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Tues 3.30 - 5 pm Wks 3, 5, 7, 8

Quantum Field Theory

Problem Set 1

Very good!!

Your solutions are longer than they need to be.

① Lagrangian ~~表~~ density ("Lagrangian")

$$\mathcal{L} = \frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi - \frac{1}{2}m^2\phi^2$$

least action principle gives the Euler-Lagrange equation (E-L equation)

(a)

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi} :$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\phi)}\left[\eta^{\sigma\nu}\partial_\sigma\phi\,\partial_\nu\phi\right]$$

$$= \frac{1}{2}\left[\eta^{\sigma\nu}\partial_\sigma\phi\,\delta^\mu_\nu + \eta^{\sigma\nu}\partial_\nu\phi\,\delta^\mu_\sigma\right]$$

$$= \frac{1}{2}\left[\eta^{\mu\sigma}\partial_\sigma\phi + \eta^{\mu\nu}\partial_\nu\phi\right]$$

$$= \eta^{\mu\sigma}\partial_\sigma\phi = \partial^\mu\phi$$

$$\therefore\ \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \partial_\mu\partial^\mu\phi$$

$$\therefore\ \partial_\mu\partial^\mu\phi = -m^2\phi \Rightarrow (\partial_\mu\partial^\mu + m^2)\phi = 0$$

→ Klein-Gordon equation

(b) $\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$ $\quad\left(\dot{\phi} = \frac{\partial\phi}{\partial t}\right)$

$$\therefore\ \pi(x) = \frac{\partial}{\partial\dot{\phi}}\left[\frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi - \frac{1}{2}m^2\phi^2\right]$$

$$= \frac{\partial}{\partial \dot\phi}\left[\frac{1}{2}\dot\phi^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{1}{2}(\partial_y\phi)^2 - \frac{1}{2}(\partial_z\phi)^2\right]$$

$$\eta = \begin{pmatrix} +1 & \\ & -1_{3} \end{pmatrix} \qquad = \dot\phi$$

(c) Hamiltonian density $\mathcal{H} = \pi\dot\phi - \mathcal{L}$

$$\mathcal{H} = \pi\dot\phi - \mathcal{L} = \dot\phi^2 - \left[\frac{1}{2}\dot\phi^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{1}{2}(\partial_y\phi)^2 - \frac{1}{2}(\partial_z\phi)^2 - \frac{1}{2}m^2\phi^2\right]$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\vec\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

(d) Transformation rules for scalar fields.

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$

under Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

The transformation of $\partial_\mu\phi(x)$ is

good

$$\partial_\mu\phi(x) \mapsto \partial_\mu(\phi(\Lambda^{-1}x)) = \Lambda^{-1\nu}{}_\mu(\partial_\nu\phi)(\Lambda^{-1}x)$$

chain rule.

* Note: $\cancel{\partial_\mu(\phi)}$ $\partial_\mu(f(x)) = \frac{\partial}{\partial x^\mu}(f(x))$

$$(\partial_\mu f)(x) = \left.\frac{\partial f}{\partial y^\mu}\right|_{y=x}$$

Now the transformed kinetic term is:

$$(\partial_\mu\phi)(\partial_\mu\phi) \longmapsto \eta^{\mu\nu}\partial_\mu(\phi(\Lambda^{-1}x))\,\partial_\nu(\phi(\Lambda^{-1}x))$$

$$= \eta^{\mu\nu}\left[(\Lambda^{-1})^{\alpha}{}_{\mu}\,(\partial_\alpha\phi)_{(\Lambda^{-1}x)}\right]\left[(\Lambda^{-1})^{\beta}{}_{\nu}(\partial_\beta\phi)_{(\Lambda^{-1}x)}\right]$$

$$= \left[(\Lambda^{-1})^{\alpha}{}_{\mu}\,(\Lambda^{-1})^{\beta}{}_{\nu}\,\eta^{\mu\nu}\right](\partial_\alpha\phi)_{(\Lambda^{-1}x)}\,(\partial_\beta\phi)_{(\Lambda^{-1}x)}$$

consider $(\Lambda^{-1})^{\alpha}{}_{\mu}\,(\Lambda^{-1})^{\beta}{}_{\nu}\,\eta^{\mu\nu}$

$$= (\Lambda^{-1})^{\alpha}{}_{\mu}\,\eta^{\mu\nu}\left[(\Lambda^{-1})^{T}\right]_{\nu}{}^{\beta}$$

$$= \underbrace{\Lambda^{-1}\cdot\eta\cdot(\Lambda^{-1})^{T}}_{\begin{pmatrix}\gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}+1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}\gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1\end{pmatrix} = \begin{pmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}} = \eta = \eta^{\alpha\beta}$$

Also let ~~$x' = \Lambda^{-1}x$, $\phi'(x) = \phi(x')$~~

$y = \Lambda^{-1}x \quad \phi'(x) = \phi(y) = \phi(y(x))$

then

~~$(\partial_\mu\phi')(\partial^\mu\phi')$~~

$$(\partial_\mu\phi')(\partial^\mu\phi') = \eta^{\alpha\beta}(\partial_\alpha\phi)_{(y)}(\partial_\beta\phi)_{(y)}$$

$$= (\partial_\alpha\phi)_{(y)}(\partial^\alpha\phi)_{(y)}$$

~~the overall~~

the overall transformed Lagrangian is

$$\mathcal{L} \mapsto \mathcal{L}'(x) = \frac{1}{2}(\partial_\alpha\phi')(\partial^\alpha\phi') - \frac{1}{2}m^2\phi'$$

$p_2 = p_4 + q$

$$= \frac{1}{2}(\partial_\alpha \phi)(y)(\partial^\alpha \phi)(y) - \frac{1}{2} m^2 \phi^2(y)$$

$$= \mathcal{L}(y)$$

$\Rightarrow \mathcal{L}'(x) = \mathcal{L}(y)$ $\to$ Lagrangian density is a scalar

The action $S = \int d^4x \, \mathcal{L}(x)$ is transformed to $S' = \int d^4x \, \mathcal{L}'(x) = \int d^4x \, \mathcal{L}(y)$

$$= \int d^4y \, \mathcal{L}(y) \, |J(y,x)|$$

where $J(y,x)$ is the Jacobian matrix of 4 vectors y and x, $|\ |$ is the determinant.

$\because y^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu$ $J(y,x) = \frac{\partial y}{\partial x}$ ~~is precis~~

$$J^{\alpha\beta} = \frac{\partial y^\alpha}{\partial x^\beta} = (\Lambda^{-1})^\alpha{}_\nu \delta^\nu_\beta = (\Lambda^{-1})^\alpha{}_\beta$$

is precisely $\Lambda^{-1} = \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\left(\begin{array}{l} \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\ \beta = \frac{v}{c} \end{array} \right)$

$$|J| = \det(J) = \det(\Lambda^{-1}) = \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = 1$$

$$\therefore S' = \int d^4y \, \mathcal{L}(y) = \int d^4x \, \mathcal{L}(x) = S$$

$\Rightarrow$ changing $\phi(x)$ to $\phi'(x) = \phi(\Lambda^{-1}x)$ leaves the action S unchanged

Well. Could be on a general form. But always $|\Lambda| = 1$ since it's a Lorentz TF.

→ if ϕ minimises the action, so does ϕ'

→ the field theory is invariant under Lorentz transformation.

we can also check that the equation of motion is invariant:

$$(\partial_\mu \partial^\mu + m^2)\phi'(x) = \cancel{[(\Lambda^{-1})^\nu{}_\mu \partial_\nu (\Lambda^{-1}x)}$$

$$\cancel{= ((\Lambda^{-1})^\mu{}_\sigma (\Lambda^{-1})} \quad = [(\Lambda^{-1})^\nu{}_\mu \partial_\nu (\Lambda^{-1})_\sigma{}^\mu \partial^\sigma + m^2]\phi(\Lambda^{-1}x)$$

$$= [(\Lambda^{-1})^\nu{}_\mu \partial_\nu (\Lambda^{-1})^{\sigma\mu} \partial_\sigma + m^2]\phi(\Lambda^{-1}x)$$

$$\cancel{= [(\Lambda^{-1})^\nu{}_\mu (\Lambda^{-1})^\mu{}_\sigma g^{\sigma\nu} \partial_\nu \partial_\sigma + m^2]\phi(\Lambda^{-1}x)}$$

$$= [\underbrace{(\Lambda^{-1})^\nu{}_\mu (\Lambda^{-1})^\sigma{}_\rho g^{\mu\rho}}_{g^{\nu\sigma}} \partial_\nu \partial_\sigma + m^2]\phi(\Lambda^{-1}x)$$

$$= [g^{\nu\sigma}\partial_\nu\partial_\sigma + m^2]\phi(\Lambda^{-1}x)$$

$$= [\partial_\sigma\partial^\sigma + m^2]\phi(\Lambda^{-1}x)$$

= the function $(\partial_\sigma\partial^\sigma + m^2)\phi$ evaluated at $\Lambda^{-1}x$, ~~be~~ but this function is identically 0 everywhere

∴ the result is 0

$\rightarrow$ Klein-Gordon equation persened.

(e)

Noether's Theorem states that ~~fa~~ for an infinitesimal transformation of field ϕ,

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha \Delta\phi(x),$$

the Lagrangian is invariant up to a 4-divergence

$$\mathcal{L}(x) \rightarrow \mathcal{L}'(x) = \mathcal{L}(x) + \alpha \partial_\mu J^\mu(x),$$

then there is a correspond conserved current

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu$$ and a conserved

charge

$$Q = \int_{\text{all space}} j^0 \, d^3x$$

Now consider infinitesimal spacetime translation,

$$x^\mu \rightarrow x'^\mu = x^\mu - a^\mu$$ (active transformation)

the transformation of field ϕ is then

$$\phi(x) \rightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

active transformation

The Lagrangian (a scalar) is also transformed

like $$\mathcal{L} \rightarrow \mathcal{L}'(x) = \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L}(x)$$

$$= \mathcal{L}(x) + a^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L})$$

Now we ~~got~~ observe the ~~corr~~ correspondence

$\Delta\phi \Leftrightarrow \partial_\nu \phi$, $\partial_\mu J^\mu \Leftrightarrow \partial_\mu(\delta^\mu_\nu \mathcal{L})$

If only the ν component of a is active (non-zero)

$\therefore$ conserved current is

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

the energy-momentum tensor. (stress-energy tensor)

Now use the Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\phi)}[\partial_\rho\phi\,\partial^\rho\phi] = \delta^\mu_\rho\,\partial^\rho\phi = \partial^\mu\phi$$

$$\therefore T^\mu_\nu = \cancel{\partial^\mu\phi\,\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}}\ \partial^\mu\phi\,\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$$

$$T^{\mu\nu} = \cancel{\partial^\mu\phi}\ \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

$$T^\mu_\nu = \partial^\mu\phi\,\partial_\nu\phi - \delta^\mu_\nu\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + \delta^\mu_\nu\frac{1}{2}m^2\phi^2$$

conserved charge $P^\nu = \begin{pmatrix} E \\ p^i \end{pmatrix}$

where

$$E = \int T^{00}\, d^3x = \int \dot\phi^2 - \mathcal{L}\; d^3x$$

$\eta^{00} = 1$

$$= \int \dot\phi^2 - (\dot\phi^2 - (\vec\nabla\phi)^2 - \tfrac{1}{2}m^2\phi^2)\, d^3x$$

$$= \int d^3x \;\; \tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2}(\vec\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2$$

$\eta^{0i} = 0$

$$P^i = \int T^{0i}\, d^3x = \int \dot\phi\, \partial^i\phi\; d^3x$$

(f) $\cancel{\partial_\mu T^{\mu\nu} =}$

$$\partial_\mu T^\mu{}_\nu = \partial_\mu(\partial^\mu\phi\,\partial_\nu\phi) - \delta^\mu{}_\nu\,\partial_\mu\left(\tfrac{1}{2}\partial_\rho\phi\,\partial^\rho\phi\right) + \delta^\mu{}_\nu\,\partial_\mu\left(\tfrac{1}{2}m^2\phi^2\right)$$

$$= (\partial_\mu\partial^\mu\phi)\,\partial_\nu\phi + \partial^\mu\phi(\partial_\mu\partial_\nu\phi) - \tfrac{1}{2}\partial_\nu(\partial_\rho\phi\,\partial^\rho\phi) + \partial_\nu(\tfrac{1}{2}m^2\phi^2).$$

$$= (\partial_\mu\partial^\mu\phi)\,\partial_\nu\phi + \cancel{\partial^\mu\phi(\partial_\mu\partial_\nu\phi)} - \cancel{(\partial_\nu\partial_\rho\phi)\partial^\rho\phi} + m^2\phi\,\partial_\nu\phi$$

$$= (\partial_\nu\phi)(\partial_\mu\partial^\mu + m^2)\phi = 0 \quad \text{by eom}$$

$\because$ ~~$\partial_\nu \phi$ is not generally~~

$\because \quad (\partial_\mu \partial^\mu + m^2)\phi = 0$ by Klein Gordon equation

$\therefore \quad \underline{\underline{\partial_\mu T^\mu{}_\nu = 0}}$

(g) $\quad P_0 = P^0 = E = \int \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\vec{\nabla}\phi|^2 + \frac{1}{2}m^2\phi^2 \, d^3x$

$\underline{\underline{= \int \mathcal{H} d^3x}}$ precisely

yes

(2)

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

(a) conjugate momenta

to ϕ :

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \frac{\partial}{\partial \dot\phi}\left[\dot\phi \dot\phi^* - \vec\nabla\phi\cdot\vec\nabla\phi^* - m^2 \phi\phi^*\right]$$

$$= \dot\phi^*$$

to ϕ^*

$$\pi^\dagger = \pi^* = \frac{\partial \mathcal{L}}{\partial \dot\phi^*} = \frac{\partial}{\partial \dot\phi^*}\left[\dot\phi\dot\phi^* - \vec\nabla\phi\cdot\vec\nabla\phi^* - m^2\phi\phi^*\right]$$

$$= \dot\phi$$

canonical commutation relations :

$$[\phi(\vec x), \pi(\vec y)] = [\phi^\dagger(\vec x), \pi^\dagger(\vec y)] = i\delta^{(3)}(\vec x - \vec y)$$

all other commutators vanish, such as

$$[\phi(\vec x), \phi(\vec y)] = [\phi(\vec x), \phi^\dagger(\vec y)] = [\phi(\vec x), \pi^\dagger(\vec y)] = 0$$

once we impose $[\phi(\vec x), \pi(\vec y)] = i\delta^{(3)}(\vec x - \vec y)$

it follows that $\phi(\vec x)\pi(\vec y) - \pi(\vec y)\phi(\vec x) = i\delta^{(3)}(\vec x - \vec y)$

taking complex conjugate (c.c) :

$$\pi^\dagger(\vec y)\phi^\dagger(\vec x) - \phi^\dagger(\vec x)\pi^\dagger(\vec y) = -i\delta^{(3)}(\vec x - \vec y)$$

$$\Rightarrow [\phi^{+}(\vec{x}), \pi^{+}(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$$

The Hamiltonian density $\mathcal{H} = \sum_i \pi_i \dot{\phi}_i - \mathcal{L}$

$$\mathcal{H} = \pi\dot{\phi} + \pi^{*}\dot{\phi}^{*} - \mathcal{L}$$

$$= \cancel{2\dot{\phi}\dot{\phi}}\ 2\dot{\phi}\dot{\phi}^{*} - \partial_\mu\phi^{*}\partial^\mu\phi^{*} + m^2\phi\phi^{*}$$

$$= 2\dot{\phi}\dot{\phi}^{*} - \dot{\phi}\dot{\phi}^{*} + \vec{\nabla}\phi\cdot\vec{\nabla}\phi^{*} + m^2\phi\phi^{*}$$

$$= \pi^{*}\pi + \vec{\nabla}\phi^{*}\cdot\vec{\nabla}\phi + m^2\phi^{*}\phi$$

$\therefore$ Hamiltonian $\quad H = \int d^3x\,\mathcal{H} = \int d^3x\,(\pi^{*}\pi + \vec{\nabla}\phi^{*}\cdot\vec{\nabla}\phi + m^2\phi^{*}\phi)$

The Heisenberg equation of motion is

$$i\frac{\partial}{\partial t}\hat{O} = [\hat{O}, \hat{H}] \quad \text{for an operator } \hat{O}.$$

Now for $\hat{\phi}(x) = \phi(x) = \phi(\vec{x}, t)$

~~$i\frac{\partial}{\partial t}\phi(x)$~~ commutation relations are applied at equal time.

$$i\frac{\partial}{\partial t}\phi(x) = [\phi(x), \int d^3x' [\pi^{*}\pi + \vec{\nabla}\phi^{*}\cdot\vec{\nabla}\phi + m^2\phi^{*}\phi]]$$

$$= \int d^3x' [\phi(x), \pi(x')]\,\pi^{*}(x')$$

$$= \int d^3x' \, i\delta^{(3)}(\vec{x}-\vec{x}') \, \pi^*(x')$$

$$= \cancel{i\pi^*(x')} \quad i\pi^*(x) = \underline{\underline{i\pi^*(x)}}$$

$$\cancel{i\frac{\partial}{\partial t}\pi^*(x) = \left[\pi^*(x), \int d^3x' \{\pi(x')\pi^*(x') + \vec{\nabla}\phi^*\cdot\vec{\nabla}\phi + m^2\phi^*\phi\}\right]}$$

$$\cancel{= \int d^3x \left[\pi(x), 2\left(\vec{\nabla}\phi^\dagger\cdot[\pi(x), \vec{\nabla}\phi(x')] + m^2\phi^\dagger[\pi(x), \phi(x')]\right)\right.}$$

$$i\frac{\partial}{\partial t}\pi^*(x) = \left[\pi^*(x), \int d^3x' \{\pi(x')\pi^*(x') + \vec{\nabla}\phi^*\cdot\vec{\nabla}\phi + m^2\phi^*\phi\}\right]$$

$$= \int d^3x' \left([\pi^*(x), \vec{\nabla}\phi^*(x')]\,\vec{\nabla}\phi + [\pi^*, \phi^*]\, m^2\phi\right).$$

$\because$ $\vec{\nabla}$ only act on x' yes

$$\therefore \quad \cancel{[\pi^*(x), \vec{\nabla}\phi} \quad \vec{\nabla}[\pi^*(x), \phi^*(x')]$$

$$= \vec{\nabla}\left(\pi^*(x)\phi^*(x') - \phi^*(x')\pi^*(x)\right)$$

$$= \pi^*(x)\,\vec{\nabla}\phi^*(x') - \left(\vec{\nabla}\phi^*(x')\right)\pi^*(x)$$

$$= [\pi^*(x), \vec{\nabla}\phi^*(x')]$$

$$\therefore\ i\frac{\partial}{\partial t}\pi^*(x) = \int d^3x' \left(\vec{\nabla}[\pi^*(x), \phi^*(x')] \cdot \vec{\nabla}\phi(x') - i\delta^{(3)}(\vec{x}-\vec{x}')\, m^2\phi(x') \right)$$

$$= -i\left[\int d^3x'\ \vec{\nabla}\delta^{(3)}(\vec{x}-\vec{x}')\cdot\vec{\nabla}\phi\right] - im^2\phi(x)$$

the integral $\int d^3x'\ \vec{\nabla}\delta^{(3)}(\vec{x}-\vec{x}')\cdot\vec{\nabla}\phi$

$$= \int d^3x'\ \vec{\nabla}_{\vec{x}'}\delta^{(3)}(\vec{x}-\vec{x}')\cdot\vec{\nabla}_{\vec{x}'}\phi$$

$\vec{\nabla}_{\vec{x}'}\delta^{(3)}(\vec{x}-\vec{x}') = -\vec{\nabla}_{\vec{x}}\delta^{(3)}(\vec{x}-\vec{x}')$

$$= -\int d^3x'\ \vec{\nabla}_{\vec{x}}\delta^{(3)}(\vec{x}-\vec{x}')\cdot\vec{\nabla}_{\vec{x}'}\phi(x')$$

$\vec{\nabla}\cdot(\rho\vec{F}) = \vec{\nabla}\rho\cdot\vec{F}$ for constant vector $\vec{F}$ in this case $\vec{F} = \vec{\nabla}_{\vec{x}'}\phi(\vec{x}')$

$$= -\vec{\nabla}_{\vec{x}}\cdot\int d^3x'\ \delta^{(3)}(\vec{x}-\vec{x}')\,\vec{\nabla}_{\vec{x}'}\phi(x')$$

$$= -\vec{\nabla}_{\vec{x}}\cdot\vec{\nabla}_{\vec{x}}\phi(\vec{x}) = -\nabla^2\phi$$

$$\therefore\ i\frac{\partial}{\partial t}\pi^*(x) = i(\nabla^2\phi - m^2\phi) = i(\nabla^2 - m^2)\phi$$

$$\therefore\ \dot{\phi} = \pi^*,\quad \dot{\pi}^* = (\nabla^2 - m^2)\phi$$

$$\therefore\ \ddot{\phi} = (\nabla^2 - m^2)\phi$$

$$\Rightarrow [(\partial_t^2 - \nabla^2) + m^2]\phi = 0 \Rightarrow (\partial^2 + m^2)\phi = 0$$

the Klein-Gordon equation

~~(a)~~ Similarly there is a ~~equa~~ Klein Gordon equation for ϕ^* :

$$(\partial^2+m^2)\phi^* = 0$$

(b) Introducing the creation and annihilation operators and $\hat{\phi}$, $\hat{\phi}^*$ ($\hat{\phi}^\dagger$) now are operators.

We have (in Schrödinger picture)

$$\hat{\phi} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(\hat{b}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + \hat{c}_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right) = \hat{\phi}(\vec{x})$$

$$\hat{\phi}^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(\hat{b}_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} + \hat{c}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \right) = \hat{\phi}^\dagger(\vec{x})$$

We expand in different creation and annihilation operators $\hat{b}_{\vec{p}}$ and $\hat{c}_{\vec{p}}$ because ~~there is no~~ ϕ is a complex field (not real), and so there is no gaurantee that operator $\hat{\phi}$ is Hermitian. yes It's not.

– Now, since $\pi = \dot{\phi}^*$ and $\pi^* = \dot{\phi}$ we need time dependent operators to get $\hat{\pi}$, $\hat{\pi}^\dagger$ from $\hat{\phi}$, $\hat{\phi}^\dagger$

This is done in the Heisenburg picture

$$\hat{\phi}(x) = \hat{\phi}(\vec{x},t) = \int \frac{d^3p}{(2\pi^3)} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(\hat{b}_{\vec{p}} e^{-ip\cdot x} + \hat{c}_{\vec{p}}^\dagger e^{ip\cdot x} \right)$$

4-moment $p = \begin{pmatrix} \omega_{\vec{p}} \\ \vec{p} \end{pmatrix}$

$$\hat{\pi}^{\dagger}(x) = \hat{\pi}^{\dagger} = \frac{\partial \phi}{\partial t} = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\vec{p}}}{2}} \left(\hat{b}_{\vec{p}} e^{-ip\cdot x} - \hat{c}_{p}^{\dagger} e^{ip\cdot x} \right)$$

turn this back into schrodinger picture

$$\hat{\pi}^{\dagger}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\vec{p}}}{2}} \left(\hat{b}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - \hat{c}_{p}^{\dagger} e^{-i\vec{p}\cdot\vec{x}} \right)$$

similarly $\hat{\pi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \, i\sqrt{\frac{\omega_{\vec{p}}}{2}} \left(\hat{b}_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}} - \hat{c}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \right)$

Now we put all operators in to the expression for Hamiltonian

$$\hat{H} = \int d^3x \; \hat{\pi}^{\dagger}\hat{\pi} + \vec{\nabla}\hat{\phi}^{\dagger}\cdot\vec{\nabla}\hat{\phi} + m^2 \hat{\phi}^{\dagger}\hat{\phi}$$

$$= \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6} \Bigg[+ \frac{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}{2} \left(\hat{b}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - \hat{c}_{\vec{p}}^{\dagger} e^{-\vec{p}\cdot\vec{x}} \right)\left(\hat{b}_{\vec{q}}^{\dagger} e^{-i\vec{q}\cdot\vec{x}} - \hat{c}_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} \right)$$

$$+ \frac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}} \left(-i\vec{p}\, \hat{b}_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}} + i\vec{p}\, \hat{c}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \right)\left(i\vec{q}\, \hat{b}_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} - i\vec{q}\, \hat{c}_{\vec{q}}^{\dagger} e^{-i\vec{q}\cdot\vec{x}} \right)$$

$$+ \frac{m^2}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}} \left(\hat{b}_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}} + \hat{c}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \right)\left(\hat{b}_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} + \hat{c}_{\vec{q}}^{\dagger} e^{-i\vec{q}\cdot\vec{x}} \right) \Bigg]$$

(use delta function is even), use $\int \frac{d^3x}{(2\pi)^3} e^{\pm i(\vec{p}\pm\vec{q})\cdot\vec{x}} = \delta^{(3)}(\vec{p}\pm\vec{q})$

$$H = \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6} \Big[+ \frac{\sqrt{\omega_{\vec p}\omega_{\vec q}}}{2} \Big(\hat b_{\vec p} \hat b^\dagger_{\vec q} e^{i(\vec p - \vec q)\cdot \vec x} - \hat b_{\vec p} \hat c_{\vec q} e^{i(\vec p + \vec q)\cdot \vec x}$$

$$- c^\dagger_{\vec p} b^\dagger_{\vec q} e^{-i(\vec p + \vec q)\cdot \vec x} + \hat c^\dagger_{\vec p} \hat c_{\vec q} e^{-i(\vec p - \vec q)\cdot \vec x} \Big)$$

$$+ \frac{1}{2\sqrt{\omega_{\vec p}\omega_{\vec q}}} \Big(\vec p\cdot\vec q\, \hat b^\dagger_{\vec p} \hat b_{\vec q} e^{-i(\vec p - \vec q)\cdot \vec x} - \vec p\cdot\vec q\, \hat b^\dagger_{\vec p} \hat c^\dagger_{\vec q} e^{-i(\vec p + \vec q)\cdot \vec x}$$

$$- \vec p\cdot\vec q\, \hat c_{\vec p} \hat b_{\vec q} e^{i(\vec p + \vec q)\cdot \vec x} + \vec p\cdot\vec q\, \hat c_{\vec p} \hat c^\dagger_{\vec q} e^{i(\vec p - \vec q)\cdot \vec x} \Big)$$

$$+ \frac{m^2}{2\sqrt{\omega_{\vec p}\omega_{\vec q}}} \Big(\hat b^\dagger_{\vec p} \hat b_{\vec q} e^{-i(\vec p - \vec q)\cdot \vec x} + \hat b^\dagger_{\vec p} c^\dagger_{\vec q} e^{-i(\vec p + \vec q)\cdot \vec x}$$

$$+ \hat c_{\vec p} \hat b_{\vec q} e^{i(\vec p + \vec q)\cdot \vec x} + \hat c_{\vec p} \hat c^\dagger_{\vec q} e^{+i(\vec p - \vec q)\cdot \vec x} \Big)$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_{\vec p}} \Big[+ \omega^2_{\vec p} \big(\hat b_{\vec p} \hat b^\dagger_{\vec p} - \hat b_{\vec p} \hat c_{-\vec p} - \hat c^\dagger_{\vec p} \hat b^\dagger_{-\vec p} + \hat c^\dagger_{\vec p} \hat c_{\vec p} \big)$$

$$+ \vec p^{\,2} \big(\hat b^\dagger_{\vec p} \hat b_{\vec p} + \hat b^\dagger_{\vec p} \hat c^\dagger_{-\vec p} + \hat c_{\vec p} b_{-\vec p} + \hat c_{\vec p} \hat c^\dagger_{\vec p} \big)$$

$$+ m^2 \big(\hat b^\dagger_{\vec p} \hat b_{\vec p} + \hat b^\dagger_{\vec p} c^\dagger_{-\vec p} + \hat c_{\vec p} \hat b_{-\vec p} + \hat c_{\vec p} \hat c^\dagger_{\vec p} \big) \Big]$$

We also impose $[\hat b_{\vec p}, \hat b^\dagger_{\vec p'}] = [\hat c_{\vec p}, \hat c^\dagger_{\vec p'}] = (2\pi)^3 \delta^{(3)}(\vec p - \vec p')$

and all $[\hat b_{\vec p}, \hat c_{\vec q}] = [\hat b_{\vec p}, \hat c^\dagger_{\vec q}] = 0$

We check this by doing

$$[\phi(\vec{x}), \pi(\vec{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{i}{2} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \Big\{ \cancel{e^{i\vec{p}\cdot\vec{x}}[}$$

$$\left[\hat{b}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + \hat{c}_p^\dagger e^{-i\vec{p}\cdot\vec{x}},\ \hat{b}_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{y}} - \hat{c}_{\vec{q}} e^{i\vec{q}\cdot\vec{y}} \right] \Big\}$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{i}{2} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \Big\{ \underbrace{[\hat{b}_{\vec{p}}, \hat{b}_{\vec{q}}^\dagger]}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})} e^{i\vec{p}\cdot\vec{x} - i\vec{q}\cdot\vec{y}} - \underbrace{[\hat{b}_{\vec{p}}, \hat{c}_{\vec{q}}]}_{0} e^{i\vec{p}\cdot\vec{x} + i\vec{q}\cdot\vec{y}}$$

$$+ \underbrace{[\hat{c}_p^\dagger, \hat{b}_{\vec{q}}^\dagger]}_{0} e^{-i\vec{p}\cdot\vec{x} - i\vec{q}\cdot\vec{y}} - \underbrace{[\hat{c}_p^\dagger, \hat{c}_{\vec{q}}]}_{-(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})} e^{-i\vec{p}\cdot\vec{x} + i\vec{q}\cdot\vec{y}}$$

$$= i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{i\vec{p}\cdot(\vec{y}-\vec{x})} \right).$$

$$= \frac{i}{2} \left(\delta^{(3)}(\vec{x}-\vec{y}) + \delta^{(3)}(\vec{y}-\vec{x}) \right) = i\, \delta^{(3)}(\vec{x}-\vec{y})$$

works

Hence:

$$\hat{H} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_{\vec{p}}} \Big(\underbrace{(\omega_{\vec{p}}^2 + \vec{p}^2 + m^2)}_{= 2\omega_{\vec{p}}^2} (\hat{b}_{\vec{p}}^\dagger \hat{b}_{\vec{p}} + \hat{c}_{\vec{p}} \hat{c}_{\vec{p}}^\dagger)$$

$$+ \omega_{\vec{p}} \Big(\underbrace{[\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}}^\dagger] - [\hat{c}_{\vec{p}}, \hat{c}_{\vec{p}}^\dagger]}_{=0} \Big) + \cancel{(\omega_{\vec{p}}}$$

$$+ \underbrace{(-\omega_{\vec{p}}^2 + \vec{p}^2 + m^2)}_{=0} \left(\hat{b}_{\vec{p}}^\dagger \hat{c}_{-\vec{p}}^\dagger + \hat{c}_{\vec{p}} \hat{b}_{-\vec{p}} \right). \Big).$$

$$\cancel{\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\omega_{\vec{p}}}}$$

$$\hat{H} = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} (\hat{b}^\dagger_{\vec{p}} \hat{b}_{\vec{p}} + \hat{c}_{\vec{p}} \hat{c}^\dagger_{\vec{p}}).$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} (\hat{b}^\dagger_{\vec{p}} \hat{b}_{\vec{p}} + \hat{c}^\dagger_{\vec{p}} \hat{c}_{\vec{p}} + \underbrace{[\hat{c}_{\vec{p}}, \hat{c}^\dagger_{\vec{p}}]}_{\delta^{(3)}(0)}).$$

eliminate zero point infinite energy (normal ordering)

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} (\hat{b}^\dagger_{\vec{p}} \hat{b}_{\vec{p}} + \hat{c}^\dagger_{\vec{p}} \hat{c}_{\vec{p}})$$

yes

Diagonalized

puh!

Now, $[\hat{H}, \hat{b}_{\vec{p}}^\dagger] = \int \frac{d^3\vec{p}'}{(2\pi)^3} \omega_{\vec{p}'} [\hat{b}^\dagger_{\vec{p}'} \hat{b}_{\vec{p}'}, \hat{b}_{\vec{p}}^\dagger]$

$$= \int \frac{d^3\vec{p}'}{(2\pi)^3} \omega_{\vec{p}'} \hat{b}^\dagger_{\vec{p}'} \underbrace{[\hat{b}_{\vec{p}'}, \hat{b}^\dagger_{\vec{p}}]}_{(2\pi)^3 \delta^{(3)}(\vec{p}' - \vec{p})}$$

$$= \omega_{\vec{p}} \hat{b}_{\vec{p}}^\dagger$$

Similar calculation shows that

$[\hat{H}, \hat{b}_{\vec{p}}] = -\omega_{\vec{p}} \hat{b}_{\vec{p}}$ $\quad$ $[\hat{H}, \hat{c}^\dagger_{\vec{p}}] = \omega_{\vec{p}} \hat{c}^\dagger_{\vec{p}}$

$[\hat{H}, \hat{c}_{\vec{p}}] = -\omega_{\vec{p}} \hat{c}_{\vec{p}}$

Now consider the vacuum state $|0\rangle$ su[ch that]
$\hat{H}|0\rangle = 0$, then

$$[\hat{H}, \hat{b}_{\vec{p}}^{\dagger}]|0\rangle = \hat{H}\hat{b}_{p}^{\dagger}|0\rangle - \hat{b}_{p}^{\dagger}\underbrace{\hat{H}|0\rangle}_{0} = \hat{H}\hat{b}_{\vec{p}}^{\dagger}|0\rangle$$

on the other hand:

$$[\hat{H}, \hat{b}_{p}^{\dagger}]|0\rangle = \omega_{\vec{p}}\hat{b}_{p}^{\dagger}|0\rangle$$

$$\therefore \hat{H}(\hat{b}_{p}^{\dagger}|0\rangle) = \omega_{\vec{p}}(\hat{b}_{p}^{\dagger}|0\rangle)$$

Similarly $\hat{H}(\hat{C}_{\vec{p}}^{\dagger}|0\rangle) = \omega_{\vec{p}}(\hat{C}_{\vec{p}}^{\dagger}|0\rangle)$

Denote
$\hat{b}_{\vec{p}}^{\dagger}|0\rangle = |\vec{P}_b\rangle$
& $\hat{C}_{\vec{p}}^{\dagger}|0\rangle = |\vec{P}_c\rangle$
for future use

$\therefore$ $\hat{b}_{\vec{p}}^{\dagger}|0\rangle$ and $\hat{C}_{\vec{p}}^{\dagger}|0\rangle$ are ~~two stat~~ two particles with same momentum $\vec{p}$ and energy $\omega_{\vec{p}}$. They thus have same mass $m = \sqrt{\omega_{\vec{p}}^2 - \vec{p}^2}$

yes!

(c) conserved charge

$$Q = \frac{i}{2}\int d^3\vec{x}\,(\phi^*\pi^* - \pi\phi).$$

Canonical Quantisation

$$\hat{Q} = \frac{i}{2}\int d^3\vec{x}\,(\phi^{\dagger}\pi^{\dagger} - \pi\phi)$$

$$\hat{Q} = \frac{i}{2}\int \frac{d^3\vec{x}\, d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6}\Big[\left(\frac{-i}{2}\right)\sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}}\left(\hat{b}^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} + \hat{c}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}}\right)\left(\hat{b}_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} - \hat{c}^\dagger_{\vec{q}} e^{-i\vec{q}\cdot\vec{x}}\right)$$

$$- \frac{i}{2}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}\left(\hat{b}^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} - \hat{c}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}}\right)\left(\hat{b}_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} + \hat{c}^\dagger_{\vec{q}} e^{-i\vec{q}\cdot\vec{x}}\right)\Big]$$

$$= \frac{1}{4}\int \frac{d^3\vec{x}\, d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6}\Big[\sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}}\Big(\hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{q}} e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} - \hat{b}^\dagger_{\vec{p}}\hat{c}^\dagger_{\vec{q}} e^{-i(\vec{p}+\vec{q})\cdot\vec{x}}$$

$$+ \hat{c}_{\vec{p}}\hat{b}_{\vec{q}} e^{i(\vec{p}+\vec{q})\cdot\vec{x}} - \hat{c}_{\vec{p}}\hat{c}^\dagger_{\vec{q}} e^{i(\vec{p}-\vec{q})\cdot\vec{x}}\Big)$$

$$+ \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}\Big(\hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{q}} e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} + \hat{b}^\dagger_{\vec{p}}\hat{c}^\dagger_{\vec{q}} e^{-i(\vec{p}+\vec{q})\cdot\vec{x}}$$

$$- \hat{c}_{\vec{p}}\hat{b}_{\vec{q}} e^{i(\vec{p}+\vec{q})\cdot\vec{x}} - \hat{c}_{\vec{p}}\hat{c}^\dagger_{\vec{q}} e^{i(\vec{p}-\vec{q})\cdot\vec{x}}\Big)\Big]$$

$$= \frac{1}{\cancel{4}}\int \frac{d^3\vec{p}}{(2\pi)^3}\cancel{2}\left(\hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}} - \hat{c}_{\vec{p}}\hat{c}^\dagger_{\vec{p}}\right) = \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3}\left(\hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}} - \hat{c}_{\vec{p}}\hat{c}^\dagger_{\vec{p}}\right)$$

$$= \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3}\left(\hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}} - \hat{c}^\dagger_{\vec{p}}\hat{c}_{\vec{p}}\right)$$ good

$\boxed{\text{normal ordering}}$

Now $\hat{Q}\,|\vec{p}_b\rangle = \hat{Q}\,\hat{b}^\dagger_{\vec{p}}|0\rangle = \frac{1}{2}\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\,\hat{b}^\dagger_{\vec{p}'}\hat{b}_{\vec{p}'}\hat{b}^\dagger_{\vec{p}}|0\rangle$

$$= \frac{1}{2}\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\,\hat{b}^\dagger_{\vec{p}'}\Big(\hat{b}^\dagger_{\vec{p}}\underbrace{\hat{b}_{\vec{p}'}|0\rangle}_{0} + \underbrace{[\hat{b}_{\vec{p}'}, \hat{b}^\dagger_{\vec{p}}]}_{(2\pi)^3\delta^{(3)}(\vec{p}\,'-\vec{p})}|0\rangle\Big)$$

$\because [H, \hat{b}_{\vec{p}}] = -\omega_{\vec{p}}\hat{b}_{\vec{p}}$

$[H, \hat{b}_{\vec{p}}]|0\rangle = H\hat{b}_{\vec{p}}|0\rangle - \hat{b}_{\vec{p}}H|0\rangle$

$= H\hat{b}_{\vec{p}}|0\rangle$

$= -\omega_{\vec{p}}\hat{b}_{\vec{p}}|0\rangle$ $\Big\} \Rightarrow \hat{b}_{\vec{p}}|0\rangle = 0$

$\because$ No negative energy.

$\therefore \hat{Q}|\vec{P}_b\rangle = \frac{1}{2}|\vec{P}_b\rangle \Rightarrow$ charge $= \frac{1}{2}$ for

Similarly $\hat{Q}|\vec{P}_c\rangle = -\frac{1}{2}|\vec{P}_c\rangle$ charge $= -\frac{1}{2}$ for c

$\Rightarrow$ opposite charges. (same mass)

particle and antiparticle.

(d) The Lagrangian of 2 complex fields should look like the following

$$\mathcal{L} = \partial_\nu \phi_1^* \partial^\nu \phi_1 + \partial_\nu \phi_2^* \partial^\nu \phi_2 - (m^2 \phi_1^* \phi_1 + m^2 \phi_2^* \phi_2)$$

Let's make a vector $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, then its hermitian conjugate is $\vec{\phi}^\dagger = (\phi_1^*, \phi_2^*)$

Also a vector $\partial_\nu \vec{\phi} = \begin{pmatrix} \partial_\nu \phi_1 \\ \partial_\nu \phi_2 \end{pmatrix}$, and ~~$\partial_\nu \vec{\phi}^\dagger = (\partial_\nu \phi_1^*$~~ &

$(\partial_\nu \vec{\phi})^\dagger = (\partial_\nu \phi_1^*, \partial_\nu \phi_2^*)$

then ~~$\mathcal{L} = (\partial_\nu \vec{\phi})^\dagger \partial_\nu$~~

$$\mathcal{L} = (\partial_\nu \vec{\phi})^\dagger (\partial^\nu \vec{\phi}) - m^2 \vec{\phi}^\dagger \vec{\phi}$$

Now consider a constant unitary 2x2 matrix U acting on $\vec{\phi}$ to get some new vector $\vec{\phi}'$

$$\vec{\phi}' = U\vec{\phi} \Leftrightarrow \phi_i' = U_{ij}\phi_j$$

Similarly $\partial_\mu \vec{\phi} \to \partial_\mu \vec{\phi}' = \partial_\mu (U\vec{\phi}) = U(\partial_\mu \vec{\phi})$
$\because$ U is constant

Now consider the effect of this transformation on $\mathcal{L}$.

$$\vec{\phi}^\dagger \vec{\phi} \to \vec{\phi}'^\dagger \vec{\phi}' = (U\vec{\phi})^\dagger (U\vec{\phi}) = \vec{\phi}^\dagger \underbrace{U^\dagger U}_{I} \vec{\phi} = \vec{\phi}^\dagger \vec{\phi}$$

where we've used unitarity condition ~~to~~ $U^\dagger U = I$

Similarly $(\partial_\mu \vec{\phi})'^\dagger (\partial^\mu \phi)' = (\partial_\mu \vec{\phi})^\dagger \underbrace{U^\dagger U}_{I} (\partial_\mu \vec{\phi}) = (\partial_\mu \vec{\phi})^\dagger (\partial_\mu \vec{\phi})$

$\therefore$ Lagrangian is invariant under this transformation $\phi_1' = \sum_{j=1}^{2} U_{1j}\phi_j$, $\phi_2' = \sum_{j=1}^{2} U_{2j}\phi_j$

Now the problem is reduced to finding the ~~independent~~ linearly independent unitary 2x2 ~~matrix~~ matrices U. ~~that sa~~

One obvious way is to consider the U(2) group. consisting of matrices

$$U = e^{-i\alpha_j \sigma_j}$$

$j = 0, 1, 2, 3$, α_j is a real parameter

$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma_{1,2,3}$ are pauli matrices.

U is Hermitian because.:.

$$U^\dagger = (e^{-i\alpha_j\sigma_j})^\dagger = e^{i\alpha_j\sigma_j} \quad \text{since all } \alpha_j \in \mathbb{R}$$

and σ_j are Hermitian

$$U^\dagger U = e^{i\alpha_j\sigma_j} e^{-i\alpha_j\sigma_j}$$

∵ $\alpha_j\sigma_j$ commutes with itself

$$\therefore U^\dagger U = e^{i\alpha_j\sigma_j - i\alpha_j\sigma_j} = I \implies U \text{ Hermitian.}$$

We thus consider 4 ~~indep~~ conserved currents generated by the symmetries

$$e^{i\alpha\sigma_0}, \quad e^{i\alpha\sigma_1}, \quad e^{i\alpha\sigma_2}, \quad e^{i\alpha\sigma_3}$$

for very small α. good

① : $e^{i\alpha\sigma_0}$

$$\vec{\phi}' = \cancel{\phi}\, e^{i\alpha\sigma_0}\vec{\phi} = (I + i\alpha\sigma_0)\vec{\phi}$$

$$\therefore \phi_1' = \phi_1 + \underbrace{i\alpha\phi_1}_{\alpha\Delta\phi_1}, \quad \phi_2' = \phi_2 + \underbrace{i\alpha\phi_2}_{\alpha\Delta\phi_2}$$

~~Apply Noether's theorem: current $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta$~~

And $\phi_1'^* = \phi_1^* - \underbrace{i\alpha\phi_1^*}_{\alpha\Delta\phi_1^*}$

$$\phi_2'^* = \phi_2^* - \underbrace{i\alpha\phi_2^*}_{\alpha\Delta\phi_2^*}$$

Noether theorem:

$$j^\mu = \sum_\phi \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu \right) = \text{current}.$$

$\because$ In all ~~are~~ our symmetries, $\mathcal{L}$ is unchanged.

$$\therefore \quad J^\mu = 0 \quad , \quad j^\mu = \sum_\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi$$

in this case

$$j_0^\mu = \sum_{k=1}^{2} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \Delta\phi_k + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k^*)} \Delta\phi_k^* \right]$$

$$= (\partial^\mu \phi_1^*)(i\phi_1) + (-i\alpha\phi_1^*)(\partial^\mu \phi_1).$$

$$+ (\partial^\mu \phi_2^*)(i\phi_2) + (-i\alpha\phi_2^*)(\partial^\mu \phi_2)$$

$$= i\left[(\partial^\mu \phi_1^*)\phi_1 - \phi_1^*(\partial^\mu \phi_1) + (\partial^\mu \phi_2^*)\phi_2 - \phi_2^* \partial^\mu \phi_2 \right]$$

$$Q_0 = \int j^0 d^3\vec{x} = i\int d^3\vec{x} \left[\pi_1 \phi_1 - \phi_1^* \pi_1^* + \pi_2\phi_2 - \phi_2^* \pi_2^* \right]$$

where we've used $\pi_{1,2} = \frac{\partial \phi_{1,2}^*}{\partial t}$, $\pi_{1,2}^* = \frac{\partial \phi_{1,2}}{\partial t}$.

Q_0 multiply by $-\frac{1}{2}$ gives the charge demanded by the Problem.

This Q_0 is the generalization of previous part.

Now consider Pauli matrices

Do transformation

$$\vec{\phi}' = U\vec{\phi} \quad \text{where} \quad U = e^{i\alpha\sigma_j} \quad (j=1,2,3)$$

then for infinitesimal α,

$$\underline{\vec{\phi}' = (I + i\alpha\sigma_j)\vec{\phi}}, \quad \cancel{\vec{\phi}'^* = (I - i\alpha\sigma_j)}\vec{\phi}^*$$

~~the vectors~~

$$\cancel{\Delta\vec{\phi} = \begin{pmatrix}\Delta\phi_1 \\ \Delta\phi_2\end{pmatrix} = i\sigma_j\vec{\phi}}$$

$$\cancel{\Delta\vec{\phi}^* = \begin{pmatrix}\Delta\phi_1^* \\ \Delta\phi_2^*\end{pmatrix} = -i\sigma_j\vec{\phi}^*}$$

$$\cancel{j_j^\nu = \sum_{k=1}^{2}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu\phi_k)}\Delta\phi_k + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_k^*)}\Delta\phi_k^*\right)}$$

$$\cancel{= \sum_{\substack{k=1 \\ b=1,2}}^{2}\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_k)}(i(\sigma_j)_{kb}\phi_b) + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_k^*)}(-i(\sigma_j)_{kb}\phi_b^*)}$$

$$\cancel{= \sum_{k,b}(\partial^\nu\phi_k^*)(i(\sigma_j)_{kb}\phi_b) - (\partial^\nu\phi_k^*)(i(\sigma_j)_{kb}\phi_b^*)}$$

$$\cancel{\therefore Q_j^0 = i\int d^3\vec{x}\sum_{k,b}\underbrace{(\partial_t\phi_k^*)}_{\pi_k}(\sigma_j)_{kb}\phi_b - (\underbrace{\partial^\nu\phi_k}_{\pi_k^*})(\sigma_j)_{kb}\phi_b^*}$$

$$\cancel{= i\int d^3\vec{x}\sum_{k,b}(\pi_k(\sigma_j)_{kb}\phi_b - \pi_k^*(\sigma_j)_{kb}\phi_b^*)}$$

$$\cancel{= -i\int d^3\vec{x}\sum_{a,b}\phi_a^*(\sigma_j)_{ab}\pi_b^* - \pi_a(\sigma_j)_{ab}\phi_b}$$

$$\therefore \cancel{\vec\phi^\dagger} \quad \vec{\phi}'^\dagger = \vec{\phi}^\dagger U^\dagger = \vec{\phi}^\dagger e^{-i\alpha\sigma_j}$$

$$\therefore \vec{\phi}'^\dagger = \vec{\phi}^\dagger (I - i\alpha\sigma_j)$$

$$\Rightarrow \quad \cancel{\Delta\vec\phi = -i\vec\phi^\dagger\sigma_j} \quad \Delta\vec{\phi} = i\sigma_j\vec{\phi} = \begin{pmatrix}\Delta\vec\phi_1 \\ \Delta\vec\phi_2\end{pmatrix}$$

$$\Delta\vec{\phi}^\dagger = -i\vec{\phi}^\dagger\sigma_j = (\Delta\vec\phi_1^*, \Delta\vec\phi_2^*)$$

$$j_j^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a^*)}\Delta\phi_a^*.$$

$$= \sum_{a,b} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \underbrace{i(\sigma_j)_{ab}\phi_b}_{\Delta\vec\phi_a} + \underbrace{(-i\phi_a^*(\sigma_j)_{ab})}_{\Delta\vec\phi^\dagger_b}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_b^*)}$$

$$= -i\sum_{a,b} \cancel{(\partial^\mu\phi_a^*)}\,\phi_a^*(\sigma_j)_{ab}(\partial^\mu\phi_b) - (\partial^\mu\phi_a^*)(\sigma_j)_{ab}\phi_b$$

$$= \cancel{\quad}$$

$$Q_j = \int d^3\vec{x}\; j_j^0 = -i\int d^3\vec{x} \sum_{a,b}\phi_a^*(\sigma_j)_{ab}\underbrace{(\partial_t\phi_b)}_{\pi_b^*} - \underbrace{(\partial_t\phi_a^*)}_{\pi_a}(\sigma_j)_{ab}\phi_b).$$

$$= -i\int d^3\vec{x} \sum_{a,b}(\phi_a^*(\sigma_j)_{ab}\pi_b^* - \pi_a(\sigma_j)_{ab}\phi_b).$$

Well done

multiply this by $-\frac{1}{2}$ gives the ~~answer we constructed~~ conserved quantity required.

$\Longleftrightarrow \partial_\mu F^{ab} + \pi^a \ldots$

consider $j, k, l \in \{1,2,3\}$

$$[Q_j, Q_k] = -\frac{1}{4}\int d^3\vec{x}\, d^3\vec{y}\, \Big[\phi_a^*(\vec{x})(\sigma_j)_{ab}\pi_b^*(\vec{x}) - \pi_a(\vec{x})(\sigma_j)_{ab}\phi_b(\vec{x}),\ \phi_c^*(\vec{y})(\sigma_k)_{cd}\pi_d^*(\vec{y}) - \pi_c(\vec{y})(\sigma_k)_{cd}\phi_d(\vec{y})\Big]$$

$$= -\frac{1}{4}\int d^3\vec{x}\, d^3\vec{y}\, \Big[\cancel{\phi_a^*(\vec{x})(\sigma_j)_{ab}\pi_b^*} (\sigma_j)_{ab}\big(\phi_a^*(\vec{x})\pi_b^*(\vec{x}) - \pi_a(\vec{x})\phi_b(\vec{x})\big),\ (\sigma_k)_{cd}\big(\phi_c^*(\vec{y})\pi_d^*(\vec{y}) - \pi_c(\vec{y})\phi_d(\vec{y})\big)\Big]$$

$$= -\frac{1}{4}\int d^3\vec{x}\, d^3\vec{y}\, (\sigma_j)_{ab}(\sigma_k)_{cd}\Big[\phi_a^*(\vec{x})\pi_b^*(\vec{x}) - \pi_a(\vec{x})\phi_b(\vec{x}),\ \phi_c^*(\vec{y})\pi_d^*(\vec{y}) - \pi_c(\vec{y})\phi_d(\vec{y})\Big]$$

$$= -\frac{1}{4}\int d^3\vec{x}\, d^3\vec{y}\, (\sigma_j)_{ab}(\sigma_k)_{cd}\Big(\underbrace{[\phi_a^*(\vec{x})\pi_b^*(\vec{x}),\ \phi_c^*(\vec{y})\pi_d^*(\vec{y})]}_{①} + \underbrace{[\pi_a(\vec{x})\phi_b(\vec{x}),\ \pi_c(\vec{y})\phi_d(\vec{y})]}_{②}. \Big)$$

$$① \Rightarrow \cancel{[\phi_a^*(\vec{x})\pi_b^*(\vec{x}),\ \phi_c^*(\vec{y})\pi_d^*(\vec{y})]}$$

$$= \cancel{[\phi_a^*(\vec{x}),\ \phi_c^*(\vec{y})]\pi_b^*(\vec{x})}$$

$\nabla_a F^{ab} = 0$ ... $_{ac} F^{cb} = 0 \Leftrightarrow \nabla_a F^{ab} = 0$

Recall

$$[AB, CD] = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]B$$

$\therefore$ ① $\Rightarrow [\phi_a^*(\vec{x})\pi_b^*(\vec{x}), \phi_c^*(\vec{y})\pi_d^*(\vec{y})]$

$$= \phi_a^*(\vec{x})\underbrace{[\pi_b^*(\vec{x}), \phi_c^*(\vec{y})]}_{-i\delta_{bc}\delta^3(\vec{x}-\vec{y})}\pi_d^*(\vec{y})$$

$$+ \phi_c^*(\vec{y})\underbrace{[\phi_a^*(\vec{x}), \pi_d^*(\vec{y})]}_{i\delta_{ad}\delta^3(\vec{x}-\vec{y})}\pi_b^*(\vec{x})$$

$$= \cancel{-i\delta_{bc}\delta^3(\vec{x}-\vec{y})}$$

$$= i\delta^3(\vec{x}-\vec{y})\left(\delta_{ad}\phi_c^*(\vec{y})\pi_b^*(\vec{x}) - \delta_{bc}\phi_a^*(\vec{x})\pi_d^*(\vec{y})\right)$$

② $\Rightarrow [\pi_a(\vec{x})\phi_b(\vec{x}), \pi_c(\vec{y})\phi_d(\vec{y})]$

$$= \pi_a(\vec{x})\underbrace{[\phi_b(\vec{x}), \pi_c(\vec{y})]}_{i\delta_{bc}\delta^3(\vec{x}-\vec{y})}\phi_d(\vec{y})$$

$$+ \cancel{\pi_c E \#}\ \pi_c(\vec{y})\underbrace{[\pi_a(\vec{x}), \phi_d(\vec{y})]}_{-i\delta_{ad}\delta^3(\vec{x}-\vec{y})}\phi_b(\vec{x})$$

$$= i\delta^3(\vec{x}-\vec{y})\left(\delta_{bc}\pi_a(\vec{x})\phi_d(\vec{y}) - \delta_{ad}\pi_c(\vec{y})\phi_b(\vec{x})\right)$$

$$\therefore [Q_j, Q_k] = \frac{-i}{4}\int d^3\vec{x}\ (\sigma_j)_{ab}(\sigma_k)_{cd}\Big(\delta_{ad}\phi_c^*(\vec{x})\pi_b^*(\vec{x}) - \delta_{bc}\phi_a^*(\vec{x})\pi_d^*(\vec{x}) + \delta_{bc}\pi_a(\vec{x})\phi_d(\vec{x}) - \delta_{ad}\pi_c(\vec{x})\phi_b(\vec{x})\Big)$$

$$= \cancel{-i\int d^3\vec{x}\,(\sigma_j)_{ab}(\sigma_k)_{cd}\,(\delta_a}$$

$$= -\frac{i}{4}\int d^3\vec{x}\ (\sigma_j)_{ab}(\sigma_k)_{ca}\,\phi_c^{*}\pi_b^{*}$$

$$- (\sigma_j)_{ac}(\sigma_k)_{cd}\,\phi_a^{*}\pi_d^{*}$$

$$+ (\sigma_j)_{ac}(\sigma_k)_{cd}\,\cancel{\phi_a}\,\pi_a\phi_d$$

$$- (\sigma_j)_{ab}(\sigma_k)_{ca}\,\pi_c\phi_b$$

$$\cancel{= -i\int d^3\vec{x}\ (-\varepsilon_{jkl})(\sigma_l)_{ab}}$$

$$\cancel{= -i\int d^3\vec{x}}\ -$$

$$= \frac{i}{4}\int d^3\vec{x}\ \left[(\sigma_j)_{ac}(\sigma_k)_{cd}\,\phi_a^{*}\pi_d^{*} - (\sigma_k)_{ac}(\sigma_k)_{cd}\,\phi_a^{*}\pi_d^{*}\right]$$

$[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$ for pauli matrices

$$- \left[(\sigma_j)_{ac}(\sigma_k)_{cd}\,\pi_a\phi_d - (\sigma_k)_{ac}(\sigma_j)_{cd}\,\pi_a\phi_d\right]$$

$$= \frac{i}{4}\int d^3\vec{x}\ 2i\varepsilon_{jkl}(\sigma_l)_{ad}\,\phi_a^{*}\pi_d^{*}$$

$$-i2\varepsilon_{jkl}(\sigma_l)_{ad}\,\pi_a\phi_d$$

$$= (i)\left(\frac{i}{2}\right) \times \varepsilon_{jkl}\int d^3\vec{x}\ \left[\phi_a^{*}(\sigma_l)_{ad}\pi_d^{*} - \pi_a(\sigma_l)_{ad}\phi_d\right]$$

$\therefore \quad [Q_j, Q_k] = i\varepsilon_{jkl} Q_l .$

$\Rightarrow$ Commutation relation of $SU(2)$

$\ast$ ~~As~~ Generalize to n identical complex scalar fields, we need the Lagrangian

$$\mathcal{L} = \sum_{a=1}^{n} (\partial_\mu \phi_a^*)(\partial^\mu \phi_a) - m^2 \phi_a^* \phi_a$$

$\because \quad \phi_a^* \phi_a = Re(\phi_a)^2 + Im(\phi_a)^2 .$

Define $Re(\phi_a) = \phi_{Ra} \quad Im(\phi_a) = \phi_{Ia}$

then $\phi_a^* \phi_a = \phi_{Ra}^2 + \phi_{Ia}^2$

Now define real vector $\vec{\phi} = \begin{pmatrix} \phi_{R1} \\ \phi_{I1} \\ \phi_{R2} \\ \phi_{I2} \\ \vdots \\ \phi_{Rn} \\ \phi_{In} \end{pmatrix}$

$$\vec{\phi}^T \vec{\phi} = \phi_{R1}^2 + \phi_{I1}^2 + \phi_{R2}^2 + \phi_{I2}^2 + \cdots + \phi_{Rn}^2 + \phi_{In}^2$$

$$= \sum_{a=1}^{n} \phi_a^* \phi_a$$

$\Rightarrow \quad \mathcal{L} = \partial_\mu \vec{\phi}^T \partial^\mu \vec{\phi} - m^2 \vec{\phi}^T \vec{\phi}$

Now we want to look at the symmetry of $\mathcal{L}$. to keep $\mathcal{L}$ invariant, we need

$\vec{\phi}' = U\vec{\phi}$ where U is orthogonal.

to see this

$$\vec{\phi}'^T\vec{\phi}' = (U\vec{\phi})^T(U\vec{\phi}) = \vec{\phi}^T \underbrace{U^T U}_{I} \vec{\phi} = \vec{\phi}^T\vec{\phi}$$

Similarly for $(\partial_\nu \vec{\phi}^T)(\partial^\nu \vec{\phi})$.

An ~~unit~~ orthogonal matrix can be generated by

$$\cancel{U = e^{iA}} \quad U = e^{i\alpha A}, \text{ where}$$

A is an antisymmetric matrix.

to see this:

$$U^\dagger U = (e^{i\alpha A})^T e^{i\alpha A}$$

$$= e^{i\alpha A^T} e^{i\alpha A} \underbrace{=}_{A^T=-A} e^{-i\alpha A} e^{i\alpha A} \underbrace{=}_{[A,-A]=0} e^{i\alpha(-A+A)} = e^{i\alpha(0)}$$

$= I$

If there are 2 complex scalar fields, $\vec{\phi}$ has dimension 4 and U is 4×4.

$\therefore$ We have 6 linearly independent generators

$$\cancel{A_1 = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}}$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad A_4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

$$A_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus we have 6 linear independent symmetries and thus 6 conserved charges

In the general case, just take the $\underline{A}_a$ valued in the lie algebra su(N).

In contrast to the case, when we use complex ~~A~~ unitary matrices to find only 4 symmetries.

To Generalise this, we say that the number of conserved charges is equal to the number of linearly independent generators ~~which~~ of the $2n\times2n$ orthogonal matrices for n complex scalar fields.

This is the same as the number of independent components in an orthogonal $2n\times 2n$ matrix.

This is ~~prec~~ precisely $\underline{\underline{n(2n-1)}}$

$\because$ we have $4n^2$ unknowns, n^2 equations, but

$\because (U^TU)^T = U^TU$ $\therefore$ the upper and lower off diagonal equations are ~~the~~ identical.

$\therefore$ we have $\underset{\text{diagonal}}{2n} + \frac{(1+2n-1)(2n-1)}{2} = 2n + \frac{2n}{2}(2n-1)$

Constraints

$\therefore$ we have $4n^2 - 2n - \frac{2n}{2}(2n-1)$

$= 2n(2n-1) - \frac{1}{2}(2n)(2n-1)$

$= \frac{1}{2}(2n)(2n-1) = \underline{\underline{n(2n-1)}}$ independent ~~compon~~ components, which corresponds to the

$$\begin{pmatrix} 0 & & x & & \cdots & x \\ & 0 & & x & & \\ & & 0 & & x & \\ & & & 0 & & x \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

the upper ~~&~~ ~~(or lower)~~ (or lower) half of the matrix without the diagonal.

You don't need to give details here. Refer to what you learn in the groups course.

for ~~A×2 A×2~~ n×n unitary matrices,

however, we only have $2n^2 - n^2 = \underline{\underline{n^2}}$

$2n^2$: number of unknowns real + ~~complex~~ imaginary $n + n$

n^2: numbers of equations, treating real and ~~complex~~ imaginary parts as separate variables.

$\underline{\underline{n^2}}$ independent parameters.

~~∴ For n=2~~

~~∴ number of symmetry real $(n(2n-1))$ → complex n^2~~

∴

number of complex fields n	real $n(2n-1)$	complex n^2
1	1	1
2	6	4
3	15	9
4	28	16
⋮	⋮	⋮

we see that at $n=2$, indeed $n(2n-1)$ gives 6 and n^2 gives 4.

Conclusion is that we have $\underline{\underline{n(2n-1)}}$ conserved ~~cha~~ charges for n complex scalar fields.

(1)(3) (a) The path integral

$$\langle q_f, t_f | q_i, t_i \rangle = \int Dq(t) \exp\left[i \int_{t_i}^{t_f} \frac{\dot{q}^2}{2} dt \right]$$

the path $q(t)$ is the path from q_i at time t_i to q_f at time t_f

the classical path should be a ~~Stri~~ straight line joining this two points in spacetime.

i.e. $\cancel{q_c(t) = q_i + \quad q_f =}$

At $t = t_i$, $q_c(t) = q_i$, At $t = t_f$ $q_c(t) = q_f$

$q_c(t)$ linear in t (straight line).

$\therefore$ let $q_c(t) = mt + b$

$$\begin{aligned} q_i &= mt_i + b \\ q_f &= mt_f + b \end{aligned} \Rightarrow m = \frac{q_f - q_i}{t_f - t_i}$$

$$b = q_i - \frac{q_f - q_i}{t_f - t_i} t_i = \frac{q_i t_f - \cancel{q_i t_i} - q_f t_i + \cancel{q_i t_i}}{t_f - t_i}$$

$$= \frac{q_i t_f - q_f t_i}{t_f - t_i}$$

$$\therefore q_c(t) = \frac{q_i t_f - q_f t_i}{t_f - t_i} + \frac{q_f - q_i}{t_f - t_i} t$$

$$= \frac{q_i (t_f - t) + q_f (t - t_i)}{t_f - t_i}$$

$$q_c(t) = \frac{q_i(t_f - t_i) + q_i t_i + q_f t - q_f t_i - q_i t}{t_f - t_i}$$

$$= q_i + \frac{q_f - q_i}{t_f - t_i}(t - t_i)$$

A general path can be represented by

$$q(t) = q_c(t) + \delta q(t)$$

where $\delta q(t_i) = \delta q(t_f) = 0$ ①

~~wh~~ From this property we can write $\delta q(t)$ as

$$\delta q(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi(t - t_i)}{t_f - t_i}\right)$$

This ensures ① and the period of $\delta q(t)$ is $2(t_f - t_i)$, so $\delta q(t)$ can take any value between $t = t_i$ and $t = t_f$.

→ by choosing different a_n's we explore all paths.

(b) By definition the action

$$S = \int_{t_i}^{t_f} \frac{1}{2}\dot{q}^2\, dt$$

$$= \int_{t_i}^{t_f} \frac{1}{2} \left|\frac{d}{dt}\left(q_c(t) + \delta q(t)\right)\right|^2 dt$$

$$= \int_{t_i}^{t_f} dt\, \frac{1}{2}\left(\frac{d}{dt}\left[q_i + \frac{q_f - q_i}{t_f - t_i}(t - t_i) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n (t-t_i)}{t_f - t_i}\right)\right]\right)^2$$

$$= \frac{1}{2}\int_{t_i}^{t_f} dt \left[\frac{q_f - q_i}{t_f - t_i} + \sum_{n=1}^{\infty} \frac{a_n \pi n}{t_f - t_i} \cos\left(\frac{\pi n (t-t_i)}{t_f - t_i}\right)\right]^2$$

$$= \frac{1}{2}\int_{t_i}^{t_f} dt \left\{ \underbrace{\left(\frac{q_f - q_i}{t_f - t_i}\right)^2}_{\boxed{1}} + 2\left(\frac{q_f - q_i}{t_f - t_i}\right)\sum_{n=1}^{\infty} \frac{a_n \pi n}{t_f - t_i} \underbrace{\cos\left(\frac{\pi n (t-t_i)}{t_f - t_i}\right)}_{\boxed{2}} + \underbrace{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n a_m \pi^2 m n}{(t_f - t_i)^2} \cos\left(\frac{\pi n (t-t_i)}{t_f - t_i}\right) \cos\left(\frac{\pi m (t-t_i)}{t_f - t_i}\right)}_{\boxed{3}} \right\}$$

look term by term

$\boxed{1}$ $$\int_{t_i}^{t_f} dt \left(\frac{q_f - q_i}{t_f - t_i}\right)^2 = \left(\frac{q_f - q_i}{t_f - t_i}\right)^2 (t_f - t_i) = \underline{\underline{\frac{(q_f - q_i)^2}{t_f - t_i}}}$$

$\boxed{2}$ $$\int_{t_i}^{t_f} \cos\left(\frac{\pi n (t-t_i)}{t_f - t_i}\right) dt \underset{\substack{\uparrow \\ \Delta t = t_f - t_i \\ T = t - t_i}}{=} \int_0^{\Delta t} dT \cos\left(\frac{n\pi T}{\Delta t}\right)$$

$$= \cancel{\sin} \; - \frac{\Delta t}{n\pi} \sin\left(\frac{n\pi T}{\Delta t}\right)\Bigg|_{T=0}^{T=\Delta t}$$

$$= -\frac{\Delta t}{n\pi}\left[\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin(0)}_{0}\right] = 0$$

$\therefore$ $\boxed{2} = 0$

$\boxed{3}$:

$$\int_{t_i}^{t_f} dt \cos\left(\frac{\pi n (t-t_i)}{(t_f-t_i)}\right)\cos\left(\frac{\pi m (t-t_i)}{(t_f-t_i)}\right)$$

$$= \int_0^{\Delta t} dT \cos\left(\frac{\pi n T}{\Delta t}\right)\cos\left(\frac{\pi m T}{\Delta t}\right)$$

If $m \neq n$

$$\text{above} = \frac{1}{2}\int_0^{\Delta t} \cos\left(\frac{\pi T}{\Delta t}(n-m)\right) + \cos\left(\frac{\pi T}{\Delta t}(n+m)\right) dT$$

$$= \frac{1}{2}\left[\frac{-\Delta t}{(n-m)\pi}\sin\left(\frac{\pi T}{\Delta t}(n-m)\right) - \frac{\Delta t}{(n+m)\pi}\sin\left(\frac{\pi T}{\Delta t}(n+m)\right)\right]_0^{\Delta t}$$

$$= 0$$

If $m = n$

$$\text{above} = \int_0^{\Delta t} \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi n T}{\Delta t}\right) dT$$

$$= \frac{1}{2}\Delta t + 0 = \frac{1}{2}\Delta t = \frac{t_f - t_i}{2}$$

$\therefore$ above $= \left(\frac{t_f - t_i}{2}\right)\delta_{mn}$

$$\boxed{3} = \sum_{m,n} \frac{a_n a_m \pi^2 mn}{(t_f - t_i)^2} \frac{(t_f - t_i)}{2} \delta_{mn}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} \frac{(n\pi)^2}{t_f - t_i} a_n^2$$

$$\therefore \quad S = \frac{1}{2}\left(\boxed{1} + \boxed{2} + \boxed{3}\right)$$

$$= \frac{1}{2}\frac{(q_f - q_i)^2}{t_f - t_i} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2}\frac{(n\pi)^2}{(t_f - t_i)} a_n^2$$

(c) the integral

$$\int da_n e^{iS} = \int da_n \exp\left\{\frac{i}{2}\frac{(q_f - q_i)^2}{t_f - t_i} + \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{2}\frac{(n\pi)^2}{(t_f - t_i)} a_n^2\right\}$$

$$= \exp\left[\frac{i}{2}\frac{(q_f - q_i)^2}{t_f - t_i}\right]\exp\left[\frac{i}{2}\sum_{k\neq n}\frac{1}{2}\frac{(k\pi)^2}{t_f - t_i} a_k^2\right] \times$$

$$\underbrace{\int da_n \exp\left\{\frac{i}{4}\frac{(n\pi)^2}{t_f - t_i} a_n^2\right\}}_{I}$$

$$I = \int_{-\infty}^{\infty} da_n \exp\left\{ \frac{i}{4} \frac{(n\pi)^2}{t_f - t_i} a_n^2 \right\}$$

$$= \int_{-\infty}^{\infty} da_n \exp\{ i\alpha^2 a_n^2 \} \qquad \alpha^2 \equiv \frac{(n\pi)^2}{4(t_f - t_i)}$$

$$= \int_{-\infty}^{\infty} \frac{dx}{\alpha} \exp\{ ix^2 \}$$

$x = \alpha a_n$
$dx = \alpha da_n$

$$= \frac{1}{\alpha} \underbrace{\int_{-\infty}^{\infty} dx\, e^{ix^2}}_{(1+i)\sqrt{\frac{\pi}{2}}} = \frac{1}{\alpha}(1+i)\sqrt{\frac{\pi}{2}}$$

$$= \frac{1}{\alpha} \sqrt{2i} \frac{\sqrt{\pi}}{\sqrt{2}} = \sqrt{\frac{i\pi}{\alpha^2}}$$

$$= \sqrt{\frac{i4\pi(t_f - t_i)}{(n\pi)^2}} = \frac{1}{n} \sqrt{\frac{i4\pi(t_f - t_i)}{\pi^2}}$$

$$= \cancel{\frac{1}{n}\sqrt{4(t_f - t_i)}} \qquad \cancel{\frac{1}{n}\sqrt{+4i(t_f - t_i)}}$$

$$= \frac{1}{n}\sqrt{4i(t_f - t_i)}$$

$$\therefore \int da_n e^{iS} = \exp\left[\frac{i}{2} \frac{(q_f - q_i)^2}{t_f - t_i} \right] \exp\left\{ \frac{i}{2} \sum_{k \neq n} \frac{1}{2} \frac{(k\pi)^2}{t_f - t_i} a_k^2 \right\} \left[\frac{1}{n}\sqrt{4i(t_f - t_i)} \right]$$

(d)

$$\int \prod_{n=1}^{\infty} da_n \, e^{iS} = \int da_1 da_2 \cdots da_n \cdots e^{iS}$$

$$= \exp\left(\frac{i}{2}\frac{(q_f-q_i)^2}{t_f-t_i}\right)\int da_1 da_2 \cdots \exp\left(\frac{i}{2}\sum_k \frac{(k\pi)^2}{t_f-t_i} a_k^2\right)$$

$$= \prod_{n=1}^{\infty}\left[\frac{1}{n}\sqrt{4i(t_f-t_i)}\right]\exp\left(\frac{i}{2}\frac{(q_f-q_i)^2}{t_f-t_i}\right) = \left(\prod_{n=1}^{\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\left[\sqrt{4i(t_f-t_i)}\right]^n\right) \times \exp\left(\frac{i}{2}\left(\frac{(q_f-q_i)^2}{t_f-t_i}\right)\right)$$

$$= \lim_{n\to\infty}\frac{\left(\sqrt{4i(t_f-t_i)}\right)^n}{n!} \times \exp\left(\frac{i}{2}\left(\frac{(q_f-q_i)^2}{t_f-t_i}\right)\right)$$

$$\cancel{= \left(\sqrt{4i(t_f-t_i)}\right)\left(\prod_{n=1}^{\infty}\frac{1}{n}\right)\exp\left(\frac{i}{2}\frac{(q_f-q_i)^2}{t_f-t_i}\right)}$$

$$\cancel{C(t_f-t_i)}$$

$$\cancel{C(t_f-t_i) \propto} \prod_{n=1}^{\infty}\frac{1}{n} = 1\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdots\cdots \to \cancel{0}$$

$$\therefore C(t_f-t_i) = \lim_{n\to\infty}\frac{\left(\sqrt{4i(t_f-t_i)}\right)^n}{n!} \to 0$$

$\therefore$ This constant is not finite.

(e) Actual path integral

$$\langle q_f, t_f | q_i, t_i\rangle = \int Dq \, e^{iS}$$

$$= \gamma \int \prod_{n=1}^{\infty} da_n \, e^{iS}$$

we require $\int dq \langle q_f, t_f | q, t\rangle\langle q, t | q_i, t_i\rangle = \langle q_f, t_f | q_i, t_i\rangle$

$$\langle q_f, t_f | q_i, t_i \rangle = \gamma c (t_f - t_i) \exp\left[\frac{i}{2}\frac{(q_f - q_i)^2}{t_f - t_i}\right].$$

$$\Leftrightarrow \langle q_f, t_f | q, t \rangle = \gamma c (t_f - t) \exp\left[\frac{i}{2}\frac{(q_f - q)^2}{(t_f - t)}\right]$$

$$\langle q, t | q_i, t_i \rangle = \gamma c (t - t_i) \exp\left[\frac{i}{2}\frac{(q - q_i)^2}{t - t_i}\right]$$

$$\int \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle \, dq$$

$$= \gamma c (t_f - t) \cdot \gamma c (t - t_i) \underbrace{\int dq \exp\left(\frac{i}{2}\frac{(q_f - q)^2}{t_f - t} + \frac{i}{2}\frac{(q - q_i)^2}{t - t_i}\right)}_{I}$$

$$= \gamma c (t_f - t) \gamma c (t - t_i) I$$

$$\cancel{I = \int dq \exp\left[\frac{i}{2}\frac{q^2 - 2q_f q + q_f^2}{t_f - t}\right] + \int dq \exp\left[\frac{i}{2}\frac{q^2 - 2q_i q + q_i^2}{t - t_i}\right]}$$

$$I = \int dq \exp\left[\frac{i}{2}\left(\frac{q^2 - 2q_f q + q_f^2}{t_f - t} + \frac{q^2 - 2q_i q + q_i^2}{t - t_i}\right)\right]$$

$$= \int dq \exp\left(\frac{i}{2}\underbrace{\left(\frac{1}{t_f - t} + \frac{1}{t - t_i}\right)}_{a} q^2 + i\underbrace{\left(-\frac{q_f}{t_f - t} - \frac{q_i}{t - t_i}\right)}_{J} q\right.$$

$$\left. + \frac{i}{2}\left(\frac{q_f^2}{t_f - t} + \frac{q_i^2}{t - t_i}\right)\right)$$

$$\exp\left(\frac{i}{2}\left(\frac{q_f^2}{t_f-t}+\frac{q_i^2}{t-t_i}\right)\right)\int dq\, \exp\left(\frac{1}{2}iaq^2+iJq\right)$$

$$=\exp\left(\frac{i}{2}\left(\frac{q_f^2}{t_f-t}+\frac{q_i^2}{t-t_i}\right)\right)\left(\frac{2\pi i}{a}\right)^{\frac{1}{2}}\exp\left(\frac{-iJ^2}{2a}\right)$$

$$=\left(\frac{2\pi i}{\left(\frac{1}{t_f-t}+\frac{1}{t-t_i}\right)}\right)^{\frac{1}{2}}\exp\left(\frac{i}{2}\left(\frac{q_f^2}{t_f-t}+\frac{q_i^2}{t-t_i}-\frac{\left[\left(\frac{q_f}{t_f-t}\right)+\left(\frac{q_i}{t-t_i}\right)\right]^2}{\left[\frac{1}{t_f-t}+\frac{1}{t_i-t}\right]}\right)\right)$$

$$=\left(\frac{2\pi i}{\left(\frac{t_f-t_i}{(t_f-t)(t-t_i)}\right)}\right)^{\frac{1}{2}}\exp\left[\frac{i}{2}\left(\frac{q_f^2}{t_f-t}+\frac{q_i^2}{t-t_i}-\frac{\left(q_f(t-t_i)+q_i(t_f-t)\right)^2}{\cancel{[(t-t_i)(t_f-t)(t_f-t_i)]}\;[(t_f-t)(t-t_i)^2+(t-t_i)(t_f-t)^2]}\right)\right]$$

$$=\left(\frac{2\pi i(t_f-t)(t-t_i)}{(t_f-t_i)}\right)^{\frac{1}{2}}\exp\left\{\frac{i}{2}\left\{\frac{q_f^2}{t_f-t}+\frac{q_i^2}{t-t_i}-\frac{q_f^2(t-t_i)^2+q_i^2(t_f-t)^2+2q_fq_i(t-t_i)(t_f-t)}{(t_f-t)(t-t_i)(t_f-t_i)}\right\}\right\}$$

$$=\left(\frac{2\pi i(t_f-t)(t-t_i)}{t_f-t_i}\right)^{\frac{1}{2}}\exp\left\{\frac{i}{2}\left\{\frac{1}{(t_f-t)(t-t_i)(t_f-t_i)}\times\right.\right.$$

$$\left.\left.\left(q_f^2(t-t_i)(t_f-t_i)+q_i^2(t_f-t)(t_f-t_i)-q_f^2(t-t_i)^2-q_i^2(t_f-t_i)^2-2q_fq_i(t-t_i)(t_f-t)\right)\right\}\right\}$$

$$= \left(\frac{2\pi i (t_f - t)(t - t_i)}{(t_f - t_i)} \right)^{\frac{1}{2}} \exp\left(\frac{i}{2} \left(\frac{1}{\cancel{(t_f - t)}\cancel{(t - t_i)}(t_f - t_i)} \times \right.\right.$$

$$\left. \left((q_f^2 + q_i^2 - 2 q_f q_i)\cancel{(t_f - t)}\cancel{(t - t_i)} \right) \right)$$

$$= \left(\frac{2\pi i (t_f - t)(t - t_i)}{(t_f - t_i)} \right)^{\frac{1}{2}} \exp\left\{ \frac{i}{2} \left(\frac{(q_f - q_i)^2}{t_f - t_i} \right) \right\}$$

$$\therefore \int dq \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle = \langle q_f, t_f | q_i, t_i \rangle$$

$$\Rightarrow \left(\frac{2\pi i (t_f - t)(t - t_i)}{(t_f - t_i)} \right)^{\frac{1}{2}} \cancel{\exp\left(\frac{i}{2} \left(\frac{(q_f - q_i)^2}{t_f - t_i} \right) \right)} \times \gamma C (t_f - t)$$

$$\times \gamma C (t - t_i)$$

$$= \gamma C (t_f - t_i) \cancel{\exp\left(\frac{i}{2} \left(\frac{(q_f - q_i)^2}{t_f - t_i} \right) \right)}$$

$$\Rightarrow \gamma C (t - t_i) \gamma C (t_f - t) \left(\frac{2\pi i (t_f - t)(t - t_i)}{t_f - t_i} \right)^{\frac{1}{2}}$$

$$= \gamma C (t_f - t)$$

$$\therefore \left[\gamma C (t - t_i) \times \sqrt{t - t_i} \right] \left[\gamma C (t_f - t) \sqrt{t_f - t} \right] = \sqrt{\frac{1}{2\pi i}} \gamma C (t_f - t_i) \sqrt{t_f - t_i}$$

$$\therefore \frac{F(t-t_i)\cdot F(t_f-t)}{F(t_f-t_i)} = \sqrt{\frac{1}{2\pi i}}$$

$\therefore$ ~~F(t)~~ $F(x) = \text{const} = \sqrt{\frac{1}{2\pi i}}$ is an obvious solution

$$\therefore \; F(T) = \gamma_C(T)\sqrt{T} = \frac{1}{\sqrt{2\pi i}}$$

$$\therefore \; \gamma_C(T) = \frac{1}{\sqrt{2\pi i T}}$$

good!

$$\therefore \gamma_C(t_f - t_i) = \frac{1}{\sqrt{2\pi i (t_f - t_i)}}$$