Groups and Representations

2016 Exam

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①

a) A representation R of a group ~~is~~ G over a vector ~~space~~ space V is a group homomorphism

$$R: G \longmapsto Aut(V)$$

A representation $R: G \to Aut(V)$ is irreducible if there does not exist a sub-vector space $U \in V$, $U \neq \{0\}$ with $R(g)U \in U \quad \forall g \in G$.

A representation R is called faithful if it is injective.

A representation R is unitary if there exist a scalar product $\langle \cdot, \cdot \rangle$ over V such that $\langle R(g)v, R(g)w \rangle = \langle v, w \rangle \quad \forall g \in G, v, w \in V$

b) Schur's Lemma: irreducible ~~repre~~ representation $R: G \to Gl(V)$, linear map $P: V \to V$

~~The~~ if ~~PR~~ $PR(g) = R(g)P$

then $P = \lambda \, id_V$

λ constant $\in K$, id_V identity map.

if G Abelian $\rightarrow$ ~~gg~~ ~~g~~

$$g_1 g_2 = g_2 g_1 \quad \forall\ g_1, g_2 \in G.$$

$$[\underbrace{R(g_1)}_{P}, R(g_2)] = 0 \quad \forall\ g_1, g_2 \in G.$$

$\Rightarrow$ $PR(g_2) = R(g_2)P$

$\Rightarrow$ By schur's lemma, ~~$P \neq \lambda(g)$~~

$$P = \lambda\, id_V = \lambda\, id_n \quad (n = \dim(V)).$$

$\Rightarrow$ $R(g_1) = \lambda id_n = \lambda(g_1)\, id_n$

Similarly $R(g_2) = \lambda(g_2)\, id_n$

Hence $R(g) = \lambda(g)\, id_n \quad \forall\ g \in G.$

this is irreducible only if $n = 1$

c) $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}.$

the multiplication, addition modulo 2, is commutative, so G is ~~ab~~ Abelian.

$\rightarrow$ irreducible representations are 1-dimensional.

the identity element is (0.0)

$\therefore$ $R(0,0) = 1$

$(1,1) \circ (1,1) = (0,0)$

$\Rightarrow R^2(1,1) = R(0,0) = 1$

$\therefore R(1,1) = \pm 1$

~~If $R(1,1) = 1$, $R(0,0) = 1$~~

$\Rightarrow (1,0) \circ (1,0) = (0,0)$

$\therefore R^2(1,0) = R(0,0) = 1 \Rightarrow R(1,0) = \pm 1$

Similarly $R(0,1) = \pm 1$

Also $R(0,1) \cdot R(1,0) = R(1,1)$
$R(1,1)\ R(1,0) = R(0,1)$
$R(1,1)\ R(0,1) = R(1,0)$

$\therefore$ All irreducible representations.

	R_1	R_2	R_3	R_4
$R(0,0)$	1	1	1	1
$R(1,0)$	1	-1	1	-1
$R(0,1)$	1	-1	-1	1
$R(1,1)$	1	1	-1	-1

Character χ_R of R is $\chi_R = tr(R(g))$

For 1-D, $\chi_R(g) = R(g)$

$\therefore$ character table:

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
χ_{R_1}	1	1	1	1
χ_{R_2}	1	-1	-1	+1
χ_{R_3}	1	+1	-1	-1
χ_{R_4}	1	-1	+1	-1

d) $V = \{ax^2 + bxy + cy^2 \mid a, b, c \in \mathbb{C}\}$

$R : \mathbb{Z}_2 \times \mathbb{Z}_2 \to Gl(V)$

$R(1,0)(P(x,y)) = P(x,-y)$

$R(0,1)(P(x,y)) = P(y,x)$

$R(1,1)(P(x,y)) = R(1,0)R(0,1)$

$= R(1,0)(P(y,x))$

$= P(y,-x)$

$R(0,0)(P(x,y)) = P(x,y)$. $\because R(e) = id.$

over the basis $\{x^2, xy, y^2\}$

$R(1,0)(x^2) = x^2 = (1)x^2 + (0)xy + (0)y^2$

$R(1,0)(xy) = -xy = (0)x^2 + (-1)xy + (0)y^2$

$R(1,0)(y^2) = y^2 = (0)x^2 + (0)xy + (1)y^2$

$$\therefore R(1,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R(0,1)(x^2) = y^2 = (0)x^2 + (0)xy + (1)y^2$

~~$R(0,1)(y^2)$~~

$R(0,1)(xy) = xy = (0)x^2 + (1)xy + (0)y^2$

$R(0,1)(y^2) = x^2 = (1)x^2 + (0)xy + (0)y^2$

$$\therefore R(0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$R(0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{obviously.}$$

$\therefore$

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
χ_R	3	1	1	-1

$$(\chi_i, \chi_j) = \frac{1}{|G|} \sum_{g \in G} \chi_i^*(g)\, \chi_j(g)$$

multiplicity of R_i in R is

$m_i = (\chi_i, \chi_R)$

$\therefore m_1 = (\chi_1, \chi_R) = \frac{1}{4}(3+1+1-1) = 1$

$m_2 = \frac{1}{4}(3-1-1-1) = 0$

$m_3 = \frac{1}{4}(3+1 \cancel{+}1+1) = 1$

$m_4 = \frac{1}{4}(3-1+1+1) = 1$

$\therefore \quad R = R_1 \oplus R_3 \oplus R_4$

$R((1,0)(0,1)) = R(1,1) = R((1,0))\,R((0,1))$

$R(1,0)\,R(0,0) = P(x,-y)\,P(x,y) = P(x,-y) = \cancel{R}\, R(1,0)$
$= R((1,0)(0,0))$.

~~R~~ similarly $\quad R(0,1)\,R(0,0) = R((0,1)(0,0))$.
$\cancel{R}\; R(0,0)(0,1) = R(0,1) \quad , R(0,0)(1,0) = R(1,0)$

$\cancel{R} \quad R(1,0)\,R(1,1) = P(x,-y)\,P(y,-x) = P(y,x)$
$= R(0,1) = R((1,0)(1,1))$.

Similarly $\quad \cancel{R(1,0)}\; R(0,1)\,R(1,1) = R((0,1)(1,1))$
$R(1,1)(0,1) = R(1,0) \quad , R(1,1)(1,0) = R(0,1)$

$R(1,1)\,R(1,1) = P(y,-x)\,P(y,-x) = P(-x,-y)$
$= P(x,y) = R(0,0) = R((1,1)(1,1))$.

quadratic

$\cancel{R}$

$\rightarrow \quad \therefore \quad R(g_1)\,R(g_2) = R(g_1 g_2) \quad \forall\, g_1, g_2 \in G$

R is a representation

(2)

quaternion group $Q = \{\pm \mathbb{1}_2, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3\}$

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

a) Conjugacy class of $g_1 \in G$ is

$[g_1] = \{ g_2 \in G \mid \exists g \in G : g_2 = g g_1 g^{-1} \}$

we know $\sigma_i \sigma_j = \delta_{ij} \mathbb{1}_2 + i \varepsilon_{ijk} \sigma_k$.

$\therefore (i\sigma_i)(i\sigma_j) = -\delta_{ij}\mathbb{1}_2 - i\varepsilon_{ijk}\sigma_k$

$\therefore (\pm \mathbb{1}_2)^{-1} = \pm \mathbb{1}_2$

$(\pm i\sigma_j)^{-1} = \mp i\sigma_j$

Conjugacy class of $\pm \mathbb{1}_2$

$[\pm \mathbb{1}_2] = \{ g_2 \in Q \mid \exists g \in Q : g_2 = g(\pm \mathbb{1}_2) g^{-1} \}$

$\Rightarrow g_2 = \pm \mathbb{1}_2 g g^{-1} = \pm \mathbb{1}_2$

$\Rightarrow [\pm \mathbb{1}_2] = \pm \mathbb{1}_2$

of $\pm i\sigma_j$:

$[\pm \sigma_j] = \{ g_2 \in Q \mid \exists g \in Q : g_2 = g(\pm i\sigma_j) g^{-1} \}$

If $g = \pm \mathbb{1}_2 \Rightarrow g_2 = \pm i\sigma_j$

If $g = \cancel{\sigma_i}\ \pm i\sigma_i$

Using $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$

$\therefore\ g_2 = g(\pm i\sigma_j)g^{-1} = \cancel{\sigma_j g g^{-1}}\ (\pm i\sigma_i)(\pm i\sigma_j)(\mp i\sigma_i)$

$= \pm\sigma_i\sigma_j\cancel{(\mp i\sigma_i) = \pm i\sigma_i\sigma_j\sigma_i}$

$\cancel{= \pm\sigma_j\sigma_i(\mp i\sigma_i) = \pm i\sigma_j\sigma_i\sigma_j}$

$\cancel{= \pm i(\sigma_j\sigma_i + [\sigma_i,\sigma_j])\sigma_i}$

$\cancel{= \pm i\sigma_j + 2i\varepsilon_{ijk}\sigma_k\sigma_i}$

$\cancel{= \pm i\sigma_j + 2i\varepsilon_{ijk}(\delta_{ki}\mathbb{1}_2 + i\varepsilon_{kij}\sigma_j)}$

$= \cancel{(\pm i\sigma_j)}\ \pm i\sigma_i\sigma_j\sigma_i$

If $i=j$ $\quad = \pm i\underbrace{\sigma_i\sigma_i}_{\mathbb{1}_2}\sigma_i = \pm i\sigma_i = \pm i\sigma_j$

If $i\neq j$ $\quad = \pm i\sigma_i\underbrace{\sigma_j\sigma_i}_{i\varepsilon_{jik}\sigma_k} = \cancel{\pm\sigma_i\sigma_k}\ \pm\sigma_i\sigma_k$

$= \pm i\sigma_j$

$\Rightarrow\ [\pm i\sigma_j] = \pm i\sigma_j$

(Note here $\pm A = \pm B$ doesn't $A=B$, $-A=-B$

It means $\cancel{+A \geq B}$ $+A = +B$ or $-B$

$-A = +B$ or $-B$)

Hence conjugacy classes are (5)

$[\mathbb{1}_2]$, $[-\mathbb{1}_2]$, $[\pm i\sigma_1]$, $[\pm i\sigma_2]$, $[\pm i\sigma_3]$

Num of conjugacy classes = 5 = number of irreducible representations of Q

~~Also~~

$$\therefore \sum_{i=1}^{5} (\dim R_i)^2 = |Q| = 8$$

$\dim R_i$ are dimensions of irr-reps. $\in \mathbb{Z}$

irr 5 positive integers (perfect squares) sum to 8

→ they are $\underline{1, 1, 1, 1, \sqrt{4} = 2}$ dimensions.

b) irreducible complex representations.

$R(g)$ \ g	$\pm \mathbb{1}_2$	$\pm i\sigma_1$	$\pm i\sigma_2$	$\pm i\sigma_3$
R_1	1	1	-1	-1
R_2	1	-1	1	-1
R_a	1	1	1	1
R_b	1	1	-1	-1
R_c	$\pm \mathbb{1}_2$	$\pm i\sigma_1$	$\pm i\sigma_2$	$\pm i\sigma_3$

the dim = 2 rep R_c is given by the question, and the 4 1D reps are obtained from the fact that $R(\mathbb{1}_2) = 1$ is the identity

R_a is the trivial rep.

For R_1, R_2, R_b, $\because$ 1-D matrices commute!

$$R(-\mathbb{1}_2)\cdot R(-\mathbb{1}_2) = R(\mathbb{1}_2) = 1$$

$$\Rightarrow \quad R(-\mathbb{1}_2) = \pm 1$$

$$R(\pm i\sigma_j)\cdot R(\pm i\sigma_j) = R(-\mathbb{1}_2) = 1$$

$$\Rightarrow R(\pm i\sigma_j) = \pm 1 \quad , \quad R(i\sigma_j) = R(-i\sigma_j)$$

$$R(\pm i\sigma_i)\, R(\pm i\sigma_j) = R(\pm i\sigma_k)$$

the solutions are R_1, R_b, R_2.

Character table: $\chi_i(g) = \mathrm{tr}(R_i(g))$

	$\mathbb{1}_2$	$-\mathbb{1}_2$	$\{\pm i\sigma_1\}$	$\{\pm i\sigma_2\}$	$\{\pm i\sigma_3\}$
χ_1	1	1	1	-1	-1
χ_2	1	1	-1	1	-1
χ_a	1	1	1	1	1
χ_b	1	1	1	-1	-1
χ_c	2	-2	0	0	0

c) $$R_4(i\sigma_1)\, R_4(i\sigma_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = -\mathbb{1}_4 = R(-\mathbb{1}_2)$$

Similarly $\quad R_4(\pm i\sigma_j)^2 = R_4(-\mathbb{1}_2)$

$$R_4(i\sigma_j)R_4(-i\sigma_j) = R_4(\mathbb{1}_2).$$

$$R_4(i\sigma_1)\,R_4(i\sigma_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix} = R_4(-i\sigma_3).$$

$$\therefore \quad R_4(i\sigma_i)\,R_4(i\sigma_j) = R_4(-i\varepsilon_{ijk}\sigma_k)$$
$$\underset{i\neq j}{\sim}$$

$\Rightarrow$ ~~the~~ multiplication structure is persevered

$\because$ $R_4 : Q \longrightarrow$ ~~Aut(R4)~~ $\mathrm{Aut}(\mathbb{R}^4)$
and is a homomorphism

$\therefore$ R_4 is a representation.

character of R_4

	$\mathbb{1}_4$	$-\mathbb{1}_2$	$\{\pm i\sigma_1\}$	$\{\pm i\sigma_2\}$	$\{\pm i\sigma_3\}$
χ_4	4	-4	0	0	0

multiplicity of irreducible representations R_i in R is

$$m_i = (\chi_R, \chi_i) = \frac{1}{|Q|}\sum_{g\in Q} \chi_R^*(g)\,\chi_i(g)$$

$\therefore$ $\cancel{\chi}$ $m_1 = m_2 = m_a = m_b = 0$

$$m_c = (\chi_4, \chi_c) = \frac{1}{8}(\cancel{4 \times} \; 4 \times 2 + (-4) \times (-2))$$

$$= \frac{1}{8} \times 16 = 2$$

$\therefore$ $R_4 = R_c \oplus R_c$

(d) $\cancel{R_4}$ $\chi_{R_4 \oplus R_4} = \chi_{R_4} + \chi_{R_4} = 2\chi_{R_4}$

$$= (8, -8, 0, 0, 0)$$

$m_1 = m_2 = m_a = m_b = 0$

$$m_c = \frac{1}{8}(2 \times 8 + (-2) \times (-8)) = 4$$

$\therefore$ $R_4 \oplus R_4 = R_c \oplus R_c \oplus R_c \oplus R_c = R_c^{\oplus 4}$

$\chi_{R_4 \otimes R_4} = \chi_{R_4} \times \chi_{R_4} = (16, 16, 0, 0, 0)$

$m_i = (\chi_{R_4 \otimes R_4}, \chi_i)$

$m_c = 0$, $m_1 = m_2 = m_a = m_b$

$$= \frac{1}{8}(16 + 16) = 4.$$

$\therefore$ $R_4 \otimes R_4 = R_1^{\oplus 4} \oplus R_2^{\oplus 4} \oplus R_a^{\oplus 4} \oplus R_b^{\oplus 4}$

③

a) Group $SU(4) = \{ U \in Aut(\mathbb{C}^4) \mid U^\dagger U = \mathbb{1},\ \det(U) = 1 \}$

Generators T : $U = \mathbb{1} + T + \dots$

$$U^\dagger U = \mathbb{1} \rightarrow (\mathbb{1} + T^\dagger)(\mathbb{1} + T) = \mathbb{1}$$

$$\rightarrow \mathbb{1} + T + T^\dagger + \dots = \mathbb{1}$$

$$\Rightarrow T = -T^\dagger \qquad \text{anti-hermitian}$$

$$\det(U) = 1 \Rightarrow 1 + tr(T) + \dots = 1$$

$$\therefore tr(T) = 0 \qquad \text{traceless}$$

$\therefore$ Lie algebra of $SU(4)$ is

$$su(4) = \mathcal{L}(SU(4)) = \{ T \in End(\mathbb{C}^4) \mid T = -T^\dagger,\ tr(T) = 0 \}$$

The Cartan-subalgebra $\underbrace{\ }_{\mathcal{H}}$ is the maximally mutual commuting (abelian) subspace of Lie-algebra $su(4)$, in this case $\mathcal{H}$ consists of the diagonal elements (so that they are Abelian) and traceless condition and antihermitian also needs to be satisfied.

$$\therefore \mathcal{H} = \{ diag(ia_1, ia_2, ia_3, ia_4) \mid a_j \in \mathbb{R},\ \sum_j a_j = 0 \}$$

A basis for the cartan subalgebra :

$$H_1 = i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad H_2 = i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad H_3 = i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

rank is $rk(su(4)) = \underline{\underline{3}}$

only imaginary

$$\dim(SU(4)) = \cancel{\frac{4(4-1)}{2}} \underbrace{2 \times \frac{4(4-1)}{2}}_{\substack{\text{real} \\ + \text{imaginary}}}\ \ \ \ + 1 \times \underbrace{(4-1)}_{\substack{\text{traceless} \\ \text{diagonal}}}$$

off diagonal

$$= 4^2 - 1 = \underline{\underline{15}}$$

~~b)~~

A simple basis

~~H_i~~ $H_i = \mathrm{diag}(0, 0, \cdots 1, 0, 0, \cdots)$

ith position.

and $\mathcal{H} = \left\{ \sum_{i=1}^{4} b_i H_i \;\middle|\; Re(b_i) = 0, \; \sum_i b_i = 0 \right\}$

b)

$H \in \mathcal{H}$, $H e_i = \sum_{j=1}^{4} b_j H_j e_i$

$H_j e_i = \delta_{ij} e_i$ (NO sum)

$= L_i(H_j) e_i$ where ~~Lif~~ $L_i \in \mathcal{H}'$ the dual of $\mathcal{H}$.

$$\therefore\ He_i = \sum_{j=1}^{4} b_j H_j e_i$$

$$= \sum_{j=1}^{4} b_j L_i(H_j)\, e_i = L_i\Big(\sum_{j=1}^{4} b_j H_j\Big) e_i$$

$$= L_i(H)\, e_i$$

For ~~#~~ each basis of cartan subalgebra.

~~He_j~~ $H_i e_j = \delta_{ij} e_j$ (No sum)

		Standard	Dynkin
$\therefore$ weight for	e_1 :	$(1\ 0\ 0\ 0)$	$(1\ 0\ 0)$
	e_2 :	$(0\ 1\ 0\ 0)$	$(-1\ 1\ 0)$
	e_3 :	$(0\ 0\ 1\ 0)$	$(0\ -1\ 1)$
	e_4 :	$(0\ 0\ 0\ 1)$	$(0\ 0\ -1)$

heighest weight.

~~The~~ ~~For~~ in the ~~4~~ $\underset{\sim}{4}$ representation

For $\underset{\sim}{\bar{4}}$ representation. $\because$ $\underset{\sim}{\bar{4}}$ is the dual of $\underset{\sim}{4}$

$\therefore$ the weights of ~~the~~ the dual ~~is~~ are the negative of the weights of the original representation.

		Standard	Dynkin
$\therefore$ weight for	$e_1 \wedge e_2 \wedge e_3$	$(1\ 1\ 1\ 0)$	$(0\ 0\ 1)$
	$e_1 \wedge e_2 \wedge e_4$	$(1\ 1\ 0\ 1)$	$(0\ 1\ -1)$
	$e_1 \wedge e_3 \wedge e_4$	$(1\ 0\ 1\ 1)$	$(1\ -1\ 0)$
	$e_2 \wedge e_3 \wedge e_4$	$(0\ 1\ 1\ 1)$	$(-1\ 0\ 0)$

(e.g. $(1\ 1\ 1\ 0) = (0\ 0\ 0\ -1) = -(0\ 0\ 0\ 1)$).

negative of lowest in $\underset{\sim}{4}$ is highest in $\underset{\sim}{\bar{4}}$

c) Young tableaux :

$$\underset{\sim}{4} \quad \ydiagram{1} \qquad\qquad \underset{\sim}{\bar{4}} \quad \ydiagram{1,1,1}$$

$$\underset{\sim}{4} \otimes \underset{\sim}{4} = \ydiagram{1} \otimes \ydiagram{1}$$

$$= \begin{ytableau} \ \\ a \end{ytableau} \oplus \begin{ytableau} \ & a \end{ytableau}$$

$$= \ydiagram{1,1} \oplus \ydiagram{2}$$

$$= \underset{\sim}{6} \oplus \underset{\sim}{10}$$

$$\underset{\sim}{4} \otimes \underset{\sim}{\bar{4}} = \ydiagram{1,1,1} \otimes \ydiagram{1}$$

$$= \begin{ytableau} \ \\ \ \\ \ \\ a \end{ytableau} \oplus \begin{ytableau} \ & a \\ \ \\ \ \end{ytableau}$$

$$= \underset{\sim}{1} \oplus \underset{\sim}{15}$$

In $\underset{\sim}{4} \otimes \underset{\sim}{4}$,

$\underset{\sim}{6} \to \ydiagram{1,1} \to \psi_{[\mu\nu]}$ ← antisymmetrized

$\underset{\sim}{10} \to \ydiagram{2} \to \psi_{(\mu\nu)}$ ← symmetrized.

d) $SU(3) \subset SU(4)$

$$U = \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix}. \qquad U_3 \in SU(3)$$

Doing this by index decomposition.

~~$\underset{\sim}{4} \to \psi_N = (\psi^\alpha, \psi^i) = (\psi^\alpha, \psi^4)$~~

~~$N = 1,2,3,4 \quad \alpha = 1,2,3, \quad i = 4$~~

$\underset{\sim}{4} \to \psi_N = (\psi_\alpha, \psi_i) = (\psi_\alpha, \psi_4)$

$N = 1,2,3,4 \qquad \alpha = 1,2,3, \quad i = 4.$

ψ_α transform under $\underset{\sim}{3}$ under $SU(3)$
ψ_α " " $\underset{\sim}{1}$ " 1
ψ_4 " " $\underset{\sim}{1}$ " $SU(3)$
ψ_4 " " $\underset{\sim}{1}$ " $SU(1)$

$$\therefore \quad \underset{\sim}{4} = (\underbrace{\underset{\sim}{3}}_{SU(3)}, \underbrace{\underset{\sim}{1}}_{1}) \oplus (\underbrace{\underset{\sim}{1}}_{SU(1)}, \underbrace{\underset{\sim}{1}}_{1})$$

$\bar{\underset{\sim}{4}} \to \psi^N = (\psi^\alpha, \psi^4)$

Similarly $\quad \bar{\underset{\sim}{4}} = (\bar{\underset{\sim}{3}}, 1) \oplus (\underset{\sim}{1}, \underset{\sim}{1})$

$4 \otimes 4 = \underset{\sim}{6} + \underset{\sim}{10}$

$\underset{\sim}{6} \to$ ~~$\psi_{\alpha T}$~~

$$\psi_{[\mu\nu]} = (\psi_{[\alpha\beta]}, \psi_{\alpha i}, \psi_{[ij]})$$

$$= \left(\begin{array}{c|c} \psi_{[\alpha\beta]} & \begin{matrix}\psi_{14}\\ \psi_{24}\\ \psi_{34}\end{matrix} \\ \hline -\psi_{14}\ -\psi_{24}\ -\psi_{34} & 0 \end{array}\right) = (\psi_{[\alpha\beta]}, \psi_{\alpha 4}, \cancel{\phi})$$

$\psi_{[\alpha\beta]}$ transforms under $\underset{\sim}{\bar{3}}$ of $SU(3)$

(It has $\begin{smallmatrix}\square\\\square\end{smallmatrix}$). and under $\underset{\sim}{1}$ of 1

$\psi_{\alpha 4}$ $(\alpha = 1,2,3)$ transforms as $\underset{\sim}{3}$ of $SU(3)$ and $\underset{\sim}{1}$ of 1

~~ψ ϕ transforms as $\underset{\sim}{1}$ of $SU(3)$ and 1~~

$\underset{\sim}{10} \to$ $\psi_{(\mu\nu)} = (\psi_{(\alpha\beta)}, \psi_{\alpha i}, \psi_{(ij)})$.

$$= \left(\begin{array}{c|c} \psi_{(\alpha\beta)} & \begin{matrix}\psi_{14}\\ \psi_{24}\\ \psi_{34}\end{matrix} \\ \hline \psi_{14}\ \psi_{24}\ \psi_{34} & \psi_{44} \end{array}\right) = (\psi_{(\alpha\beta)}, \psi_{\alpha 4}, \psi_{44}).$$

$\psi_{(\alpha\beta)}$ transforms as $\square\square = \underset{\sim}{6}$ of $SU(3)$ and $\underset{\sim}{1}$ of 1

$\psi_{\alpha 4}$ transforms under $\underset{\sim}{3}$ of $SU(3)$ and $\underset{\sim}{1}$ of 1

ψ_{44} transforms under $\underset{\sim}{1}$ of $SU(3)$ and 1

$$\therefore \underset{\sim}{4} \otimes \underset{\sim}{4} = \underset{\sim}{10} + \underset{\sim}{6} = (\bar{\underset{\sim}{3}}, \underset{\sim}{1}) \oplus (\underset{\sim}{3}, \underset{\sim}{1}) \oplus (\underset{\sim}{6}, \underset{\sim}{1}) \oplus (\underset{\sim}{3}, \underset{\sim}{1}) \oplus (\underset{\sim}{1}, \underset{\sim}{1})$$

$\downarrow$ $SU(4)$ $\quad\quad$ $\downarrow$ $SU(4)$ $\quad\quad$ $\downarrow$ $(SU(3), 1)$

$$= \underset{\sim}{6} \oplus \bar{\underset{\sim}{3}} \oplus \underset{\sim}{3} \oplus \underset{\sim}{3} \oplus \underset{\sim}{1} \quad (SU(3))$$

(4)

a) $SO(7) = \{ O \in Aut(\mathbb{R}^n) \mid O^T O = \mathbb{1}$
$\det(O) = 1 \}$

$O = \mathbb{1} + T + \cdots$

$\downarrow$

generators

$O^T O = \mathbb{1} \Rightarrow \mathbb{1} = (\mathbb{1} + T^T)(\mathbb{1} + T)$
$= \mathbb{1} + T + T^T + \cdots$

$\Rightarrow \quad T = -T^T \quad \Rightarrow$ antisymmetric.

$\det(O) = 1 \Rightarrow \det(\mathbb{1} + T + \cdots) = 1$

$\therefore \quad 1 + \mathrm{tr}(T) + \cdots = 1 \quad \Rightarrow \mathrm{tr}(T) = 0$
$\Rightarrow$ traceless.

$\therefore$ Lie algebra

$so(7) = \mathcal{L}(SO(7)) = \{ T \in End(\mathbb{R}^n) \mid T = -T^T \}$

Basis of $so(7)$ is

$(\sigma_{\mu\nu})^\rho{}_\sigma = \delta^\rho{}_\mu \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta^\rho{}_\nu$

$\rightarrow \quad [\sigma_{\alpha\beta}, \sigma_{\gamma\delta}] = \delta_{\alpha\delta} \sigma_{\beta\gamma} + \delta_{\alpha\gamma} \sigma_{\delta\beta} + \delta_{\beta\delta} \sigma_{\gamma\alpha} + \delta_{\beta\gamma} \sigma_{\alpha\delta}$

so if $\alpha \neq \gamma$ and $\alpha \neq \delta$ and $\beta \neq \gamma$ and $\beta \neq \delta$.

then $[\sigma_{\alpha\beta}, \sigma_{\gamma\delta}] = 0$

Hence the cartan-subalgebra ~~is~~ for $so(7)$ ~~is~~ (B_3) is.

$$\mathcal{H} = \{ \sigma_{12}, \sigma_{34}, \sigma_{56} \}$$

$$\therefore \quad rk(so(7)) = \dim(\mathcal{H}) = 3$$

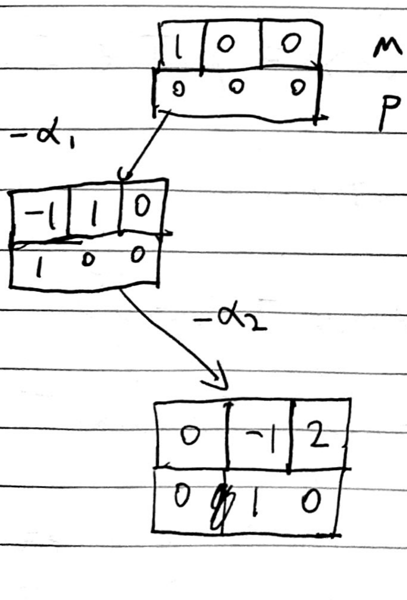
$$\dim(SO(7)) = \frac{7(7-1)}{2} = 21$$

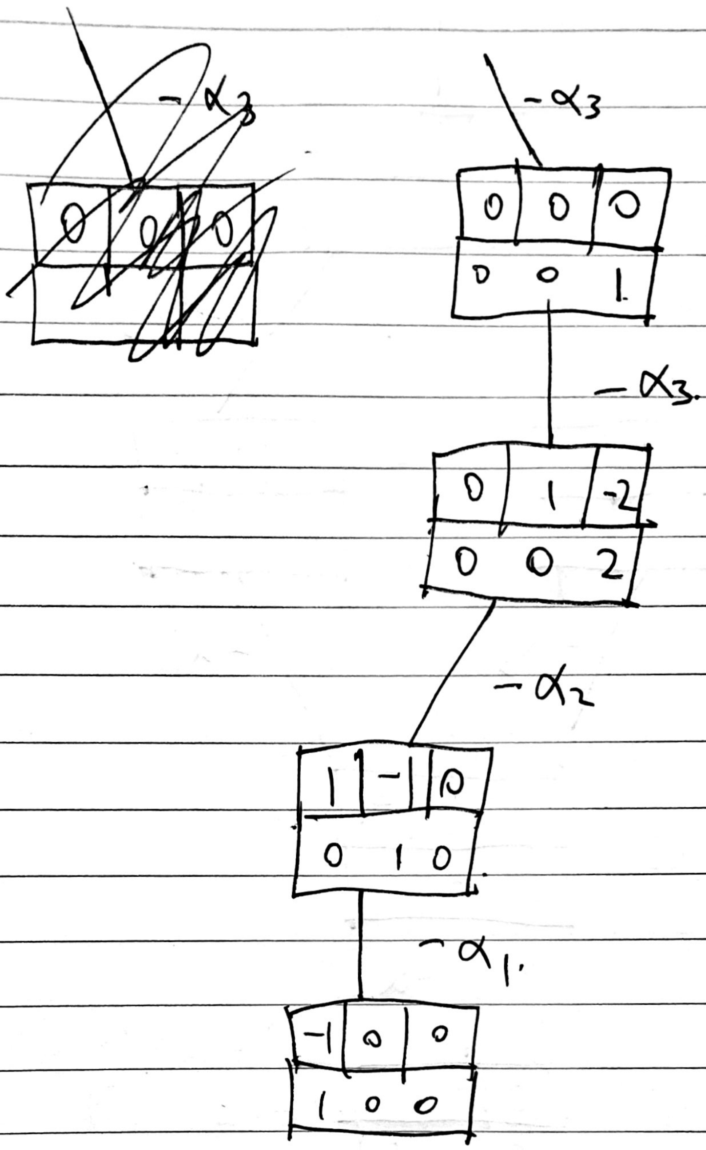
b)

$$A(B_3) = \begin{array}{cc} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} & \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \end{array}$$

$M = (1\ 0\ 0)$ $\qquad$ $P = (0\ 0\ 0)$

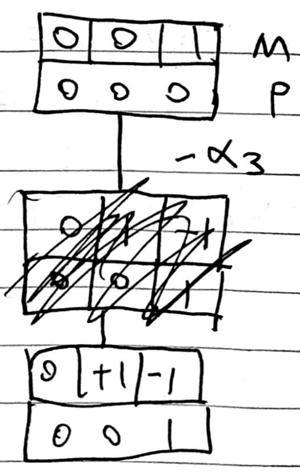
need $m + p > 0$ for a subtraction to be allowed.

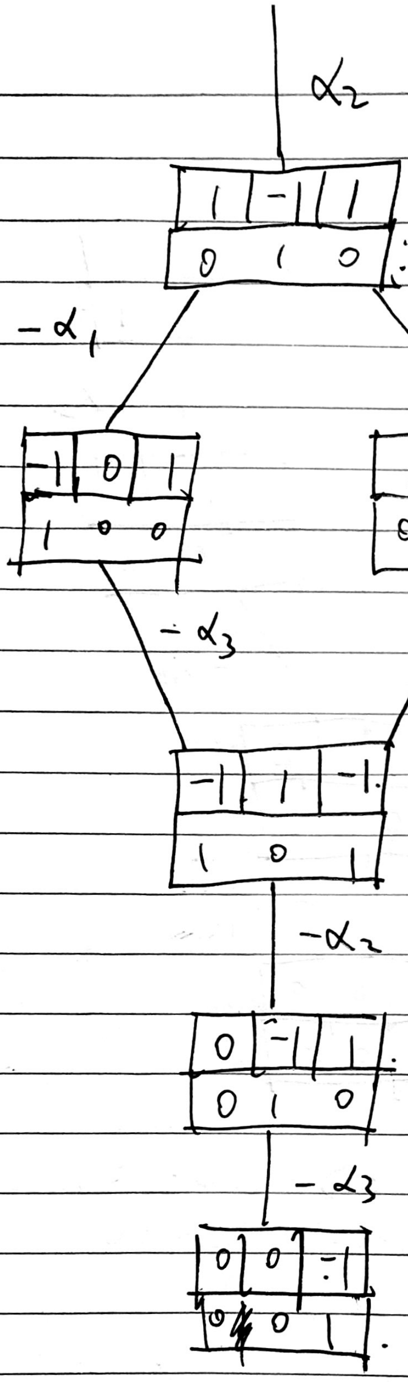




$\therefore$ (1 0 0) (-1 1 0) (0 -1 2) (0 0 0)
(0 1 -2) (1 -1 0) (-1 0 0)

c) For highest weight (0 0 1)





$\therefore$ $(0\ 0\ 1)$ $(0\ 1\ -1)$ $(1\ -1\ 1)$ $(-1\ 0\ 1)$

$(1\ 0\ -1)$ $(-1\ 1\ -1)$ $(0\ -1\ 1)$ $(0\ 0\ -1)$

d)

$$P(SU(4) \subset SO(7)) = P(A_3 \subset B_3)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Action of P

$$SO(7) \xrightarrow{P} SU(4).$$

$(0\ 0\ 1) \longrightarrow (0\ 0\ 1)$

$(0\ 1\ {-1}) \longrightarrow (1\ 0\ 0)$

$(1\ {-1}\ 1) \longrightarrow (-1\ 1\ 0)$

$(-1\ 0\ 1) \longrightarrow (0\ {-1}\ 1)$

$(1\ 0\ {-1}) \longrightarrow (0\ 1\ {-1})$

$(-1\ 1\ {-1}) \longrightarrow (1\ {-1}\ 0)$

$(0\ {-1}\ 1) \longrightarrow (-1\ 0\ 0)$

$(0\ 0\ {-1}) \longrightarrow (0\ 0\ {-1})$

$\therefore$ (001) of $SO(7)$ branches into reps of $SU(4)$ with weights.

$(1\ 0\ 0)$	$(0\ 0\ 1)$
$(-1\ 1\ 0)$	$(0\ 1\ {-1})$
$(0\ {-1}\ 1)$	$(1\ {-1}\ 0)$
$(0\ 0\ {-1})$	$(-1\ 0\ 0)$
$\underset{\sim}{4}$	$\underset{\sim}{\bar{4}}$

$$\Rightarrow \quad (0\,0\,1)_{SO(7)} \to \underset{\sim}{4} + \underset{\sim}{\bar{4}} \quad (SU(4))$$