C7.5: General Relativity I
Problem Sheet 2

Please return solutions (collection boxes in the basement of the Mathematical Institute) no later than Wednesday 1st of November, 1pm.

1. Uniform Acceleration and the Equivalence Principle

Let us start from a global inertial frame $\mathcal{O}$ in Minkowski space with coordinates $x^a = (t, x, y, z)$. Now consider the transformation to a *non-inertial* frame $\mathcal{O}'$ with coordinates $x'^a = (t', x', y', z')$ such that

$$
\begin{aligned}
t &= \left(\frac{1}{g} + z'\right) \sinh\left(gt'\right) \\
z &= \left(\frac{1}{g} + z'\right) \cosh\left(gt'\right) - \frac{1}{g} \\
x &= x' \\
y &= y'
\end{aligned}
$$

where g is a constant with units of acceleration.

1. For $t' \ll 1/g$ show that this transformation corresponds to a uniformly accelerated reference frame in Newtonian mechanics.
2. Plot the trajectory of the point $z' = 0$ in the inertial frame $\mathcal{O}$.
3. Show that a clock at rest at $z' = h$ runs fast compared to a clock at rest at $z' = 0$ by the factor $(1 + gh)$.
4. Use the equivalence principle to interpret this result in terms of gravitational time dilation.
5. What is the line element ds^2 of a uniform gravitational field?

2. Flat Space in Polar Coordinates

Flat $\mathbb{R}^2$ has coordinates x^1 and x^2 and a metric with components $g_{11} = g_{22} = 1$ as well as $g_{12} = g_{21} = 0$. Changing to polar coordinates defined by

$$
\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} .
$$

1. Find the components of the metric.

2. Find the Christoffel symbols using

   (a) the geodesic equation as derived from the point-particle Lagrangian.

   (b) the expression of the Christoffel symbols in terms of the metric

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

   (c) the transformation behavior of the Christoffel symbols under a change of coordinates.

3. Show that straight lines in $\mathbb{R}^2$ solve the geodesic equation in polar coordinates.

3. The Covariant Derivative

Together with the usual properties of a derivative, the covariant derivative is defined to map (p, q) tensors to $(p, q+1)$ tensors. The action of the covariant derivate on a vector (i.e. $(1, 0)$ tensor) with components v^a can be written as

$$\nabla_b v^a = \partial_b v^a + \Gamma^a{}_{bc} v^c \, .$$

1. Prove that $\nabla_b v^a$ transforms as a $(1, 1)$ tensor under general coordinate transformations provided the Christoffel symbols transform as

$$\Gamma'^a{}_{bc} = \frac{\partial x^p}{\partial x'^b} \frac{\partial x^q}{\partial x'^c} \left(\frac{\partial x'^a}{\partial x^r} \Gamma^r{}_{pq} - \frac{\partial^2 x'^a}{\partial x^p \partial x^q} \right)$$

2. Show that the above transformation behavior is implied by

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

3. What is the action of the covariant derivative ∇_a on a scalar ? Use this to show how ∇_a must act on a one-form ω_a.

4. What is the action of the covariant derivative ∇_a on a (p, q) tensor ?

4. 2d de Sitter Space

Consider the two-dimensional de Sitter metric

$$ds^2 = -du^2 + \cosh^2 u \, d\varphi^2$$

where $\infty - < u < \infty$ and $0 \leq \varphi \leq 2\pi$.

1. What is the proper length of the space-like curve defined by $u = u_c$?

2. Write down the lagrangian for (affinely parametrized) geodesics and, using Lagrange's equations, compute the non-vanishing Christoffel symbols:

$$\Gamma^{u}_{\varphi\varphi} \qquad \Gamma^{\varphi}_{u\varphi} \qquad \Gamma^{\varphi}_{\varphi u} \,.$$

3. Show that

$$J = \cosh^2 u\, \dot{\varphi} \qquad E = \dot{u}^2 - \cosh^2 u\, \dot{\varphi}^2$$

are both conserved along geodesics. *Hint: for an elegant derivation of the second conserved quantity, you may want to compute the hamiltonian and explain why it is conserved.*

4. Consider a geodesic with initial condition $\dot{\varphi}(0) = 0$. Show that $J = 0$ and $E = \dot{u}^2$.

5. Now consider the case $J \neq 0$. Introduce the variable $v = \tanh u$ and show that

$$\left(\frac{dv}{d\varphi}\right)^2 = \left(\frac{E}{J^2} + 1\right) - v^2 \,.$$

Write down the most general solution $v(\varphi)$ and discuss its behaviour as a function of the ratio E/J^2.

5. Infinitesimal Symmetries

The Lie derivative of a type $(2,0)$ tensor T_{ab} with respect to a vector field X^a is defined by

$$\mathcal{L}_X T_{ab} = X^c \partial_c T_{ab} + (\partial_a X^c) T_{cb} + (\partial_b X^c) T_{ac} \,.$$

1. Show that you can replace ∂_a with any covariant derivative ∇_a in this expression and so argue that the Lie derivative transforms as a tensor.

2. Consider the infinitesimal coordinate transformation $\delta x^a = \epsilon K^a$ generated by a vector field K^a. Show that the action of an (affinely parametrized) geodesic

$$S = \int_{s_1}^{s_2} g_{ab}(x(s)) \frac{dx^a}{ds} \frac{dx^b}{ds}$$

is invariant under this transformation if

$$\mathcal{L}_K g_{ab} = 0 \,.$$

Show that this can be equivalently expressed as

$$\nabla_{(a} K_{b)} = 0$$

where ∇_a is the covariant derivative associated to the metric g_{ab}. A vector field with this property is known as a 'Killing vector' and generates an infinitesimal symmetry of the geometry defined by the metric g_{ab}.

3. Show that the inner product $g_{ab}K^a\dot{x}^b$ is conserved along the geodesic.

4. Consider the two-dimensional de Sitter metric from problem 4. Show that the vector field K^a with components $K^u = 0$ and $K^\varphi = 1$ is a Killing vector and the corresponding conserved quantity is J.

5* Show that the sum $X^a + Y^a$ and commutator $[X,Y]^a = X^b\nabla_b Y^a - Y^b\nabla_b X^a$ of two Killing vectors are also Killing vectors. Together with the Jacobi identity, this means that Killing vectors generate a Lie algebra.

6.* Maxwell Equations in General Coordinates

In a general spacetime, the sourceless Maxwell equations are given by

$$\nabla_a F^{ab} = 0, \;\; \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0$$

where $F_{ab} = -F_{ba}$. The Maxwell energy-momentum tensor is as

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F^c{}_b + \frac{1}{4}F^{cd}F_{cd}g_{ab}\right).$$

1. Show that

$$\nabla^a T_{ab} = 0.$$

2. Show that $\Gamma^b{}_{ab} = \partial_a \log\sqrt{-|g|}$, where $|g|$ is the determinant of the metric.

3. Show that the first Maxwell equation can be written as

$$\partial_a\left(\sqrt{|g|}F^{ab}\right) = 0.$$

4. Show that we can substitute covariant derivatives by partial derivatives in the second Maxwell equation.