Groups and Representations

2017 Exam

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①

a) a subset $H \subset G$ is called a subgroup of group G iff

(i) $e \in H$
(ii) $\forall\ h \in H \Rightarrow h^{-1} \in H$
(iii) $\forall\ h_1, h_2 \in H \Rightarrow h_1 \cdot h_2 \in H$.

A subgroup $H \subset G$, a group, is called normal iff

$$gH = Hg \quad \forall\ g \in G.$$

~~qs~~ quotient $\frac{G}{H} = \{gH \mid g \in G\}$

Consider $g_1, g_2 \in G$ and multiplication defined by

$$(g_1 H)(g_2 H) = g_1 g_2 H$$

to show the well-definedness of this multiplication,

If $g_1, \tilde{g_1} \in g_1 H$, $g_2, \tilde{g_2} \in g_2 H$

then $\because$ coset is an equivalence class,

$$g_1 H = \tilde{g_1} H, \quad g_2 H = \tilde{g_2} H$$

And $\because \tilde{g_1} \in g_1 H \therefore \tilde{g_1} = g_1 h_1$ for some $h_1 \in H$

similarly $\tilde{g_2} = g_2 h_2$

for some $h_2 \in H$

$\therefore \tilde{g}_1 \tilde{g}_2 H = g_1 h_1 g_2 h_2 H$

If H is normal, $gH = Hg \ \forall g \in G$

$\therefore \tilde{g}_1 \tilde{g}_2 H = \cancel{g_1 h_1 g} \ g_1 h_1 h_2 g_2 H = g_1 h_3 g_2 H$

$h_3 = h_1 h_2 \in H$

$= g_1 g_2 h_3 H = g_1 g_2 H$

$\Rightarrow$ ~~produc~~ multiplication well-defined.

Identity: $eH = H$ is the ~~d~~ identity

$\because eH \, gH = eg H = gH$

Inverse: $\cancel{g^{-1}} \ g^{-1}H$ is the inverse of gH.

Associativity: $\cancel{g g_2} \ g_1 H (g_2 H \, g_3 H)$

$= g_1 H \, g_2 g_3 H = g_1 (g_2 g_3) H$

$= (g_1 g_2) g_3 H. = g_1 g_2 H \, g_3 H. = (g_1 H \, g_2 H) g_3 H$

Closure: ~~of con~~ of course $g_1 g_2 H$ is a coset since $g_1 g_2 \in G \ \forall g_1, g_2 \in G$.

All group axioms satisfied

$\rightarrow \frac{G}{H}$ is a group.

b)

$G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$

the group is Abelian $\rightarrow$ all complex irreducible representations are 1-dimensional.

the identity is 0

$\therefore \quad R(0) = 1$

$R(1^4) = \cancel{R(1+1+1+1)} = \cancel{R}\ R(1 \cdot 1 \cdot 1 \cdot 1).$

$= R(4 \bmod 4) = R(0) = 1$

Also $R(1^4) = R(1)^4$

$\Rightarrow R(1)^4 = 1 \quad \therefore R(1) = e^{\frac{2\pi i q}{4}} = e^{\frac{\pi i q}{2}}$

for $q = 0, 1, 2, 3$

$R(2) = R(1)^2 = e^{\pi i q}$, $R(3) = \cancel{R}\, e^{\frac{3\pi i q}{2}}$

$\therefore$ ~~repre~~ 4 representations (irreducible + complex).

$R_q(0) = 1$, $R_q(1) = e^{\frac{\pi i q}{2}}$, $R_q(2) = e^{\pi i q}$, $R_q(3) = e^{\frac{3\pi i q}{2}}$

for $q = 0, 1, 2, 3$ $\qquad R_q(g) = e^{\frac{i\pi q g}{2}}$

For 1-D representations character

$$\chi_k(g) = tr(R(g)) \underset{1D}{=} R(g)$$

character table for $\chi_q(g)$ and $g \in G = 0,1,2,3$.

$\chi_q(g)$ \ g	0	1	2	3
χ_0	1	1	1	1
χ_1	1	$e^{\frac{\pi i}{2}}$	-1	$e^{\frac{3\pi i}{2}}$
χ_2	1	-1	1	-1
χ_3	1	$e^{\frac{3\pi i}{2}}$	-1	$e^{\frac{9\pi i}{2}}$

$$\chi_q(g) = e^{\frac{i\pi q g}{2}}$$

$$\therefore (\chi_p, \chi_q) = \frac{1}{\underset{|G|}{4}} \sum_{g=0}^{g=3} \chi_p^*(g)\, \chi_q(g)$$

$$= \frac{1}{4} \sum_{g=0}^{3} e^{-\frac{i\pi p g}{2}} e^{\frac{i\pi q g}{2}}$$

$$= \frac{1}{4} \sum_{g=0}^{3} e^{\frac{i\pi (q-p) g}{2}}$$

~~$p \neq q$~~

$$(\chi_p, \chi_q) = \frac{1}{4}\left(\cancel{e^{i\pi k}}\ 1 + e^{\frac{i\pi k}{2}} + e^{i\pi k} + e^{\frac{3i\pi k}{2}}\right)$$

$$\cancel{= \begin{cases} \frac{1}{4}\left(1 + \cancel{\ \ } \right) & 1 + i - 1 - i \end{cases}}$$

$$\cancel{= \frac{1}{4}\left(1 + e^{i\pi k} + e\right)}$$

$$= \begin{cases} \frac{1}{4}(1 + i - 1 - i) & k = 1 \mod 4. \\ \frac{1}{4}(1 \cancel{+} - 1 + 1 - 1) & k = 2 \mod 4. \\ \frac{1}{4}(1 - i - 1 + i) & k = 3 \mod 4. \\ \frac{1}{4}(1 + 1 + 1 + 1) & k = 0 \mod 4. \end{cases}$$

$$= \begin{cases} 1 & k = 0 \mod 4. \\ 0 & \text{otherwise.} \end{cases}$$

$\because\ k = q - p,\quad |k| = |q - p|\qquad q = 0, 1, 2, 3$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ p = 0, 1, 2, 3$

$\therefore\ |k| = |q - p|$ cannot be 4 or bigger.

$\therefore\ |k| = 0 \rightarrow k = 0$ is the only case when $k = 0 \mod 4$

$$\therefore\quad (\chi_p, \chi_q) = \begin{cases} 1 & p = q \\ 0 & \text{otherwise} \end{cases} = \delta_{pq}$$

orthonormal.

c) $H = \{0, 2\}$

of G.

(i) identity 0 is in H.

(ii) inverse of 2 is 2. inverse of 0 is 0, both in H.

(iii) $0 \cdot 2 = 2 = 2 \cdot 0$ $\quad$ $0 \cdot 0 = 0$

$2 \cdot 2 = 0$

$\Rightarrow$ closure

$\rightarrow$ $H \subset G$ is a subgroup of G.

restriction of G to H:

rep of H:

$R_{H_1}(0) = 1$ $\quad$ ~~R_{H_2}~~ $R_{H_1}(2) = 1$

or $R_{H_2}(0) = 1$ $\quad$ $R_{H_2}(2) = -1$

Rep of G $\quad$ ~~Branch~~ Branching to rep of H

$$\begin{array}{ccc} R_0 & & R_{H1} \\ R_1 & \xrightarrow{\text{Branching}} & R_{H2} \\ R_2 & H \subset G. & R_{H_1} \\ R_3 & & R_{H_2} \end{array}$$

* R_{H_1}, R_{H_2} are representations of $\mathbb{Z}_2$ group.

d)

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$M^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$M^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_4$$

$M^5 = M$ ....

$R : G \to Gl(\mathbb{C}^4)$, $R(g) = M^g$

$\cancel{R(0) = R(4) = R}$
$\cancel{4 \equiv 0 \bmod 4.}$

$R(0) = R(1 \cdot 1 \cdot 1 \cdot 1) = R(1)^4 = M^4 = \mathbb{1}_4$
$R(1) = M$
$R(2) = M^2$
$R(3) = M^3$

$$R(g_1)\,R(g_2) = M^{g_1} M^{g_2} = M^{g_1+g_2} \cancel{= M^{g_1+g_2}}$$
$$= M^{(g_1+g_2 \bmod 4)} M^{4 \cdot k} = M^{(g_1+g_2 \bmod 4)} \mathbb{1}_4$$
$$(k \in \mathbb{Z}) = M^{g_1+g_2 \bmod 4}$$
$$= R(g_1+g_2 \bmod 4)$$

$= R(g_1 g_2)$

$\Rightarrow$ R is a representation.

character table for R of G:

$\chi_R(g)$ \ g	0	1	2	3
χ_R	4	0	0	0

$\chi_R(g) = \mathrm{tr}(R(g))$

multiplicity of irreducible representations R_q in R is given by

$$m_q = (\chi_R, \chi_q) = \frac{1}{4} \sum_{g \in G} \chi_R^*(g)\, \chi_q(g)$$

$\Rightarrow$ $m_0 = m_1 = m_2 = m_3 = 1$

$\therefore$ $\underline{R = R_0 \oplus R_1 \oplus R_2 \oplus R_3}$

$\therefore$ R is equivalent to

$R(0) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\quad \cancel{R(1) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & e^{\frac{i\pi}{2}} & \\ & & & e^{-\frac{i\pi}{2}} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & i & \\ & & & -i \end{pmatrix}}$

$$R(1) = \begin{pmatrix} 1 & & & \\ & i & & \\ & & -1 & \\ & & & -i \end{pmatrix} \qquad R(2) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$R(4) = \begin{pmatrix} 1 & & & \\ & -i & & \\ & & -1 & \\ & & & i \end{pmatrix}$$

looking at $R(0)$ and $R(2)$

we see that

if R_H is the Branching of R under restriction of G to $H \subset G$,

then $R_H = R_{H_1}^{\oplus 2} \oplus R_{H_2}^{\oplus 2}$

② $\alpha = e^{\frac{2\pi i}{5}}$ $\quad T = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ $\quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$G = \{ T^k, \sigma T^k \mid k = 0, \dots, 4 \}$.

$T^k = \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix}$ $\qquad \alpha^5 = \alpha^0 = 1$

$T^5 = T^0 = \mathbb{1}_2$.

$\therefore T^k = \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix}$ $\quad (k = 0, \dots, 4)$.

$$\sigma T^k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \alpha^{-k} \\ \alpha^k & 0 \end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$$

$$T^k \sigma = \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha^k \\ \alpha^{-k} & 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & \alpha^{k-5} \\ \alpha^{5-k} & 0 \end{pmatrix}}_{\alpha^5 = 1} = \sigma T^{5-k}$$

For $k > 4$, $T^k = T^{k \bmod 5} \in G$.

For $k_1, k_2 < 5$ $\quad (= 0, \dots, 4)$

$\sigma T^{k_1} T^{k_2} = \sigma T^{k_1+k_2 \bmod 5} \in G$

$T^{k_2} \sigma T^{k_1} = \sigma T^{5-k_2} T^{k_1} = \sigma T^{5-k_2+k_1 \bmod 5} \in G$

$\sigma T^{k_2} \sigma T^{k_1} = \cancel{\sigma\sigma}\ \underbrace{\sigma^2}_{\sigma^2 = \mathbb{1}_2} T^{5-k_2+k_1} = T^{5-k_2+k_1 \bmod 5} \in G$

$\Rightarrow$ closure satisfied. ✓

$k=0$ $\quad T^0 = \mathbb{1}_2 = $ identity ✓

$$\left.\begin{aligned} T^{5-k} &= (T^k)^{-1} \\ (\sigma T^k) &= (\sigma T^k)^{-1} \end{aligned}\right\} \Rightarrow \text{inverse ✓}$$

matrix multiplication satisfies associativity automatically ✓

$\Rightarrow$ G is a group

$\because$ T^k, σT^k $\quad k = 0, \dots, 4$ are all distinct 2×2 matrices

$\therefore$ $|G| = 10$

Conjugacy class of $g_1 \in G$ is

$[g_1] = \{ \cancel{g g_1 g^{-1}} \ g_2 = g g_1 g^{-1} \mid g \in G \}$.

~~If~~ conjugacy classes are ~~is~~ an equivalence classes and thus they partition the group.

$[T^0 = \mathbb{1}_2] = \{ g_2 = gg^{-1} \mid g \in G \} = \{ g_2 = \mathbb{1}_2 \}$

$= \{\mathbb{1}_2\}$

$[T] = \{ g_2 = gTg^{-1} \mid g \in G \}$

$= \{ T^k T T^{5-k} \} \cup \{ \sigma T^k T \sigma T^k \}$

$= \{ T^6 = T^1 = T \} \cup \{ \sigma T^{k+1} \sigma T^k \}$

$= \{T\} \cup \{ \cancel{\sigma} T^{5-k-1} \sigma^2 T^k \}$

$= \{T\} \cup \{T^4\} = \{T, T^4\}$

$[T^2] = \{ g_2 = gT^2 g^{-1} \mid g \in G \}$

$= \{ T^k T^2 T^{5-k} \} \cup \{ \sigma T^k T^2 \sigma T^k \}$

$= \{ T^7 = T^2 \} \cup \{ \cancel{\sigma} T^{5-2-k} \sigma^2 T^k \}$

$= \{T^2, T^3\}$

~~$[\sigma T^k] =$~~

~~$\{\sigma T\}$~~

$[\sigma] = \{ g_2 = g \sigma g^{-1} \mid g \in G \}$

$= \{ T^k \sigma T^{5-k} \} \cup \{ \sigma T^k \sigma \sigma T^k \}$

$= \{ \sigma T^{10-2k \bmod 5} \} \cup \{ \sigma T^{2k} \}$

$= \{ \cancel{\sigma T^0}, \sigma, \sigma T^3, \sigma T, \sigma T^4, \sigma T^2 \} \cup$
$\{ \sigma, \sigma T^2, \sigma T^4, \sigma T, \sigma T^3 \}$

$= \{ \sigma T^k \mid k = 0, \cdots, 4 \}$

representation $\quad R(T^k) = T^k$

$\qquad\qquad\qquad\quad R(\sigma T^k) = \sigma T^k$

If $R(g)$ is ~~irred~~ reducible then $R(g)$ can be put into the following form by a suitable basis transformation. $\forall g$.

$$S R(g) S^{-1} = \begin{pmatrix} a(g) & 0 \\ 0 & b(g) \end{pmatrix}.$$

where $a(g), b(g) \in \mathbb{C}$

this is because every reducible representation of a finite group is completely (fully) reducibles. And $\because$ $\dim(R) = 2$, if it is fully reducible it can only be decomposed into the direct sum of 2 1-dimensional representations

for $g \in \{\sigma T^k\}$, $\quad g = \begin{pmatrix} 0 & \alpha^{-k} \\ \alpha^k & 0 \end{pmatrix}.$

$\therefore \det(g - \lambda I) = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1.$

eigenvalues $= \pm 1$.

$\lambda = 1 \qquad g\begin{pmatrix} 1 \\ \alpha^k \end{pmatrix} = (1)\begin{pmatrix} 1 \\ \alpha^k \end{pmatrix}.$

$\lambda = -1 \qquad g\begin{pmatrix} 1 \\ -\alpha^{-k} \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -\alpha^{-k} \end{pmatrix}.$

$\therefore$ the eigenvectors depend on k

$\therefore$ ~~&~~ matrix S depends on k.

$\therefore$ No single matrix S can transform R(g) to ~~[…]~~ diagonal form ~~for ev~~ $\forall g$

$\rightarrow$ R is irreducible

b) there are 4- conjugacy classes

number of irreducible representations
= number of conjugacy classes. = 4.

Also $\sum_{i=1}^{4} \dim(R_i)^2 = |G| = 10$

$\therefore$ ~~4 pe~~ 4 perfect squares sum to 10.

$\therefore$ we have $1+1+1+1+1+1+4$ (i)

$1+1+4+4$ (ii)

$1+9$ (iii)

case (iii) is impossible because we ~~to~~ already know from ~~e~~ a) that there is at least one 2-dimensional irreducible rep-resentations.

to determine whether case (i) or (ii) is correct, let's see how many 1-dimensional irreducible representations do we have is it 6 or 2 ?

↓ (i) ↓ (ii)

For 1-D representations $R_q(g)$

$R_q(\mathbb{1}_2) = 1$

$\therefore$ ~~R 1 = R~~

$R_q(\sigma) R_q(\sigma) = R_q(\sigma^2) = R_q(\mathbb{1}_2) = 1$

$\therefore R_q(\sigma) = \pm 1$.

$R_q(T)^5 = R_q(T^5) = R_q(\mathbb{1}_2) = 1$

$\therefore R_q(T)^5 = e^{\frac{2\pi i}{5} q} \quad (q = 0,1,2,3,4) = \alpha^q$

$R_q(T^k) = R_q(T)^k = e^{\frac{2\pi i}{5} qk} \quad (k = 0, \cdots, 4) = \alpha^{qk}$

$R_q(\sigma T^k) = (\pm 1) R_q(T^k) = (\pm 1) e^{\frac{2\pi i qk}{5}} = \pm \alpha^{qk}$

$R_q(\sigma T^k) R_q(\sigma T^k) = R_q(\sigma T^k \sigma T^k) = R_q(\mathbb{1}_2) = 1$

$\therefore R_q(\sigma T^k) = \pm 1$

$\therefore \pm \alpha^{kq} = \pm 1. \Rightarrow q = 0$ is allowed.

2 1-D reps:

$R_{q_1}(T^k) = 1 \qquad R_{q_1}(\sigma T^k) = 1$

or $R_{q_2}(T^k) = 1 \qquad R_{q_2}(\sigma T^k) = -1$

OR

$\therefore$ 4 conjugacy classes.

$$\sum_{i=1}^{4} \dim(R_i)^2 = |G| = 10$$

$\dim(R_i) \in \mathbb{Z}$ $\therefore$ $1 + 1 + 2^2 + 2^2 = 10$

2 1-D reps, 2 2-D reps.

there are 4 complex irreducible representations of G.

two 1-D (R_{q_1}, R_{q_2})
two 2-D (R, R_2)

As shown before

$$\chi_{R_{q_1}}(T^k) = tr(R_{q_1}(T^k)) = R_{q_1}(T^k) = 1$$

$\chi_{R_{q_1}}(\sigma T^k) = 1$

$\chi_{R_{q_2}}(T^k) = 1$ $\chi_{R_{q_2}}(\sigma T^k) = -1$

R is given

$\therefore$ ~~character table is~~

$$\chi_R(T^k) = \alpha^k + \alpha^{-k} = e^{\frac{2\pi ik}{5}} + e^{-\frac{2\pi ik}{5}}$$

$$= 2\cos\left(\frac{2\pi k}{5}\right)$$

$\chi_R(\sigma T^k) = 0$ (traceless)

χ \ Conjugacy class	$\{\mathbb{1}_2\}$	$\{T, T^4\}$	$\{T^2, T^3\}$	$\{\sigma T^k\}$
degeneracy	1	2	2	5
$\chi_{R_{q_1}}$	1	1	1	1
$\chi_{R_{q_2}}$	1	1	1	-1
χ_R	2	$2\cos(\frac{2\pi}{5})$	$2\cos(\frac{4\pi}{5})$	0
χ_{R_2}	2	$2\cos(\frac{4\pi}{5})$	$2\cos(\frac{2\pi}{5})$	0
		$= a$	$= b$	$= c$

The final known 2-D rep $R_2 \Rightarrow \chi_{R_2}(\mathbb{1}_2) = 2$ is obvious $\because$ $\dim(R_2) = 2$.

let $\chi_{R_2}(T) = a$ $\quad \chi_{R_2}(T^2) = b$ $\quad \chi_{R_2}$

use orthogonality of characters

$$(\chi_i, \chi_j) = \delta_{ij} = \frac{1}{|G|} \sum_{g \in G} \chi_i^*(g)\, \chi_j(g).$$

Assume real trace for R_2, $a, b, c \in \mathbb{R}$.

$$\therefore \quad \left.\begin{array}{l} 2(1+a+b) + 5c = 0 \\ 2(1+a+b) - 5c = 0 \end{array}\right\} \Rightarrow c = 0$$

$$2(1+a+b) = 0 \quad \cancel{\Rightarrow b = a} \Rightarrow a+b = -1$$

$$2(2 + 2a\cos(\tfrac{2\pi}{5}) + 2b\cos(\tfrac{4\pi}{5})) = 0$$

$$= 2(\cancel{1 + 2a + \cos\tfrac{2\pi}{5} + \cos\tfrac{4\pi}{5}})$$

Also $(\chi_{R2}, \chi_{R2}) = 1$

$\therefore \quad 4 + 2a^2 + 2b^2 = 10 \quad , \quad a+b = -1.$

$\therefore \quad a^2 + b^2 = 3$

$\therefore \quad a^2 + (a+1)^2 = 3 \quad \Rightarrow \quad a^2 + a - 1 = 0$

$\therefore a = \frac{-1 \pm \sqrt{5}}{2} \quad , \quad b = \frac{-1 \mp \sqrt{5}}{2}.$

If $\quad a = \frac{-1+\sqrt{5}}{2} = 2\cos(\frac{2\pi}{5})$

$\quad b = \frac{-1-\sqrt{5}}{2} = 2\cos(\frac{4\pi}{5})$

this is χ_R

So for χ_{R2}, $\quad a = \frac{-1-\sqrt{5}}{2} = 2\cos(\frac{4\pi}{5})$

$\quad b = \frac{-1+\sqrt{5}}{2} = 2\cos(\frac{2\pi}{5})$

check $(\chi_R, \chi_{R2}) = 4\left(1 + a\cos(\frac{2\pi}{5}) + b\cos(\frac{4\pi}{5})\right)$

$= 4\left(1 + 2\cos(\frac{4\pi}{5})\cos(\frac{2\pi}{5}) + 2\cos(\frac{2\pi}{5})\cos(\frac{4\pi}{5})\right)$

$= 4\left(1 + 4\cos(\frac{2\pi}{5})\cos(\frac{4\pi}{5})\right)$

$= 4\left(1 + \cancel{4}\,(-\frac{1}{2} + \frac{\sqrt{5}}{2})(-\frac{1}{2} - \frac{\sqrt{5}}{2})\right)$

$= 4\left(1 + (\frac{1}{4}) - (\frac{5}{4})\right) = 0$

$\therefore$ character table is as shown.

c) character of $R \otimes R$ is

$$\chi_{R\otimes R} = \chi_R \times \chi_R$$

$$= \left(4,\ 4\cos^2\left(\tfrac{2\pi}{5}\right),\ 4\cos^2\left(\tfrac{4\pi}{5}\right),\ 0\right)$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\{\mathbb{1}_2\} \qquad \{T, T^4\} \qquad \{T^2, T^3\} \qquad \{\sigma T^k\}$$

multiplicity $m_i = (\chi_{R\otimes R}, \chi_i)$.

$$m_{q_1} = (\chi_{R\otimes R}, \chi_{q_1})$$

$$= \frac{4}{10}\left(1 + 2\cos^2\left(\tfrac{2\pi}{5}\right) + 2\cos^2\left(\tfrac{4\pi}{5}\right)\right)$$

$$= \frac{2}{5}\left(1 + \cancel{2} 1 + \cos\left(\tfrac{4\pi}{5}\right) + 1 + \underbrace{\cos\left(\tfrac{8\pi}{5}\right)}_{=\cos\left(\frac{2\pi}{5}\right)}\right)$$

$$= \frac{2}{5}\left(\cancel{1} 3 + \underbrace{(\cos\tfrac{4\pi}{5}) + \cos\left(\tfrac{2\pi}{5}\right)}_{-\frac{1}{2}}\right)$$

$$= \frac{2}{5}\left(\frac{5}{2}\right) = 1 \quad /\!/$$

$$m_{q_2} = m_{q_1} = 1 \quad /\!/$$

$$m_R = (\chi_{R\otimes R}, \chi_R)$$

$$= \frac{1}{10}\left(8 + 16\cos^3\left(\tfrac{2\pi}{5}\right) + 16\cos^3\left(\tfrac{4\pi}{5}\right)\right)$$

$$= \frac{1}{10}\left(8 + 8\cos\left(\tfrac{2\pi}{5}\right)\left(1 + \cos\tfrac{4\pi}{5}\right) + 8\cos\left(\tfrac{4\pi}{5}\right) \times \left(1 + \cos\tfrac{2\pi}{5}\right)\right)$$

$$= \cancel{\tfrac{1}{10}}\ \frac{4}{5}\left(1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + 2\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\right)$$

$$= \frac{4}{5}\left(1 + \left(-\frac{1}{2}\right) + 2\left(-\frac{1}{4}\right)\right) = 0$$

to keep the dimension $= 4$ we need ~~dim~~ $m_2 = 1$, but we prove it explicitly

$$m_2 = (\chi_{R\otimes R}, \chi_2).$$

$$= \frac{1}{10}\left(8 + 16\cos^2\left(\frac{2\pi}{5}\right)\cos\frac{4\pi}{5} + 16\cos^2\left(\frac{4\pi}{5}\right)\cos\frac{2\pi}{5}\right)$$

$$= \frac{1}{10}\left(8 + 8\cos\frac{4\pi}{5}\left(1 + \cos\frac{4\pi}{5}\right) + 8\cos\frac{2\pi}{5}\left(1 + \cos\frac{2\pi}{5}\right)\right)$$

$$= \frac{4}{5}\left(1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos^2\frac{2\pi}{5} + \cos^2\frac{4\pi}{5}\right)$$

$$= \frac{4}{5}\left(1 - \frac{1}{2} + \frac{\left(1 + \cos\frac{4\pi}{5}\right)}{2} + \frac{\left(1 + \cos\frac{2\pi}{5}\right)}{2}\right)$$

$$= \frac{4}{5}\left(1 - \frac{1}{2} + 1\ \cancel{0} - \frac{1}{4}\right).$$

$$= \frac{4}{5}\left(\frac{2}{4} + \frac{3}{4}\right) = \frac{4}{5}\cdot\left(\frac{5}{4}\right) = 1$$

$$\therefore\ R\otimes R = R_{q_1} \oplus R_{q_2} \oplus R_2$$

d) $\alpha^* = \alpha^{-1}$

$$\therefore \quad T^* = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \qquad \sigma^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$(T^k)^* = \begin{pmatrix} \alpha^{-k} & 0 \\ 0 & \alpha^k \end{pmatrix} = (T^*)^k.$$

$$(\sigma T^k)^* = \begin{pmatrix} 0 & \alpha^k \\ \alpha^{-k} & 0 \end{pmatrix} = \sigma^* (T^*)^k.$$

let $S = \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S^{-1}$

$$R^*(T^k) = (T^*)^k = (T^k)^* = \begin{pmatrix} \alpha^{-k} & 0 \\ 0 & \alpha^k \end{pmatrix}.$$

$$S R(T^k) S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha^k \\ \alpha^{-k} & 0 \end{pmatrix} = \begin{pmatrix} \alpha^{-k} & 0 \\ 0 & \alpha^k \end{pmatrix}$$

$$R^*(\sigma T^k) = \begin{pmatrix} 0 & \alpha^k \\ \alpha^{-k} & 0 \end{pmatrix}.$$

$$S R(\sigma T^k) S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha^{-k} \\ \alpha^k & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^{-k} & 0 \\ 0 & \alpha^k \end{pmatrix} = \begin{pmatrix} 0 & \alpha^k \\ \alpha^{-k} & 0 \end{pmatrix}.$$

$\Rightarrow$ we have

$$R^* = S R S^{-1} \quad \text{with } S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\therefore \quad R^* \cong R.$

③ $SU(6) = \{ \mathcal{U} \in \mathrm{Aut}(\mathbb{C}^6) \mid \mathcal{U}^\dagger \mathcal{U} = \mathbb{1},\ \det(\mathcal{U}) = 1 \}$

a) Expand $\mathcal{U} \in SU(6)$ around ~~a~~ identity.

$\mathcal{U} = 1 + T + \cdots$

$\rightarrow$ generators.

$\mathcal{U}^\dagger \mathcal{U} = \mathbb{1} \Rightarrow \mathbb{1} = \mathbb{1} + T + T^\dagger + \cdots$

$\Rightarrow T = -T^\dagger$

$\det(\mathcal{U}) = 0 \Rightarrow 1 = 1 + tr(T) + \cdots$

$\Rightarrow tr(T) = 0$

$\therefore$ Lie algebra of $SU(6)$:

$su(6) = \{ T \in \mathrm{End}(\mathbb{C}^6) \mid T = -T^\dagger,\ tr(T) = 0 \}$

the Cartan subalgebra $\mathcal{H}$ is the maximal Abelian sub-algebra of $su(6)$

$\therefore \mathcal{H} = \{ \mathrm{diag}(ia_1, \cdots, ia_6) \mid a_i \in \mathbb{R},\ \sum_{i=1}^{6} a_i = 0 \}$

$\Rightarrow rk(su(6)) = \dim(\mathcal{H}) = 6 - 1 = 5$

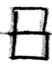
$\Rightarrow \dim(su(6)) = 2 \times \frac{6(6-1)}{2} + 6 - 1$

$= (6+1)(6-1) = 6^2 - 1 = 35$

b)

$(10000) \rightarrow$  $\rightarrow \psi_\mu \rightarrow \underline{6}$

$(00001) \rightarrow$  $\rightarrow \phi^\mu \rightarrow \underset{\sim}{\bar{6}}$

$(01000) \rightarrow$  $\rightarrow$

$\chi_{[\rho\sigma]} \rightarrow \underset{\sim}{15}$

c) singlets.

$\cancel{\varepsilon^{\mu\nu\rho} \psi_\mu \tilde{\psi}_\sigma \tilde{\tilde{\psi}}_\rho} \quad \cancel{\varepsilon_{\mu\nu\rho} \phi^\mu \tilde{\phi}^\nu \tilde{\tilde{\phi}}^\rho}$

$\cancel{\varepsilon^{\mu\nu\rho\sigma} \psi_\mu \tilde{\psi}_\nu \chi_{[\rho\sigma]}}$

$\cancel{\varepsilon^{\mu\nu\rho\sigma\tau} \psi_\mu \chi_{[\nu\rho]} \tilde{\chi}_{[\sigma\tau]}}$

$\varepsilon^{\mu\nu\rho\sigma\tau\lambda} \chi_{[\mu\nu]} \tilde{\chi}_{[\rho\sigma]} \tilde{\tilde{\chi}}_{[\tau\lambda]}$ //

$\phi^\mu \tilde{\phi}^\nu \chi_{[\mu\nu]}$ //

(ε needs to have 6 indices for SU(6))

d) Branching

$$\psi_\mu = (\psi_\alpha, \psi_i) = (\psi_\alpha, \psi_6)$$

$\mu = 1,\dots,6$ $\quad \alpha = 1,\dots,5$ $\quad i = 6$

$\psi_\alpha \to \underset{\sim}{5}$, $\quad \psi_6 \to \underset{\sim}{1}$

$$\Rightarrow \quad \underset{\sim}{6} = \underset{\sim}{5} + \underset{\sim}{1}$$

U(1) matrix of SU(6), is $U = \begin{pmatrix} e^{i\alpha} &&&&& \\ & e^{i\alpha} &&&& \\ && e^{i\alpha} &&& \\ &&& e^{i\alpha} && \\ &&&& e^{i\alpha} & \\ &&&&& e^{-5i\alpha} \end{pmatrix}$

$$\psi_\mu' = U\psi_\mu = \begin{pmatrix} e^{i\alpha} && \\ & \ddots & \\ && e^{i\alpha} \\ &&& e^{-5i\alpha} \end{pmatrix} \begin{pmatrix} \left.\begin{matrix}\psi_1 \\ \vdots \\ \psi_5\end{matrix}\right\} \psi_\alpha \\ \psi_6 \end{pmatrix}.$$

$$= \begin{pmatrix} e^{i\alpha}\psi_1 \\ \vdots \\ e^{i\alpha}\psi_5 \\ e^{-5i\alpha}\psi_6 \end{pmatrix} = \begin{pmatrix} e^{i\alpha}\psi_\alpha \\ e^{-5i\alpha}\psi_6 \end{pmatrix}.$$

$$\Rightarrow \quad \underset{\sim}{6} = \underset{\sim}{5}(1) + \underset{\sim}{1}(-5)$$

$$\phi^\nu = (\phi^\alpha, \phi^i) = (\phi^\alpha, \phi^6) \qquad \nu = 1,\dots,6$$
$$\alpha = 1,\dots,5.$$

$\Rightarrow \quad \bar{\underset{\sim}{6}} = \bar{\underset{\sim}{5}} + \underset{\sim}{1}$

Similarly $\quad \bar{\underset{\sim}{6}} = \bar{\underset{\sim}{5}}(1) + \underset{\sim}{1}(-5)$

$\underset{\sim}{15}$: $\chi_{[\mu\nu]} = (\chi_{[\alpha\beta]}, \chi_{[\alpha i]}, \cancel{\chi} 0)$

$\chi_{[\alpha\beta]} \to \underset{\sim}{10}$, $\quad \chi_{[\alpha i]} \to \underset{\sim}{5}$

$$= \left(\begin{array}{c|c} \chi_{[\alpha\beta]} & \chi_{\alpha 6} \\ \hline -\chi_{\alpha 6} & 0 \end{array} \right).$$

$$\chi'_{[\mu\nu]} = \begin{pmatrix} e^{i\alpha} & & & \\ & \ddots & & \\ & & e^{i\alpha} & \\ & & & e^{-i5\alpha} \end{pmatrix} \left(\begin{array}{c|c} \chi_{[\alpha\beta]} & \chi_{\alpha 6} \\ \hline -\chi_{\alpha 6} & 0 \end{array} \right) \begin{pmatrix} e^{i\alpha} & & & \\ & \ddots & & \\ & & e^{i\alpha} & \\ & & & e^{-5i\alpha} \end{pmatrix}$$

$$= \left(\begin{array}{c|c} e^{2i\alpha} \chi_{[\alpha\beta]} & e^{-4i\alpha} \chi_{\alpha 6} \\ \hline -e^{-4i\alpha} \chi_{\alpha 6} & 0 \end{array} \right).$$

$\Rightarrow \quad \underset{\sim}{15} = \underset{\sim}{10}(2) + \underset{\sim}{5}(-4)$

e) ~~$\psi \chi$~~ index decomposition.

$$\psi_\mu = (t_\alpha, h_6) \qquad \phi^\nu = (d^\alpha, L^6)$$

$$\chi_{[\mu\nu]} = (u_{[\alpha\beta]}, Q_{\alpha 6}, \underbrace{-Q_{\alpha 6}}_{Q_{6\alpha}}, 0).$$

~~$\varepsilon^{\mu\nu\rho} \psi_\mu \tilde{\psi}_\nu \tilde{\tilde{\psi}}_\rho =$~~

$$\varepsilon^{\mu\nu\rho\sigma\tau\kappa} \chi_{[\mu\nu]} \tilde{\chi}_{[\rho\sigma]} \tilde{\tilde{\chi}}_{[\tau\kappa]}$$

$$= 2 Q_{6\alpha} \tilde{u}_{[\beta\gamma]} \tilde{\tilde{u}}_{[\delta\varepsilon]} \varepsilon^{6\alpha\beta\gamma\delta\varepsilon} + 2 u_{[\alpha\beta]} \tilde{Q}_{6\gamma} \tilde{\tilde{u}}_{[\delta\varepsilon]} \varepsilon^{\alpha\beta 6\gamma\delta\varepsilon} + 2 u_{[\alpha\beta]} \tilde{u}_{[\gamma\delta]} \tilde{\tilde{Q}}_{6\varepsilon} \varepsilon^{\alpha\beta\gamma\delta 6\varepsilon}$$

$$= 2\varepsilon^{\alpha\beta\gamma\delta\varepsilon} \left(Q_{6\alpha} \tilde{u}_{[\beta\gamma]} \tilde{\tilde{u}}_{[\delta\varepsilon]} + u_{[\alpha\beta]} \tilde{Q}_{6\gamma} \tilde{\tilde{u}}_{[\delta\varepsilon]} + u_{[\alpha\beta]} \tilde{u}_{[\gamma\delta]} \tilde{\tilde{Q}}_{6\varepsilon} \right)$$

(factor 2 comes from $\varepsilon^{6\alpha\cdots} = -\varepsilon^{\alpha 6\cdots}$ and $Q_{6\alpha} = -Q_{\alpha 6}$)

($\varepsilon^{6\alpha\beta\gamma\delta\varepsilon} = \varepsilon^{\alpha\beta 6\gamma\delta\varepsilon} = \varepsilon^{\alpha\beta\gamma\delta 6\varepsilon} = \varepsilon^{\alpha\beta\gamma\delta\varepsilon}$

$\because$ $\varepsilon^{612345} = 1 = \varepsilon^{12345}$ and all 6-D ε's requires even number of swaps to get $\varepsilon^{6\alpha\beta\gamma\delta\varepsilon}$ and this $= \varepsilon^{\alpha\beta\gamma\delta\varepsilon}$).

~~$\phi^\mu \tilde{\phi}^\nu \chi_{[\mu\nu]} = d^\alpha \tilde{d}^\beta \chi_{[\alpha\beta]}$~~

~~$d^\alpha \tilde{d}^\beta u_{[\alpha\beta]} + \tilde{d}^\alpha d^\beta u_{[\alpha\beta]}$~~

$$\phi^\mu \tilde{\phi}^\nu \chi_{[\mu\nu]} = d^\alpha \tilde{d}^\beta U_{\alpha\beta} + d^\alpha \tilde{L}^6 Q_{\alpha 6}$$
$$- L^6 \tilde{d}^\alpha Q_{\alpha 6}$$

(No $L^6 \tilde{L}^6$ term $\because \chi_{[66]} = 0$.)
($Q_{\alpha 6} = - Q_{6\alpha}$)

④ For Lie algebra G_2,

$$A = \begin{array}{cc} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} & \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} \end{array}$$

a) $(1\ 0)$

$$\begin{tikzcd}[column sep=small]
 & \begin{matrix}\boxed{1 \;\; 0} \\ 0 \;\; 0\end{matrix} \arrow[d, "-\alpha_1"] & \\
 & \begin{matrix}\boxed{-1 \;\; 3} \\ 1 \;\; 0\end{matrix} \arrow[d, "-\alpha_2"] & \\
 & \begin{matrix}\boxed{0 \;\; 1} \\ 0 \;\; 1\end{matrix} \arrow[d, "-\alpha_2"] & \\
 & \begin{matrix}\boxed{1 \;\; -1} \\ 0 \;\; 2\end{matrix} \arrow[dl, "-\alpha_1"'] \arrow[dr, "-\alpha_2"] & \\
\begin{matrix}\boxed{-1 \;\; 2} \\ 1 \;\; 0\end{matrix} \arrow[dr, "-\alpha_2"'] & & \begin{matrix}\boxed{2 \;\; -3} \\ 0 \;\; 3\end{matrix} \arrow[dl, "-\alpha_1"] \\
 & \begin{matrix}\boxed{0 \;\; 0} \\ 1 \;\; 1\end{matrix} \arrow[dl, "-\alpha_1"'] \arrow[dr, "-\alpha_2"] & \text{degeneracy} = 2 \\
 {} & & {}
\end{tikzcd}$$

$-\alpha_1$ $\qquad\qquad$ $-\alpha_2$

$\boxed{-2 \quad 3}$ $\qquad\qquad$ $\boxed{1 \quad -2}$

$2 \quad 0$ $\qquad\qquad\qquad$ $0 \quad 2$

$-\alpha_2$ $\qquad\qquad$ $-\alpha_1$

$\boxed{-1 \quad 1}$

$1 \quad 1$

$\downarrow -\alpha_2$

$\boxed{0 \quad -1}$

$0 \quad 2$

~~dim(0,1) = 2 dim(1,2) = 12~~

$\downarrow -\alpha_2$

$\boxed{1 \quad -3}$

$0 \quad 3$

$\downarrow -\alpha_1$

$\boxed{-1 \quad 0}$

$1 \quad 0$

$\dim(1,0) = 12 \times 1 + 2 = \underline{\underline{14}}$

$(0\,1)$

$$\boxed{0 \quad 1}$$
$0 \quad 0$

$\downarrow \; -\alpha_2$

$$\boxed{1 \quad -1}$$
$0 \quad 1$

$\downarrow \; -\alpha_1.$

$$\boxed{-1 \quad 2}$$
$1 \quad 0$

$\downarrow \; -\alpha_2$

$$\boxed{0 \quad 0}$$
$0 \quad 1$

$\downarrow \; -\alpha_2$

$$\boxed{1 \quad -2}$$
$0 \quad 2$

$\downarrow \; -\alpha_1$

$$\boxed{-1 \quad 1}$$
$1 \quad 0$

$\downarrow \; -\alpha_2$

$$\boxed{0 \quad -1}$$
$0 \quad 1$

$\dim(0,1) = 7$

b) Quadratic casimir :

$$C(\vec{a}) = \vec{a}^T G (\vec{a} + 2\vec{\delta})$$

where $\vec{a}$ is the highest dynkin weight

of the representation and $\vec{\delta} = (1,1,\cdots,1)^T$.
$= (1,1)^T$ for G_2

$$C((1,0)) = \frac{1}{3}(1,0)\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

$$= \frac{1}{3}(1,0)\begin{pmatrix} 24 \\ 13 \end{pmatrix}$$

$$= \frac{24}{3} = \underline{\underline{8}}$$

$$C((0,1)) = \frac{1}{3}(0,1)\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$= \frac{1}{3}(0,1)\begin{pmatrix} 21 \\ 12 \end{pmatrix}$$

$$= \frac{12}{3} = \underline{\underline{4}}$$

c) $P = P(A_2 \subset G_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$$G_2 \xrightarrow[P]{} A_2 \quad (SU(3))$$

$(1\ 0)$	$\underset{\sim}{14}$	$(1\ 1)$	$\underset{\sim}{8}$
$(-1\ 3)$	$\underset{\sim}{14}$	$(-1\ 2)$	$\underset{\sim}{8}$
$(0\ 1)$	$\underset{\sim}{14}, \underset{\sim}{7}$	$(0\ 1)$	$\underset{\sim}{\bar{3}}$
$(1\ -1)$	$\underset{\sim}{14}, \underset{\sim}{7}$	$(1\ 0)$	$\underset{\sim}{3}$
$(-1\ 2)$	$\underset{\sim}{14}, \underset{\sim}{7}$	$(-1\ 1)$	$\underset{\sim}{3}$
$(2\ -3)$	$\underset{\sim}{14}$	$(2\ -1)$	$\underset{\sim}{8}$

$(0\ 0)$	$\underset{\sim}{14}, \underset{\sim}{7}$	$(0\ 0)$.	$\underset{\sim}{8}\ \underset{\sim}{1}$
$(-2\ 3)$	$\underset{\sim}{14}$	$(-2\ 1)$	$\underset{\sim}{8}$
$(1\ -2)$	$\underset{\sim}{14}, \underset{\sim}{7}$	$(1\ -1)$	$\underset{\sim}{\bar{3}}$
$(-1\ 1)$	$\underset{\sim}{14}, \underset{\sim}{7}$	$(-1\ 0)$	$\underset{\sim}{\bar{3}}$
$(0\ -1)$	$\underset{\sim}{14}, \underset{\sim}{7}$	$(0\ -1)$	$\underset{\sim}{3}$
$(1\ -3)$	$\underset{\sim}{14}$	$(1\ -2)$	$\underset{\sim}{8}$
$(-1\ 0)$	$\underset{\sim}{14}$	$(-1\ -1)$	$\underset{\sim}{8}$

(Note, in $\underset{\sim}{14}$ of G_2, (0.0) has degeneracy 2
$\rightarrow$ $\underset{\sim}{8}$ of A_2

in $\underset{\sim}{7}$ of G_2, $(0,0)$ has degeneracy 1
$\rightarrow$ $\underset{\sim}{1}$ of A_2)

d)

$\therefore$ Branching from $G_2 \rightarrow A_2$

$\underset{\sim}{14} \rightarrow \underset{\sim}{8} \oplus \underset{\sim}{3} \oplus \underset{\sim}{\bar{3}}$ //

$\underset{\sim}{7} \rightarrow \underset{\sim}{3} \oplus \underset{\sim}{\bar{3}} + \underset{\sim}{1}$ //

b)

$(1\,0\,0\,0\,0) \rightarrow$  $\rightarrow \psi_\mu \rightarrow \underset{\sim}{6}$

$(0\,0\,0\,0\,1) \rightarrow$  $\rightarrow \phi^\mu \rightarrow \underset{\sim}{\bar{6}}$

$(0\,1\,0\,0\,0) \rightarrow$  $\rightarrow \chi_{[\rho\sigma]} \rightarrow \underset{\sim}{15}$

c) singlets.

$\cancel{\varepsilon^{\mu\nu\rho}\psi_\mu\tilde{\psi}_\nu\tilde{\tilde{\psi}}_\rho}$ $\cancel{\varepsilon_{\mu\nu\rho}\phi^\mu\tilde{\phi}^\nu\tilde{\tilde{\phi}}^\rho}$

$\cancel{\varepsilon^{\mu\nu\rho\sigma}\psi_\mu\tilde{\psi}_\nu\chi_{[\rho\sigma]}}$

$\cancel{\varepsilon^{\mu\nu\rho\sigma\tau}\psi_\mu\chi_{[\nu\rho]}\tilde{\chi}_{[\sigma\tau]}}$

$\varepsilon^{\mu\nu\rho\sigma\tau\lambda}\chi_{[\mu\nu]}\tilde{\chi}_{[\rho\sigma]}\tilde{\tilde{\chi}}_{[\tau\lambda]}$

$\phi^\mu\tilde{\phi}^\nu\chi_{[\mu\nu]}$

(ε needs to have 6 indices for SU(6)).