General Relativity I - Problem Sheet 3

Please return solutions (collection boxes in the basement of the Mathematical Institute) no later than 15th of November, 1pm.

1. Independent Components of The Riemann Tensor

The Riemann tensor $R_{abc}{}^d$ associated to any covariant derivative ∇_a obeys the following algebraic identities:

1. $R_{abc}{}^d = -R_{bac}{}^d$
2. $R_{abc}{}^d + R_{bca}{}^d + R_{cab}{}^d = 0$

What is the additional algebraic identity obeyed when ∇_a is the Levi-Civita convariant derivative assoicated to a metric g_{ab}?

In this case, show that the Riemann tensor in four dimensions has, in general, 20 independent components.

Optional: How many independent components does it have in D dimensions?

2. Riemann tensor in two dimensions

1. Prove that in two dimensions, the Riemann tensor has the form

$$R_{abcd} = \frac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc}) \,.$$

   where R is the Ricci scalar. *Hint: express the Ricci scalar R in terms of the one independent component of the Riemann tensor, say R_{1212}.*

2. Show that Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ vanishes automatically in two dimensions.

3. Now consider the two-dimensional de Sitter metric

$$ds^2 = -du^2 + \cosh^2 u \, d\varphi^2$$

   where $\infty - < u < \infty$ and $0 \leq \varphi \leq 2\pi$. Using your results from question 4 of exercise sheet 2, compute the one independent component of the Riemann tensor, say $R_{u\varphi u\varphi}$.

4. Compute the Ricci scalar R of two-dimensional de Sitter metric and show that indeed $R_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$.

3. Newtonian Limit of Geodesic Deviation

The relative acceleration of a one-parameter family of geodesics $x^a_s(\tau) = x^a(\tau, s)$ is determined by the geodesic deviation equation

$$\frac{D^2 X^d}{D\tau^2} = -R_{abc}{}^d V^a X^b V^c$$

where

$$V^a = \frac{dx^a}{d\tau} \qquad X^b = \frac{dx^b}{ds}$$

and $D/D\tau = V^a \nabla_a$ is the covariant derivative in the direction of V^a.

In the Newtonian limit, show that this reduces to

$$\frac{\partial^2 X^i}{\partial t^2} = -\sum_j \frac{\partial^2 \phi}{\partial x^i \partial x^j} X^j$$

where ϕ is the gravitational potential.

4. Schwarzschild Solution

Everyone should check that the Schwarzschild solution satisfies the vacuum field equations $R_{ab} = 0$. You should do this carefully, in your own time, and it is not neccessary to hand this calculation in.

5. Circular Orbits

The lagrangian for affinely parametrized geodesics in the Schwarzschild solution is

$$L = -(1 - 2M/r)\dot{t}^2 + (1 - 2M/r)^{-1}\dot{r}^2 + r^2(\,\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\,)$$

Show that

$$E = (1 - 2M/r)\dot{t} \qquad J = r^2 \sin^2\theta\,\dot{\phi}$$

are conserved. What symmetries do they correspond to? Explain why the lagrangian itself is conserved and why $L = -1$ for time-like geodesics parametrized by the proper time. Show that without loss of generality, you can always restrict attention to time-like geodesics lying in the equatorial plane, $\theta = \pi/2$. Now consider a circular orbit in the equatorial plane at coordinate distance R. Show that this is a time-like geodesic if $R > 3M$ and find the conserved quantities (E, J).