+ \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)$$

~~* n=3,~~ =



* $n=5$, there is no fully contracted term due to odd number of fields

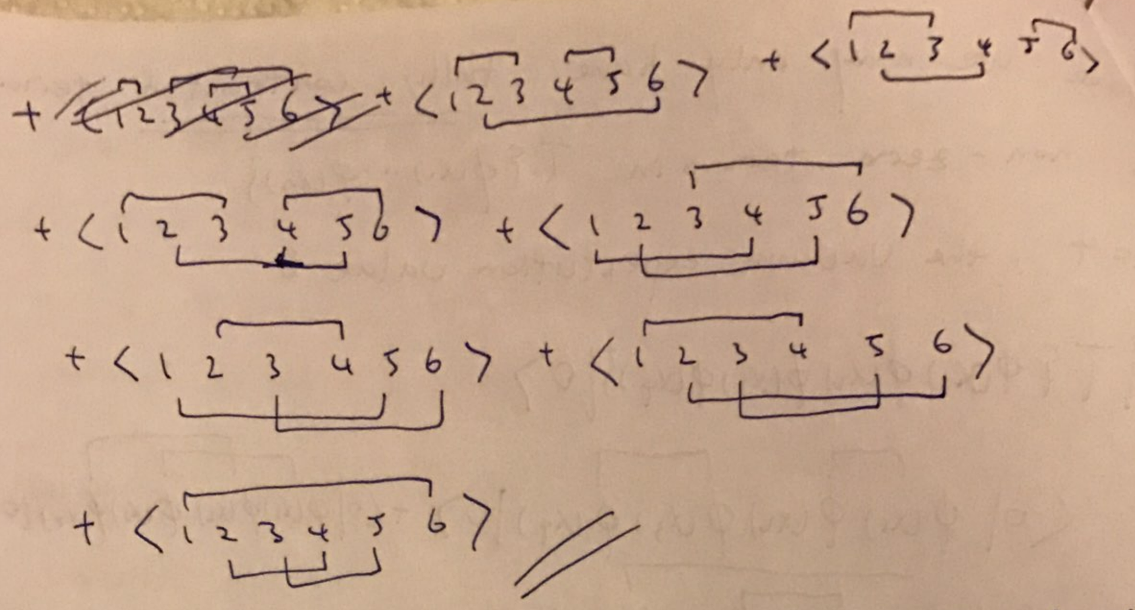
$\therefore \langle 0|\phi(x_1)\cdots\phi(x_5)|0\rangle = 0$

* $n=6$

$$\langle 0|\phi(x_1)\cdots\phi(x_6)|0\rangle = \langle\wick{\c1 1\c1 2\c2 3\c2 4\c3 5\c3 6}\rangle + \langle\wick{\c1 1\c1 2\c2 3\c3 4\c2 5\c3 6}\rangle$$

$$+ \langle\wick{\c1 1\c1 2\c2 3\c3 4\c3 5\c2 6}\rangle + ~~3\,4~~\ \langle\wick{\c1 1\c2 2\c2 3\c1 4\c3 5\c3 6}\rangle + \langle\wick{\c1 1\c2 2\c2 3\c3 4\c1 5\c3 6}\rangle$$

$$+ \langle\wick{\c1 1\c2 2\c2 3\c3 4\c3 5\c1 6}\rangle + \langle\wick{\c1 1\c3 2\c2 3\c2 4\c1 5\c3 6}\rangle + \langle\wick{\c1 1\c3 2\c2 3\c2 4\c3 5\c1 6}\rangle$$



There are 15 ways, this can be verified by counting

$$N = \# \text{ of contractions} = \frac{\binom{n}{2}\cdot\binom{n-2}{2}\cdots\binom{2}{2}}{(\frac{n}{2})!} = \frac{n!}{2^{n/2}(\frac{n}{2})!}$$

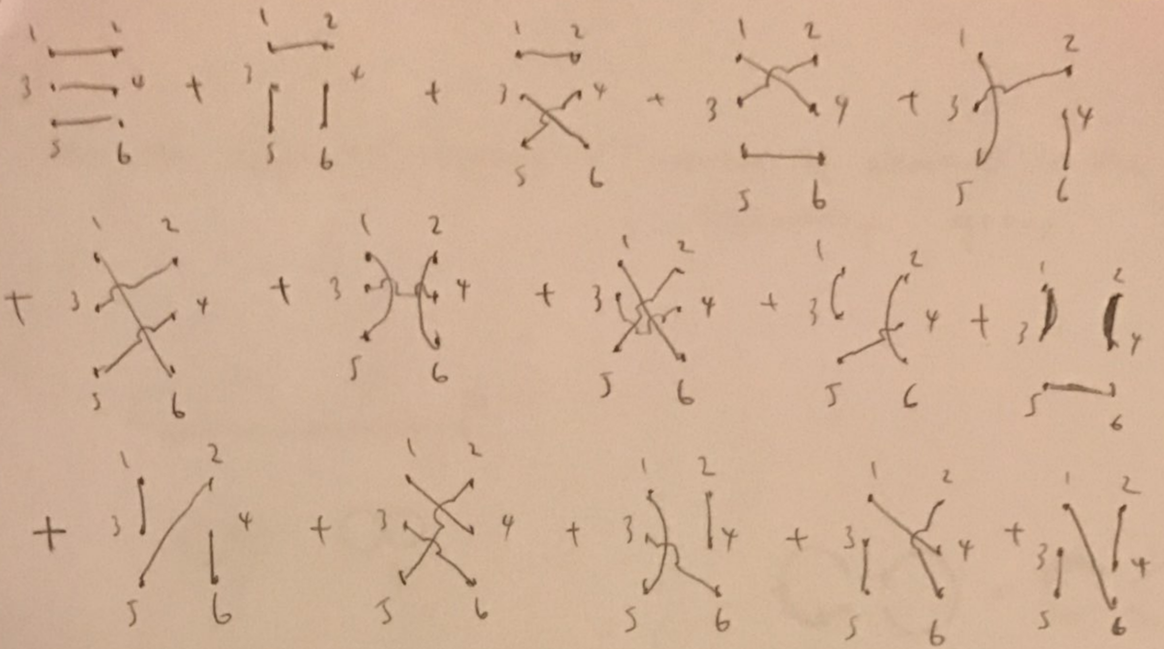
we choose a pair, having $\binom{n}{2}$ choices first time, $\binom{n-2}{2}$ choices by 2nd time, etc... and since we don't care the order we divide by $(\frac{n}{2})!$ □

$$n=4 \Rightarrow \quad N = \frac{\binom{4}{2}}{2!} = \frac{4\times3}{2\times2} = \underline{\underline{3}} = \frac{4!}{4\times2}.$$

$$n=6 \Rightarrow \quad N = \frac{\binom{6}{2}\binom{4}{2}}{3!} = \frac{\frac{30}{2}\,\frac{12}{2}}{3\times2} = \underline{\underline{15}} = \frac{6!}{2^3\times3\times2}$$

✓

the diagrams...



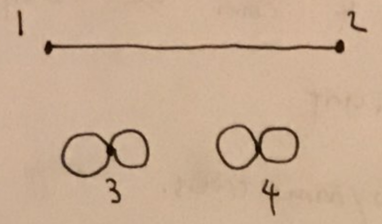
□

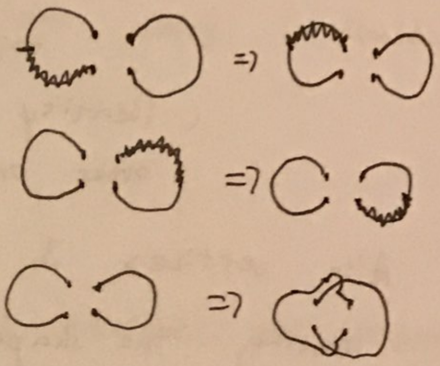
good

⑤

(a) the symmetry factor = $\left(\text{number of elements in the symmetry group}\right)^{-1}$

a)





- Each "double bubble" 3 or 4 has 3 possible independent swapps that keeps the shape invariant

this gives $2\times2\times2 = 8$ symmetries

- there are 2 ~~bubbles~~ double bubbles ⇒ $8\times8 = 64$ symmetries

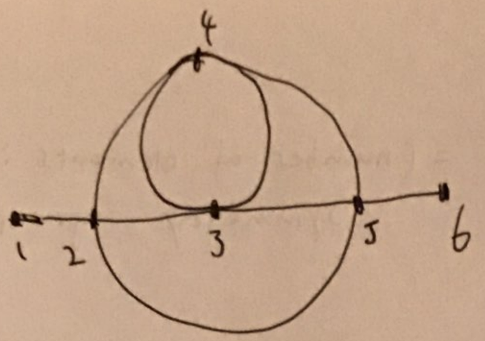
- bubbles 3 and 4 can swap leave the total shape invariant

⇒ $2\times 64 = 128$ symmetries.

External points 1, 2 are fixed.

∴ $$W_G = \frac{1}{128}$$

b)



- two lines joining 3 and 4 can be swapped leaves the total shape invariant

$\Rightarrow$ 2 symmetries.

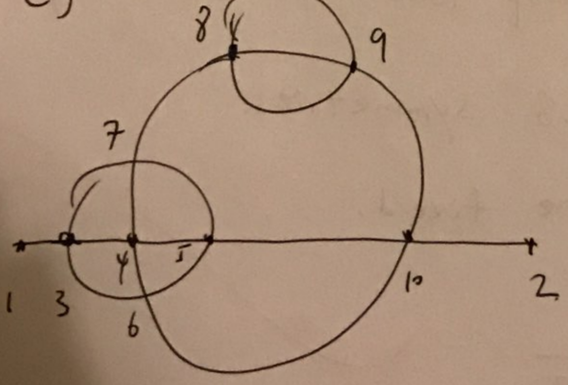
(identity is 1 symmetry, swapped is the other one)

Also vertices 3 and 4 can be exchanged leaving the shape invariant

$2 \times 2 = 4$ symmetries

$\therefore W_G = \frac{1}{4}$

c)

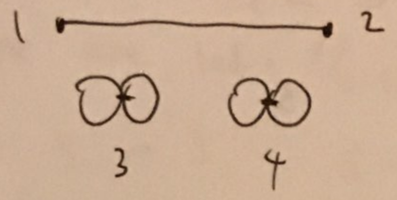


The only symmetry comes from connection between 8 and 9 the 3 lines can be permuted to give a total of $3! = 6$ symmetries.

$W_G = \frac{1}{6}$

) $W_G = \frac{1}{2^{S+D} 3!^T N_{IVP}}$ (I'm using "—" to refer to "is connected to").

a)



~~Point~~ Vertex 3 gives 2 self connections and 1 double connection

Vertex 4 gives same as 3

Vertices 3, 4 gives 2 identical vertices permutations

$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$

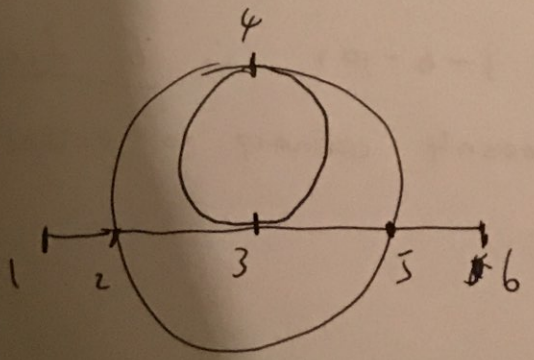
∴ $S = 2 + 2 = 4$

$D = 1 + 1 = 2$

$N_{IVP} = 2$

∴ $W_G = \frac{1}{2^{2+4} \cdot 2} = \frac{1}{2^7} = \frac{1}{128}$

b)



1, 6 fixed

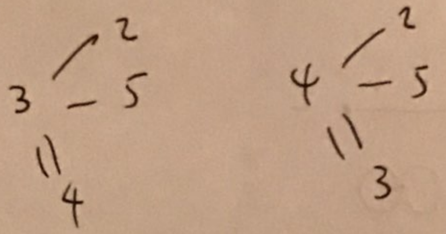
∵ 2 connects 1

5 connects 6

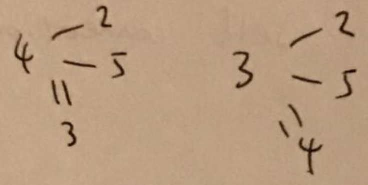
∴ 2, 5 fixed

Vertices 3 , 4 gives a double connection. => ~~5~~ D = 1

Originally ,



If we permute (3, 4) to (4, 3)

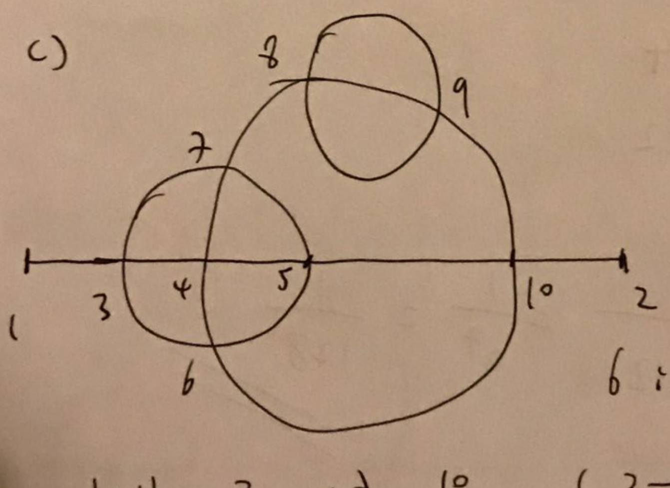


connections are the same => invariant

∴ $N_{svp} = 2$

$$W_G = \frac{1}{2^1 \cdot 2} = \frac{1}{4}$$

c)



1, 2 fixed

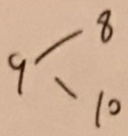
~~∵ 3→1 — 10→2~~

∴ ∵ 3 — 1 , 10 — 2

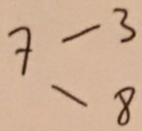
∴ 3, 10 fixed

6 is the only point connects to both 3 and 10 (3 − 6 − 10) ∴ 6 fixed

8 is the only point doesn't connect to either 3 or 10 ∴ 8 fixed.

9 is only point 

∴ 9 fixed

7 is only point 

∴ 7 fixed

∴ we are left with 4 and 5.

But 4 − 3 , 5 − 10 so 4, 5 cannot be permuted either

⇒ ~~[illegible]~~ whole diagram is fixed (points are fixed).

Only symmetry is the triple connection between 8 and 9 ∴ $T = 1$,

$$W_G = \frac{1}{(3!)^1} = \frac{1}{6}$$

⑥

$$\langle\phi(x)\phi(y)|\rangle = \frac{\langle 0|T\{\phi_I(x)\phi_I(y)\exp(-i\int_{-\infty}^{\infty}dt\, H_I(t))\}|0\rangle}{\langle 0|T\{\exp(-i\int_{-\infty}^{\infty}dt\, H_I(t))\}|0\rangle}$$

(a)

Denominator

$$\langle 0|T\{\exp(-i\int_{-\infty}^{\infty}dt\, H_I(t))\}|0\rangle$$

$$= \langle 0|\ 1 + (-i)\int_{-\infty}^{\infty}dt_1 \underbrace{H_I(t_1)}_{\lambda} + \frac{(-i)^2}{2!}\int_{-\infty}^{\infty}dt_1 dt_2 \underbrace{T\{H_1(t_1)H_2(t_2)\}}_{\lambda^2}$$

$$+ \cdots\ |0\rangle$$

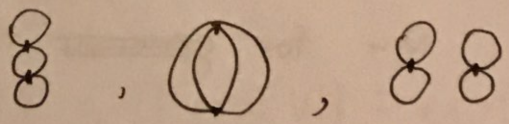
$$\because H_I(t) = \cancel{\int d^3z} \int d^3\vec{z}\, \frac{\lambda}{4!}\phi_I^4(\vec{z},t) = \int d^3z\, \frac{\lambda}{4!}\phi_I^4(z)$$

By ~~we~~ Wick's Theorem, the non-zero $\langle 0|\cdots|0\rangle$ terms are from those with all ϕ_I's contracted with some other field. In Denominator only $\phi_I(z)$ are present

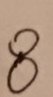
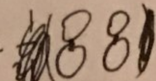
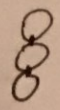
- So For ~~X~~ order λ :

 is the only contribution.

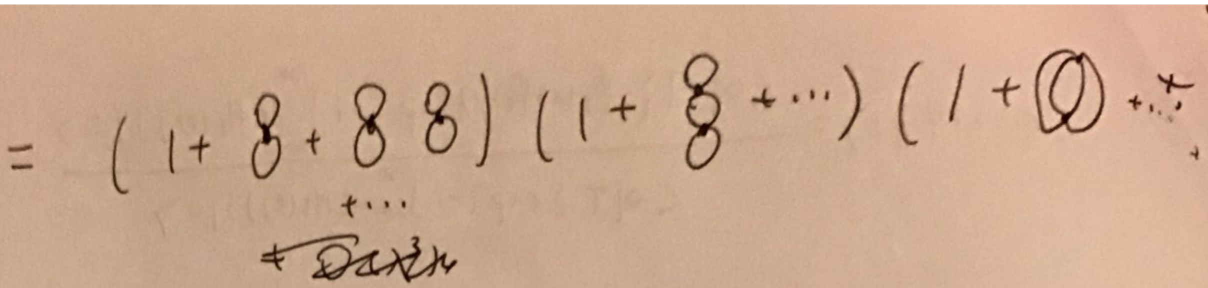
- For order λ^2 :

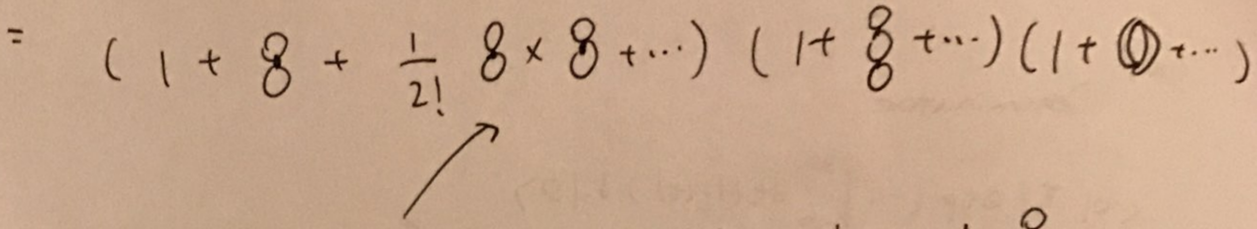


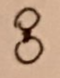
So up to λ^2 .

~~As~~ Denominator = 1 +  +  +  + 

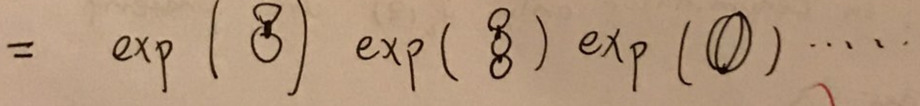
+ $O(\lambda^3)$

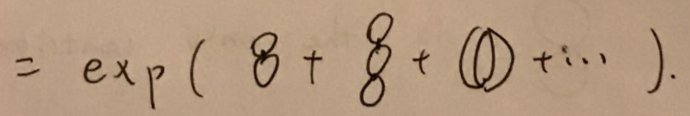




this is because the two identical  introduces an extra identical vertices permutation, so yes!

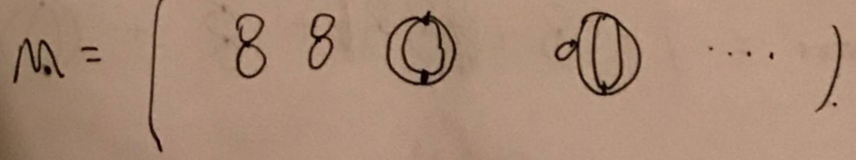
multiply by $\frac{1}{2!}$. N disconnected identical diagrams will give ~~a~~ extra identical vertices permutation of N! so should multiply $\frac{1}{N!}$.



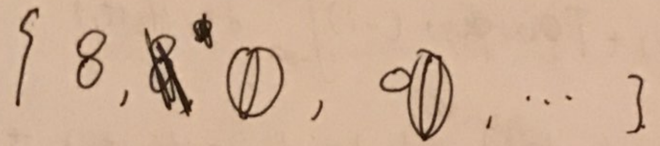


We've verified that the equation is true for up to order λ^2. Now for ~~general~~ the exact case!

Consider a particular diagram in denominator



It can be broken into disconnected pieces. ~~to~~ labelled by V_i

$V_i \in$ 

If M has n_i pieces of the form V_i and let V_i denote the value of diagram V_i, then

$$M = \prod_i \underbrace{\frac{1}{n_i!}}(V_i)^{n_i}$$

$\longrightarrow$ comes from extra identical vertices permutation from having ~~ident~~ n_i identical ~~diagr~~ disconnected pieces.

Denominator = sum of all M

$$= \sum_{\{n_i\}} \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

$\longleftarrow$ M is defined by the set $\{n_i\} = \{n_1, n_2, \cdots\}$

$$= \left(\sum_{n_1} \frac{1}{n_1!} V_1^{n_1}\right)\left(\sum_{n_2} \frac{1}{n_2!} V_2^{n_2}\right)\cdots$$

$$= \prod_i \left(\sum_{n_i} \frac{1}{n_i!} V_i^{n_i}\right) = \prod_i \exp(V_i) = \exp\left(\sum_i V_i\right)$$

$= \exp($ all vacuum bubbles $)$ ✓

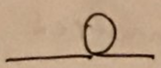
(b) numerator $= \langle 0 | T\{ \phi_I(x)\,\phi_I(y)\, \exp[i\int_{-\infty}^{\infty} dt\, H_I(t)] \} |0\rangle$

$= \langle 0 | T\phi(x)\phi(y) + T\phi(x)\phi(y)\,(-i)\int_{-\infty}^{\infty} dt_1\, H_I(t_1)$

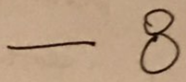
$+ T\phi(x)\phi(y)\,\frac{(-i)^2}{2!}\int_{-\infty}^{\infty} dt_1 dt_2\, H_I(t_1)H_I(t_2) + \cdots |0\rangle$

= [all diagrams with 2 external vertices $\phi(x)$ and ~~ϕ~~ $\phi(y)$

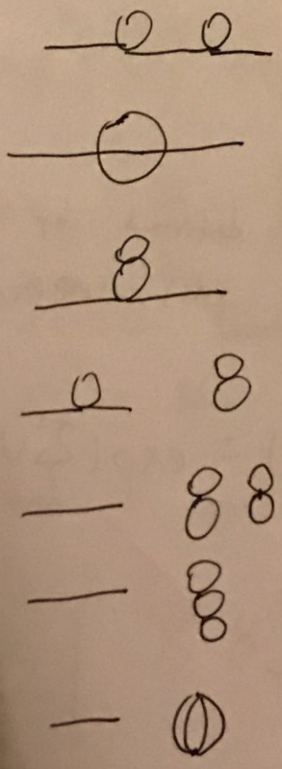
- For order λ :



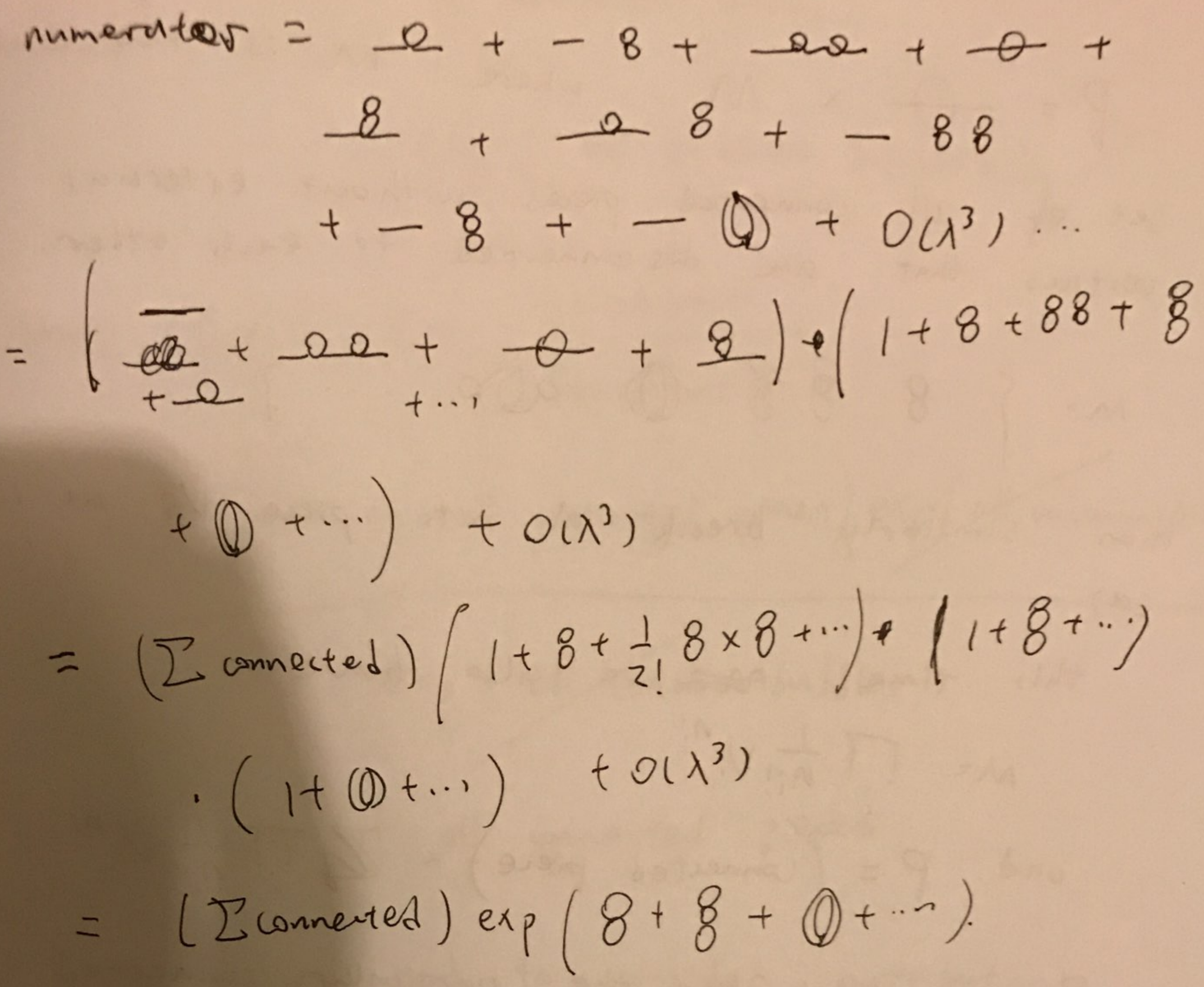
and



- For order λ^2 :

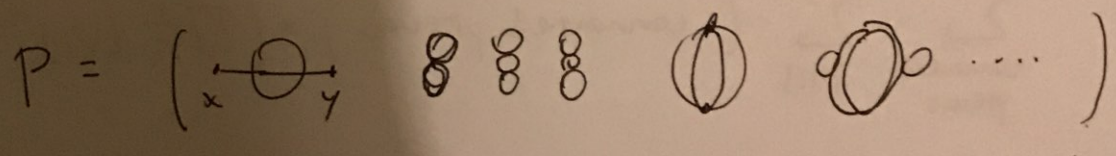


so up to order λ^2 we have



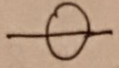
up to order λ^2

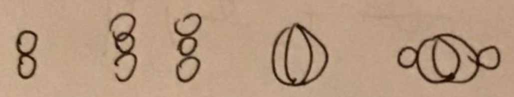
In general, consider a particular diagram



~~If~~ since each vertex has a even number of lines coming into it, x and y must be in the same ~~connect~~ connected piece.

We can take this piece out and

$P =$  $\times M$ where M is the set of all connected pieces without external vertices that are disconnected to each other.

$M = \{$  $\dots \}$

then similarly break M into piece V_i as in (a)

this time ~~AA~~ we also have

$$M = \prod_i \frac{1}{n_i!} V_i^{n_i}$$

and $P = \left(\text{connected piece}\right) \times \prod_i \frac{1}{n_i!} V_i^{n_i}$

P is defined by the particular connected piece and the set $\{n_i\} = \{n_1, n_2, \cdots\}$ that describes M.

$\therefore$ Numerator $=$ sum of all P

$$= \sum_{\text{connected pieces}} \sum_{\{n_i\}} \left(\text{connected pieces}\right) \times \left(\prod_i \frac{1}{n_i} (V_i)^{n_i}\right)$$

$$= \left(\sum \text{connected pieces}\right) \underbrace{\sum_{\{n_i\}} \prod_i \frac{1}{n_i!} (V_i)^{n}}_{\text{By (a)} \ = \ \exp\left(\sum_i V_i\right)}$$

 = (sum of connected pieces)

Numerator

× exp (all vacuum bubbles). ✓

□

Now $\langle \phi(x)\phi(y)\rangle = \dfrac{\text{Numerator}}{\text{Denominator}}$

$$= \frac{(\text{sum of all connected pieces})\,\cancel{\exp(\text{all vacuum bubbles})}}{\cancel{\exp(\text{all vacuum bubbles})}}$$

$$= \text{Sum of all connected pieces}$$

We only need to consider contributions from connected pieces. □ Good!

⑦ (a)

Lagrangian (Euclidean space)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}M^2\chi^2 + \frac{1}{2}\lambda\phi^2\chi$$

with ~~both~~ ϕ, χ both real fields.

we can write the (Minkowski) canonical quantisation of ϕ and χ as operators

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(\hat{a}_{\vec{p}}e^{-ip\cdot x} + \hat{a}_{\vec{p}}^\dagger e^{ip\cdot x}\right)$$

~~$\chi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\Omega_p}}$~~

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\Omega_{\vec{p}}}}\left(\hat{b}_{\vec{p}}e^{-ip\cdot x} + \hat{b}_{\vec{p}}^\dagger e^{ip\cdot x}\right)$$

where $\omega_{\vec{p}}^2 = \vec{p}^2 + m^2$ and $\Omega_{\vec{p}}^2 = \vec{p}^2 + M^2$

Hence the ~~vacuum expectation value the~~ free propagators are

$$\langle\phi(x)\phi(y)\rangle_0 = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$$

$$= \Delta_{F(m)}(x-y) = \int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

$\Rightarrow$ in Euclidean space

$$\langle\phi(x)\phi(y)\rangle_0 = \int\frac{d^Dp}{(2\pi)^D}\frac{e^{ip\cdot(x-y)}}{p^2+m^2}$$

$\downarrow$

Wick rotation

$d^Dp \to id^Dp$

$p_0 \to -ip_0$

$p^2 \to -p^2$

$p\cdot x \to -p_\mu x^\mu$

$\Rightarrow$

Similarly. $\langle \chi(x)\,\chi(y)\rangle_0 = \int \frac{d^Dp}{(2\pi)^D}\,\frac{e^{ip(x-y)}}{p^2+M^2}$

the exponentials will become the momentum conserving delta function when integrated over vertice positions

so we are left with $\frac{1}{p^2+m^2}$ and $\frac{1}{p^2+M^2}$ representing each line in the Feynmann diagram.

Also $\langle \phi(x)\,\chi(y)\rangle_0 = \langle 0|\,T\{\phi(x)\,\chi(y)\}\,|0\rangle$

$$= \overline{\phi(x)\,\chi(y)} \underset{\text{assuming } x^0>y^0}{=} [\phi^+(x),\ \chi^-(y)]$$

But ∵ $[a_{\vec p},\ b^\dagger_{\vec q}]=0$

∴ $\langle \phi(x)\,\chi(y)\rangle_0 = 0$ ∴ ~~ϕ cannot~~ there is <u>no</u> propagation from ϕ to χ.

The Feynmann Rules in momentum space are.

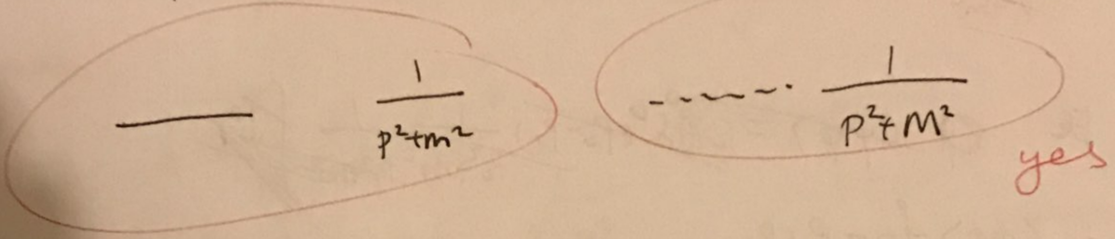
1. Draw all topologically distinct diagrams
2. Assign momenta flowing through each line so that momentum is conserved at each ~~ver~~ vertex.
3. To each vertex associate a factor $-\lambda$
4. To $\underbrace{\text{each line}}_{\text{between } \phi \text{ and } \phi}$ ~~ast~~ associate a factor $\frac{1}{p^2+m^2}$ ✓

5. To each line between χ and χ associate a factor $\frac{1}{p^2+M^2}$

6. Integrate over loop momenta

7. Multiply by the symmetry factor.

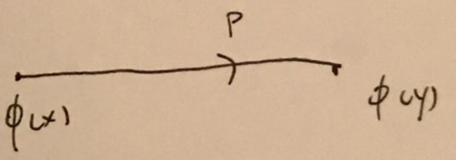
8. Multiply by momentum conserving ~~deltat~~ delta function



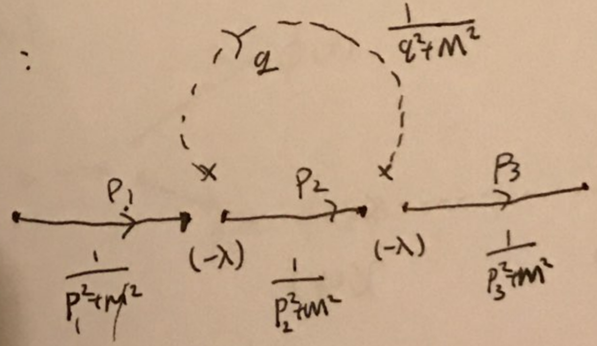
(b)

$\langle\phi\phi\rangle = \langle\Omega|\phi(x)\phi(y)|\Omega\rangle$

tree level :



loop level :



$\times$ is a interaction potential $\frac{\lambda}{2}\phi^2\chi$

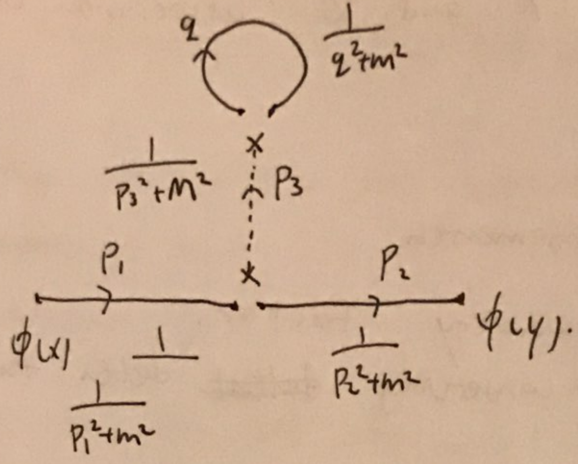
~~P_1+~~ $P_1 = P_2 + q = P_3$ $\Rightarrow$ $P_3 = P_1$. $\Rightarrow (2\pi)^D\delta^D(P_3-P_1)$

$P_2 = P_1 - q$.

Apply Feynmann Rule !

$$\langle\phi\phi\rangle = (-\lambda)^2(2\pi)^D\delta^D(P_3-P_1)\frac{1}{P_1^2+m^2}\frac{1}{P_3^2+m^2}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^2+M^2}\frac{1}{(P_1-q)^2+m^2}$$

(symmetry factor $W_G = 1$)



$P_3 + q - q = 0 \Rightarrow P_3 = 0$

$W_G = \frac{1}{2}$ ~~upon ext~~

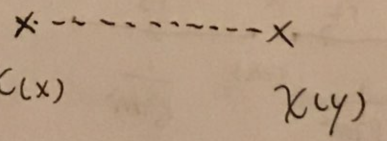
∵ 1 self connection.

Ele ~~$\langle\phi\phi\rangle = (2\pi)\delta^D(p_2-p_1)\frac{1}{p_2^2+m^2}\frac{1}{p_1^2+m^2}\int d^Dq$~~

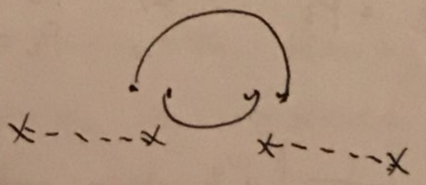
$$\therefore \langle\phi\phi\rangle = \left[(2\pi)^D\delta^D(p_2-p_1)\frac{1}{p_2^2+m^2}\frac{1}{p_1^2+m^2}\frac{1}{+M^2}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^2+m^2}\right]\times\frac{(-\lambda)^2}{2}$$

$\langle\chi\chi\rangle$:

tree level :

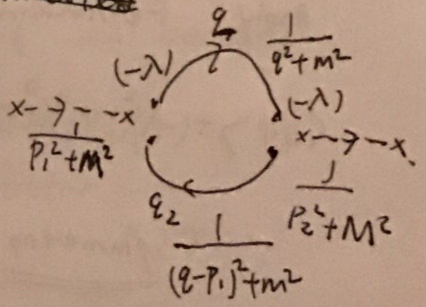


loop level :



=

~~disconnected~~



~~q_1+q_2~~ $q_1 - q_2 = P_1 = P_2$

$\therefore q_2 = q - P_1$

$$\therefore \langle \chi\chi \rangle = (2\pi)^D \delta^D(p_2 - p_1) \frac{(-\lambda)^2}{2} \left[\int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2+m^2} \frac{1}{(q-p_1)^2+m^2} \right] \frac{1}{p_1^2+M^2} \frac{1}{p_2^2+M^2}$$

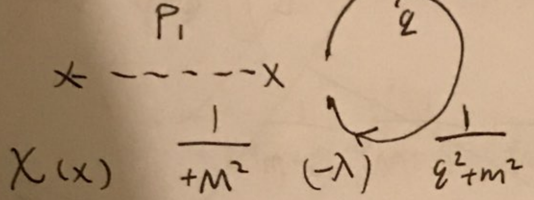
$\because$ 1 double connection

$\therefore$ $W_G = \frac{1}{2}$

$\langle \chi \rangle$:

No tree level diagram

1-loop :



$p_1 + q - q = 0$

$\therefore$ $p_1 = 0$

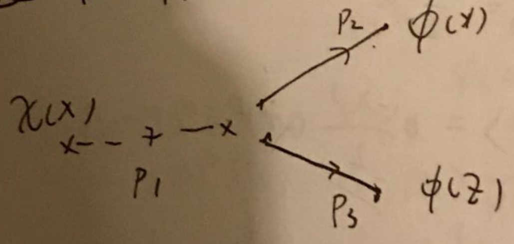
$W_G = 2$

$\because$ 1 self connection.

$$\therefore \langle \chi \rangle = (2\pi)^D \delta^D(p_1) \frac{1}{+M^2} \frac{(-\lambda)}{2} \int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2+m^2}.$$

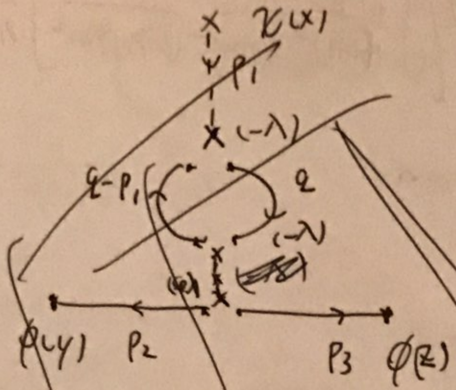
$\langle \chi\phi\phi \rangle$:

tree level.



~~$\langle \chi\phi\phi$~~

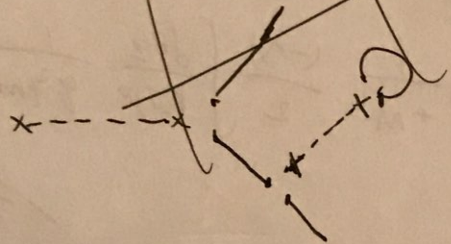
1-loop level : (contribution only from connected diagrams).

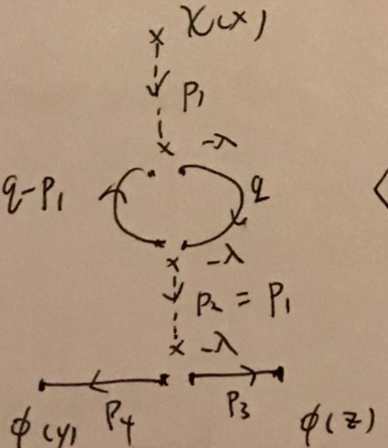


~~$W_G = \frac{1}{2}$~~

~~$$\langle \chi\phi\phi \rangle = (2\pi)^D \delta(p_1 - p_2 - p_3) \frac{(-\lambda)^2}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+m^2} \frac{1}{(q-p_1)^2+m^2}$$~~

~~$$\frac{1}{p_1^2+M^2} \frac{1}{p_2+m^2} \frac{1}{p_3+m^2}$$~~



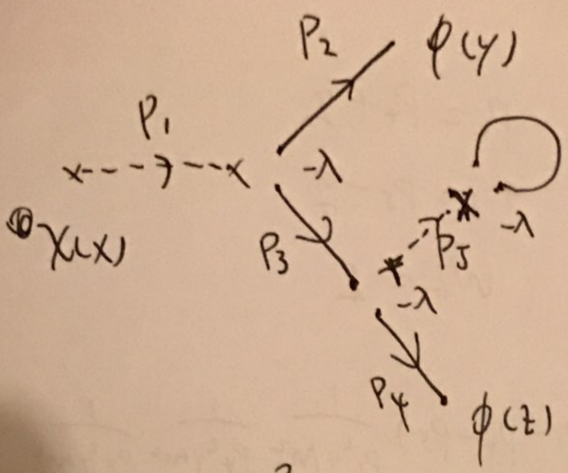


$W_G = \frac{1}{2}$

$$\langle \chi\phi\phi \rangle = \frac{(-\lambda)^3}{2} (2\pi)^D \delta^D(p_1 - p_3 - p_4) \times$$

$$\frac{1}{p_1^2+M^2} \frac{1}{p_1^2+M^2} \frac{1}{p_3^2+m^2} \frac{1}{p_4^2+m^2} \times$$

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+m^2} \frac{1}{(q-p_1)^2+m^2}$$



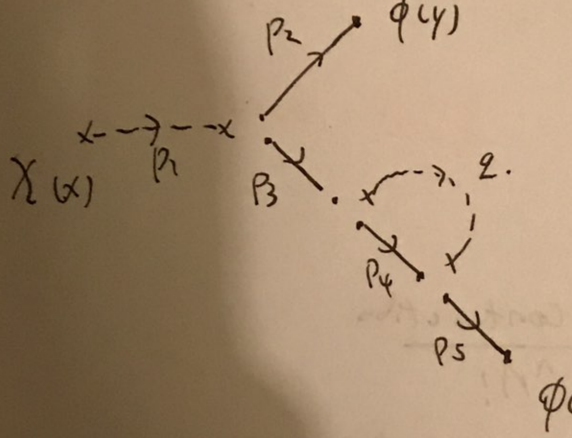
$P_5 = 0.$

$P_3 = P_4.$

$W_G = \frac{1}{2}.$

$$\langle \chi \phi\phi \rangle = \frac{(-\lambda)^3}{2} \frac{1}{P_1^2+M^2} \frac{1}{P_2^2+m^2} \frac{1}{P_4^2+m^2} \frac{1}{P_4^2+m^2} (2\pi)^D \delta^D(P_1-P_2-P_4)$$

$$\times \frac{1}{+M^2} \int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2+m^2}$$



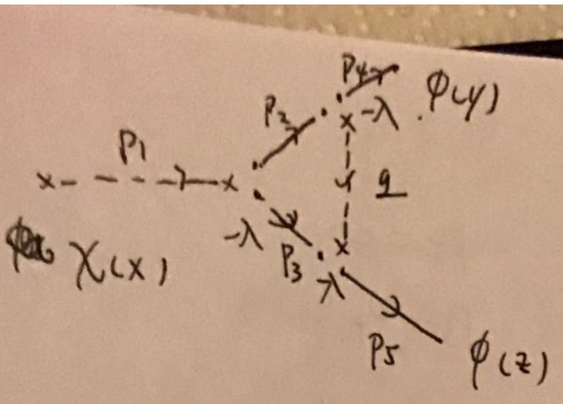
$P_4 = P_3 - q. = P_5 - q$

$P_3 = P_5$

$W_G = 1$

$$\langle \chi \phi\phi \rangle = \frac{(-\lambda)^3}{1} (2\pi)^D \delta^D(P_1-P_2-P_5) \frac{1}{P_1^2+M^2} \frac{1}{P_2^2+m^2} \frac{1}{P_5^2+m^2} \frac{1}{P_5^2+m^2} \times$$

$$\int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2+M^2} \frac{1}{(P_5-q)^2+m^2}$$



$P_2 = P_4 + q$

$P_3 = P_5 - q$

$W_G = 1$

$$\therefore \langle \chi\phi\phi \rangle = (-\lambda)^3 (2\pi)^D \delta^D(P_1 - P_4 - P_5) \frac{1}{P_1^2+M^2} \frac{1}{P_4^2+m^2} \frac{1}{P_5^2+m^2} \times$$

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+M^2} \frac{1}{(P_4+q)^2+m^2} \frac{1}{(P_5-q)^2+m^2}$$

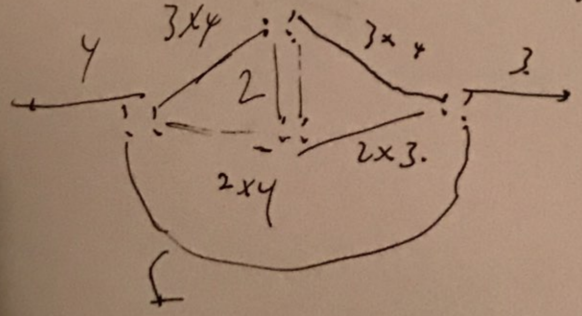
yes

Well done

$$W_G = \frac{\#\text{ Wick contraction}}{(4!)^n n!}$$

$$= \frac{A!}{N_{2vp}} \frac{\#\text{ ways to connected points}}{D!}$$

$$= \frac{1}{|S_G|} \rightarrow$$ # of elements of symmetry group.



$W_G =$