C7.5: General Relativity I

Problem Sheet 1

Please return solutions (collection boxes in the basement of the mathematical institute) no later than 17th of October, 1pm.

1. Practice with Tensors

Imagine we have a tensor X^{ab} and a vector V^a with components

$$X^{ab} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \quad V^a = (-1, 2, 0, -2)$$

Find $X^a{}_b, X_a{}^b, X^{(ab)}, X_{[ab]}, X^\lambda{}_\lambda, V^aV_a, V_aX^{ab}$, using $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

2. Summation Convention

Explain why each equation below is ambiguous or inconsistent. Provide a possible correct version of each one.

1. $x'^a = L^{ab}x^b$
2. $x'^a = L^b{}_c M^c{}_d x^d$
3. $\delta^a_b = \delta^a_c \delta^c_d$
4. $x'^a = L^a{}_c\, x^c + M^c{}_d x^d$
5. $x'^a = L^a{}_c\, x^c + M^{ad} x^d$
6. $\phi = (X^a A_a)(Y^a B_a)$

3. Operations on Tensors

Consider two general coordinate systems $\{x^a\}$ and $\{x'^a\}$ on overlapping regions of flat Minkowski spacetime. How do the components of a (p, q) tensor transform under the change of coordinates from $\{x^a\} \to \{x'^a\}$?

What form form does the Jacobian matrix $\delta x'^a/\delta x^b$ take for a Lorentz transformation between two inertial frames?

- Show that the sum $S^a{}_b + T^a{}_b$ of two $(1, 1)$ tensors is also a $(1, 1)$ tensor.
- Show that the tensor product $S^a{}_b T^c$ of a $(1, 1)$ tensor and an $(1, 0)$ tensor is a $(2, 1)$ tensor.
- Show that the contraction $S^{ac}{}_{bc}$ of a $(2, 2)$ tensor is a $(1, 1)$ tensor.
- Show that the partial derivatives $\partial_a S^b$ of a $(1, 0)$ tensor transforms under as a $(1, 1)$ tensor under Lorentz transformations between inertial frames but not under general coordinate transformations.

4. Levi-Civita tensor

For any tensor $T^{a_1,\cdots,a_q}$, we may define its symmetrization

$$T^{(a_1,\cdots,a_q)} \equiv \sum_{\sigma\in\mathcal{S}_q} T^{\,a_{\sigma(1)},\cdots,a_{\sigma(q)}}$$

and anti-symmetrization

$$T^{[a_1,\cdots,a_q]} \equiv \sum_{\sigma\in\mathcal{S}_q} \mathrm{sig}(\sigma)\; T^{\,a_{\sigma(1)},\cdots,a_{\sigma(q)}}$$

Here $\mathcal{S}_q$ is the group of permutations $\{\sigma\}$ of q elements and $\mathrm{sig}(\sigma)$ denotes the sign of a permutation σ.

Prove that a completely anti-symmetric $(0, m)$ tensor in n dimensions vanishes unless $m \leq n$. How many independent components does such a tensor have?

In a four dimensional spacetime with metric g_{ab}, the Levi-Civita tensor ϵ_{abcd} is defined by two properties:

1. It is completely anti-symmetric: $\epsilon_{abcd} = \epsilon_{[abcd]}$.
2. $\epsilon_{0123} = \sqrt{-g}$ in a right-handed coordinate system $\{x^0, x^1, x^2, x^3\}$ where g is the determinant of the metric.

Show that $\epsilon_{0123} = 1$ in a right-handed inertial frame. Prove that ϵ_{abcd} transforms as a $(0, 4)$ tensor under general coordinate transformations

$$x \to x'(x)\,.$$

5. Maxwell's Equations in an Inertial Frame

Show that

$$\partial_{[a}F_{bc]} = 0 \qquad \Leftrightarrow \qquad \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0$$

whenever $F_{ab} = -F_{ba}$.

The electromagnetic field is encoded in an anti-symmetric type $(0,2)$ tensor field, F_{ab}. The electric and magnetic fields measured by an observer with 4-velocity V^a are extracted from F_{ab} by

$$E_a = F_{ab}V^b \qquad B_a = -\frac{1}{2}\epsilon_{abcd}F^{bc}V^d \, .$$

where ϵ_{abcd} is the Levi-Civita tensor. By contracting with the 4-velocity V^a, explain why E_a and B_a each have only 3 independent components. For an observer at rest in an inertial frame, $V^a = (1,0,0,0)$, show that

$$E_a = (0, \vec{E}) \qquad \text{where} \qquad E_i = F_{i0}$$

$$\text{and} \qquad B_a = (0, \vec{B}) \qquad \text{where} \qquad B_i = \frac{1}{2}\epsilon_{ijk}F^{jk} \, .$$

Hence show that

$$\partial_a F^{ab} = -4\pi J^b \qquad \partial_{[a}F_{bc]} = 0$$

reproduce Maxwell's equations for the electromagnetic fields $(\vec{E}, \vec{B})$. The vector field J^a has components $(\rho, \vec{J})$ where ρ is the electric charge density and $\vec{J}$ is the electric current density measured by an observer at rest in the inertial frame. Show that $\partial_a J^a = 0$ and explain the physical meaning of this equation.

The contribution to the energy-momentum tensor from the electromagnetic field is

$$T^{ab} = \frac{1}{4\pi}\left[F^{ac}F^{b}{}_{c} - \frac{1}{4}(F^{cd}F_{cd})\eta^{ab}\right]$$

Assuming $J^a = 0$, show that this energy momentum tensor is conserved $\partial_a T^{ab} = 0$. What happens when $J^a \neq 0$?

6. Geodescics and Motion in an EM field

Consider a curve $x^a(s)$ in flat Minkowski space parametrized by a real parameter $s_1 \leq s \leq s_2$. What is the condition for this curve to be time-like?

The proper time of a time-like curve is the time experienced by an observer moving along it. In an inertial reference frame, this is given by the functional

$$\Delta\tau = \int_{s_1}^{s_2} ds \sqrt{-\eta_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds}}$$

where $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$. Consider the variational problem for this functional and show that the curve of extremal proper time between two points $x^a(s_1)$ and $x^a(s_2)$ is a straight line. Is this a minimum or a maximum?

Show that one may always reparametrize the curve such that $\sqrt{-\eta_{ab}\,\dot{x}^a\dot{x}^b}$ is constant. What is the parametrization which achieves this ? Such a parameter is called an *affine* parameter.

Why is extremizing the functional

$$S = -\frac{1}{2}\int_{s_1}^{s_2} \eta_{ab}\,\frac{dx^a}{ds}\frac{dx^b}{ds}$$

equivalent to extremizing the proper time when using an affine parameter ? What is left of the reparametrization freedom $s \to s'(s)$ when working with this action, i.e. what are the relations between choices of affine parameters ?

Now consider the modified functional

$$S = -\int_{s_1}^{s_2} \left[\frac{m}{2}\,\eta_{ab}\,\frac{dx^a}{ds}\frac{dx^b}{ds} - qA_a\frac{dx^a}{ds} \right] .$$

Show that the solution of the variational problem is

$$\frac{d^2x^a}{ds^2} = \frac{q}{m}F^a{}_b\frac{dx^b}{ds} \qquad \text{with} \qquad F_{ab} = \partial_a A_b - \partial_b A_a \,.$$

This equation describes the motion of a particle of mass m and electric charge q in an electromagnetic field F_{ab}. Contract the equation of motion with $\dot{x}^a$ and therefore show that $\sqrt{-\eta_{ab}\,\dot{x}^a\dot{x}^b}$ remains constant in the presence of an electromagnetic field.

7. Energy momentum tensor of a perfect fluid

Consider some distribution of matter with energy momentum tensor T^{ab}. What is the 4-momentum per-unit-volume and directional pressure measured by an observer with 4-velocity V^a?

The energy momentum tensor of a perfect fluid is given by

$$T^{ab} = (\rho + P)\,U^a U^b + P\eta^{ab}$$

where η^{ab} is the inverse metric in Minkowski space and U^a is the 4-velocity of the fluid. By considering an observer at rest with respect to the motion of the fluid, explain the physical meaning of ρ and P.

The equation of motion of a perfect fluid in an inertial frame is

$$\partial_a T^{ab} = 0 \,.$$

In the remainder of the question you will show how the equations of fluid mechanics are compactly encoded in this single expression.

First show that the tensor

$$h^a{}_b = \delta^a{}_b + U^a U_b$$

obeys

1. $h^a{}_b U^b = 0$

2. $h^a{}_b h^b{}_c = h^a{}_c$

3. $h^a{}_a = 3$

and therefore explain why $h^a{}_b$ is a projector onto the 3-dimensional hypersurfaces perpendicular to the fluid's 4-velocity U^a. What is the meaning of the symmetric tensor $h_{ab} = \eta_{ac} h^c{}_b$?

By projecting the equation of motion parallel and perpendicular to the 4-velocity of the fluid U^a, derive the equations

$$\partial_a(\rho U^a) + P\partial_a U^a = 0 \qquad (\rho + P)\,\frac{dU^a}{d\tau} + h^{ab}\partial_b P = 0 \tag{1}$$

where τ is the proper time of a particle moving with the fluid. The equations are the relativistic versions of the continuity and Euler equations of fluid mechanics. Show that the fluid particles move along geodesics when $P = 0$.

We now consider the non-relativistic approximation to equations (1). In the non-relativistic approximation you will need to assume that

1. $U^a = (1, \vec{u}) \qquad |\,\vec{u}\,| \ll 1$

2. $P \ll \rho$

3. $|\,\vec{u}\,|\,\partial_t P \ll |\,\vec{\nabla} P\,|$.

What is the physical intuition behind each of the approximations? (It may be helpful to restore the speed of light c in the equations.) Using the approximation, show that

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \qquad \rho\left(\partial_t + \vec{u}\cdot\vec{\nabla}\right)\vec{u} = -\vec{\nabla} P\,.$$