General Relativity I - Problem Sheet 4

Please return solutions (collection boxes in the basement of the Mathematical Institute) no later than 28th of November, 1pm.

1. Kepler's 3rd Law

What is the physical meaning of the coordinate time t in the Schwarzschild solution ?

Show that the proper-time τ of an observer on a circular orbit at radius R and the coordinate time t are related by

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{1 - 3M/R} \, .$$

Using this result, show that

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{M}{R^3}$$

for a circular orbit at radius R.

2. Stability of Circular Orbits

Consider time-like geodesic in the Schwarzschild geometry that is a small perturbation from a circular orbit at radius R in the equatorial plane

$$r(\tau) = R + \varepsilon(\tau) \, .$$

Show that the perturbation must solve an equation of the form

$$\ddot{\varepsilon} + f(R)\varepsilon = 0$$

and find the function $f(R)$. Plot this function, showing clearly its asymptote and intercept. Hence, re-derive the fact that circular orbits only exist if $R > 3M$ and show that these orbits are stable if $R > 6M$.

3. Meaning of E

Consider a stationary observer at radius R in the Schwarzschild geometry and a massive test particle moving on a time-like geodesic $x^a(\tau)$ that intersect at some point P. Show that the stationary observer measures the energy per-unit-rest mass of the test particle to be

$$\sqrt{1-\frac{2M}{R}}\,\dot{t}\,.$$

Let the test particle and the stationary observer have relative velocity v at the point P. Explain why

$$\gamma(v)=\sqrt{1-\frac{2M}{R}}\,\dot{t}\,.$$

Now derive an expression for the conserved quantity E. Expanding this expression for large distances ($R \gg 2M$) and small velocities ($v \ll 1$), show that E is approximately the sum of the rest mass, kinetic energy and potential energy.

This is an approximate characterization of the conserved quantity E. To find the precise meaning, suppose the stationary observer converts the energy measured into a photon and sends it out to another stationary observer at $r \to \infty$. What is the energy of the photon measured by a stationary observer at infinity?

4. Acceleration

Observers not freely falling experience acceleration forces. This is encoded in the acceleration 4-vector $A^a = V^b \nabla_b V^a$ where V^a is the 4-velocity. This measures the failure of the corresponding curve to be a geodesic.

Consider a stationary observer at radius R in the Schwarzschild geometry and show that his acceleration four-vector is

$$A^a = (0, -m/R^2, 0, 0)\,.$$

You will need to compute the Christoffel symbol Γ^r_{tt} for the Schwarzschild metric from the r-equation of motion. Now compute the proper acceleration $a = (g_{ab}A^aA^b)^{1/2}$. Show that this agrees with the Newtonian expectation for $r \gg 2M$ and that stationary observers can only exists for $R > 2M$.

5. Capture by a Black Hole

For geodesics in the Schwarzschild solution

$$\frac{E^2-\kappa}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

where

$$V(r) = -\frac{\kappa M}{r} + \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}$$

with $\kappa = 0$ for null geodesics and $\kappa = 1$ for time-like geodesics

In this question we are interested in when incoming geodesics will be captured by the black hole. For such problems it is convenient to define the 'impact parameter' b by

$$b \equiv \frac{J}{\sqrt{E^2 - \kappa}}\,.$$

a) Massless Particle

First consider an incoming null geodesic. Show that a massless particle is captured by the black hole if the impact parameter b is smaller than a critical value b_c. Show that the capture cross-section $\sigma \equiv \pi b_c^2$ is

$$\sigma = 27\pi M^2.$$

b) Non-relativistic Massive Particle

Now consider an incoming time-like geodesic. We will assume that the massive particle starts at $r \to \infty$ with non-relativistic velocity $v \ll 1$ measured by a stationary observer. Explain why

$$b = \frac{J}{v} + \mathcal{O}(v)$$

and draw a diagram explaining the physical significance of the impact parameter in this case.

Show that the massive particle will be captured by the black hole if the impact parameter b is smaller than a critical value b_c. Show that the capture cross-section $\sigma \equiv \pi b_c^2$ is approximately

$$\sigma = 16\pi M^2/v^2.$$

6. Einstein's Equations in Cosmology

You do not have to hand in this question, but you should perform the computations carefully in your own time.

Using the cosmological metric from question 2) in coordinates (τ, r, θ, ϕ) show that Einstein's equation in the presence of a perfect fluid imply

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3P)\,.$$

Multiply the first equation by a^2, differentiate with respect to τ and eliminate a'' using the second equation to show that

$$\rho' + 3\frac{a'}{a}(\rho + P) = 0\,.$$

Derive the same equation directly from the local conservation of energy and momentum $\nabla^a T_{ab} = 0$.

7.* Cosmological Constant

A cosmological constant Λ modifies Einstein's equations as follows

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 8\pi T_{ab}\,.$$

Show that the cosmological constant is mathematically equivalent to a perfect fluid with density $\rho_\Lambda = \Lambda/8\pi$ and pressure $P_\Lambda = -\Lambda/8\pi$. Hence show that for cosmological solutions we have

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}\,.$$

In an expanding universe with contributions from pressureless matter, radiation and a cosmological constant, which contribution will dominate the energy density at *a*) early times and *b*) late times? Consider a universe with a positive cosmological constant: how does the scalar factor $a(\tau)$ behave at late times? Does it have a future horizon?