General Relativity II

problem set 4

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$\boxed{1}$

Schwarzschild Geodesics ($\theta = \frac{\pi}{2}$, $\dot\theta = 0$)

$$\left(1 - \frac{2M}{r}\right)\dot t = E$$

$$-\left(1 - \frac{2M}{r}\right)\dot t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot r^2 + r^2\dot\phi^2 = -\sigma$$

$$r^2\dot\phi = h$$

E, h constants, and $(g_{\mu\nu}\dot x^\mu \dot x^\nu = -\sigma)$

where
$\sigma = 1 \rightarrow$ time like
$\sigma = 0 \rightarrow$ null
$\sigma = -1 \rightarrow$ spacelike

$\therefore$ result is $\frac{1}{2}\dot r^2 + V(r) = \frac{1}{2}E^2$

and $V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\sigma + \frac{h^2}{r^2}\right)$

We deal with radial geodesics $\Rightarrow h = 0$

$\therefore$ $\dot r^2 + \left(1 - \frac{2M}{r}\right)\sigma = E^2$

$\therefore$ $\underline{\dot r^2 - \frac{2M}{r}\sigma = E^2 - \sigma}$

$\rightarrow$ space-like radial geodesic $\sigma = -1$: $\dot r^2 + \frac{2M}{r} = E^2 + 1$

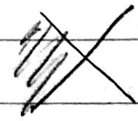
further require $E = 0 \Rightarrow \dot r^2 + \frac{2M}{r} = 1$

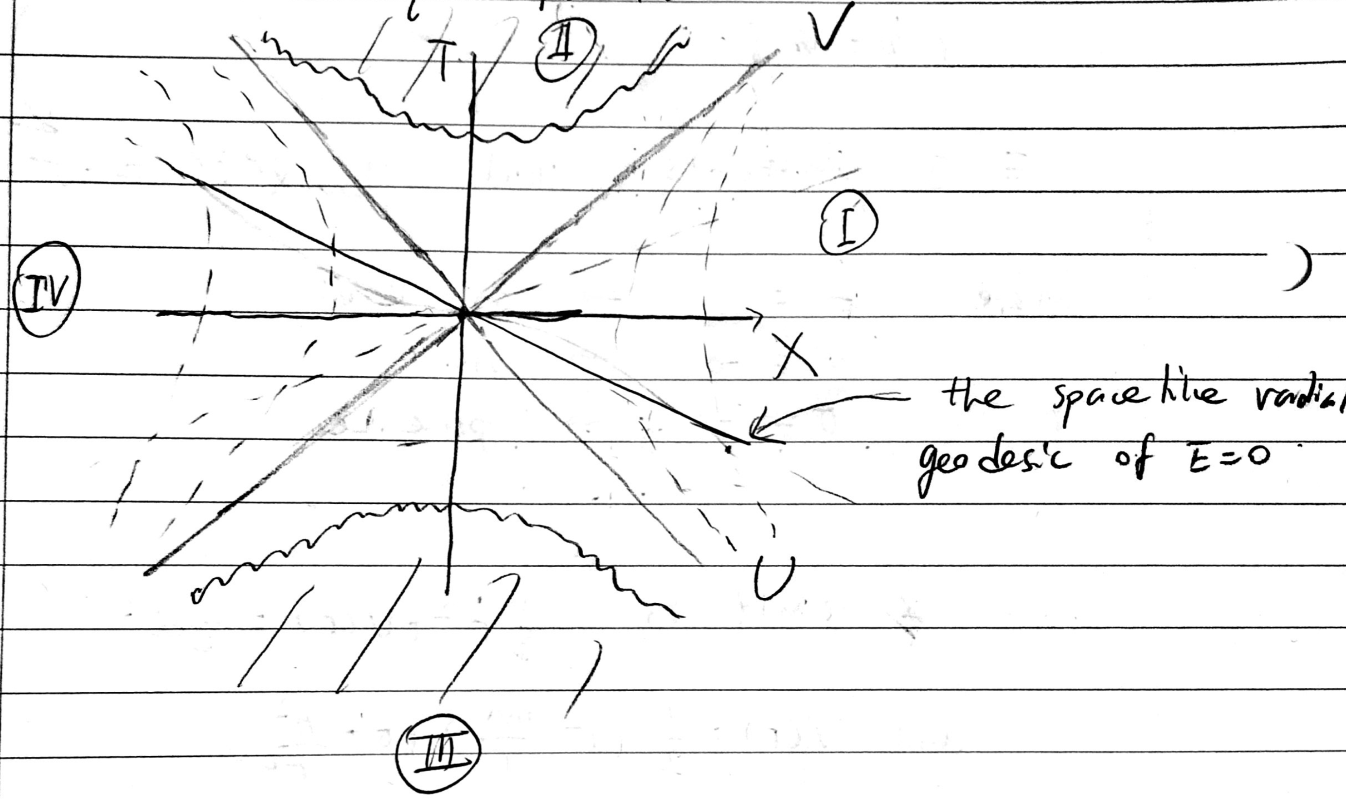
$\Rightarrow 1 - \frac{2M}{r} \geq 0 \Rightarrow$ $\underline{r \geq 2M}$

and White Hole).

this means that this geodesic does not go into the Black Hole of both universes in Kruskal ~~space~~ space time.

$\therefore$ to ensure the geodesic is always spacelike.





$E=0$ means $r \geq 2M \Rightarrow$ geodesic passes through the ~~war~~ worm hole to the other universe III

→ time-like geodesic $\sigma = 1$

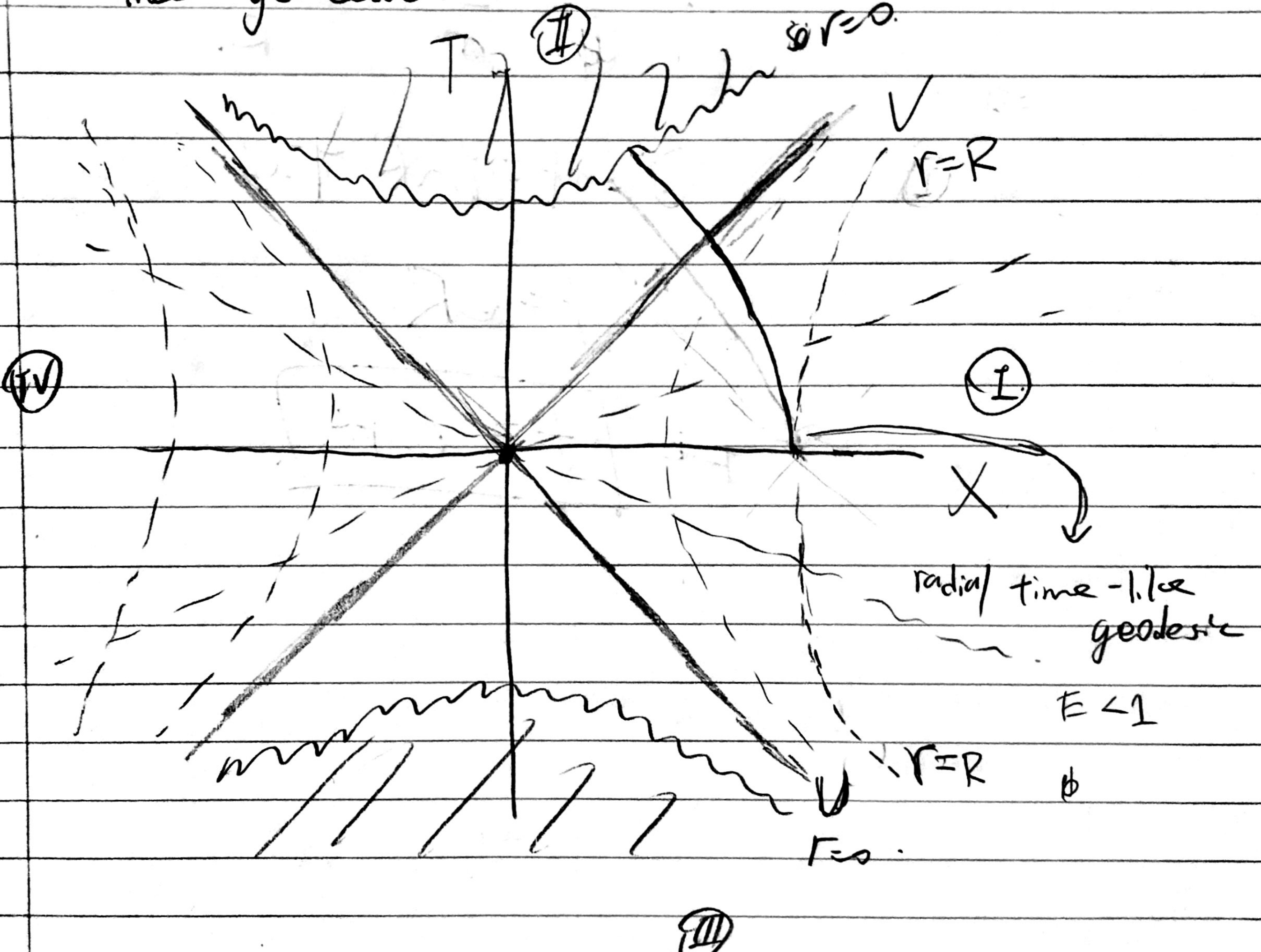
$\therefore \underline{\dot{r}^2 - \frac{2M}{r} = E^2 - 1}$ $\qquad \because 0 < E < 1 \quad \therefore$ ~~$E^2 <$~~ $\therefore 0 < E^2 < 1 \Rightarrow \underline{1 - E^2 > 0}$

$\therefore 0 \leq \dot{r}^2 = \frac{2M}{r} - (1 - E^2)$

$\therefore \frac{2M}{r} \geq (1 - E^2) \qquad r \leq \frac{2M}{1 - E^2} \equiv R$

$\therefore r$ is bounded by some maximum value $R \equiv \frac{2M}{1-E^2}$,

→ this means that the particle will fall into the Black Hole ~~and~~ and eventually reach the singularity following this time like geodesic.



In above diagram, particle dropped from rest at $r=R$ and falls into the Black Hole.

→ Now for time-like geodesic with $E=0$, the maximum ~~radius~~ r value for radial geodesics is $R_{(E=0)} = \frac{2M}{1-0} = \underline{\underline{2M}}$

and thus particle geodesics start and end both <u>within the Black Hole</u> i.e. region II

$\boxed{2}$ The 2 basis Killing vectors of Kerr metric ~~is~~ are $K^\mu = \partial_t = (1,0,0,0)$ and

$L^\mu = \partial_\phi = (0,0,0,1)$

Write a linear combination, $M^\mu = K^\mu + \cancel{\Omega} L^\mu \omega$
$= (1,0,0,\cancel{\Omega}\omega)$
so that M^μ is also a Killing vector.

$\rightarrow$ tangent vector U of the observer's world line is the observer's 4-velocity (up to a multiplicative scalar function)

$\because$ U is also a Killing vector $\therefore U^\mu \propto M^\mu$ (up to some scalar function)

$\therefore$ we see that $U^r = U^\theta = 0$

So $\frac{dr}{d\tau} = 0$, $\frac{d\theta}{d\tau} = 0$ (τ = proper time of observer).

$\therefore$ r, θ coordinates remain unchanged for this observer.

$\because$ In Kerr metric, $g_{\mu\nu}$ only depends on r, θ

$\Rightarrow$ $g_{\mu\nu}$, metric, is unchanged for this ~~ob~~ observer.

Write $U = U^t(1,0,0,\cancel{\Omega}\omega)$ so ~~...~~ $U^t = U^\phi/\omega$

$\therefore \omega \cancel{\Omega} = \frac{U^\phi}{U^t} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{d\phi}{dt}$ ~~has~~ is the coordinate angular velocity.

∵ U is 4-velocity we require $U^\mu U_\mu = \sigma$

and ∵ U is timelike, $\sigma = -1$ ∴ $U^\mu U_\mu = -1$

$$\Rightarrow g_{\mu\nu} U^\mu U^\nu = -1 \Rightarrow (g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2)(U^t)^2 = -1$$

For real U^t to exist we need

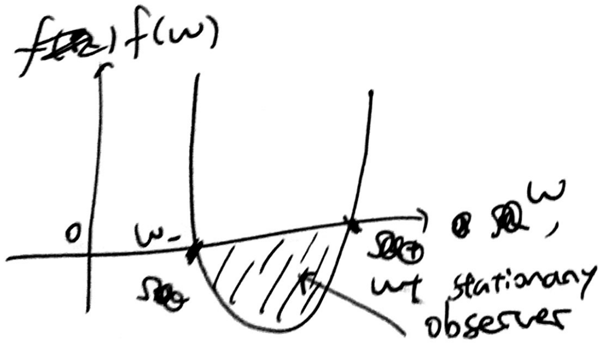
$$\underline{f(\omega) = g_{\phi\phi}\omega^2 + 2g_{t\phi}\omega + g_{tt} < 0}$$

$$\because \quad g_{\phi\phi} = \frac{1}{\Sigma}\sin^2\theta\,(\Delta\Sigma + 2Mr(r^2+a^2))$$

$$= \sin^2\theta\left(r^2+a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right) \geq 0$$

$(\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2Mr)$

∴ this is an upward parabola for ω



the <u>discriminant</u> is

$D = 4(g_{t\phi}^2 - g_{\phi\phi}g_{tt})$, we have

$$\frac{1}{4}D = g_{t\phi}^2 - g_{\phi\phi}g_{tt} = \left(-\frac{2Mar}{\Sigma}\sin^2\theta\right)^2 - \left(-\left(1-\frac{2Mr}{\Sigma}\right)\right)\left[\frac{1}{\Sigma}\sin^2\theta \times (\Delta\Sigma + 2Mr(r^2+a^2))\right]$$

$$= \frac{4M^2a^2r^2\sin^4\theta}{\Sigma^2} + \left(1-\frac{2Mr}{\Sigma}\right)(\sin^2\theta)\left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\Sigma}\right)$$

$$= \cancel{\frac{4m^2a^2r^2\sin^4\theta}{\Sigma^2}} + \sin^2\theta\left(r^2+a^2+\frac{2mra^2\sin^2\theta}{\Sigma}\right)$$

$$- \frac{2mr}{\Sigma}\sin^2\theta\,(r^2+a^2) - \cancel{\frac{4m^2r^2a^2\sin^4\theta}{\Sigma}}$$

$$= \left[\left(1-\frac{2mr}{\Sigma}\right)(r^2+a^2) + \frac{2mra^2\sin^2\theta}{\Sigma}\right]\sin^2\theta$$

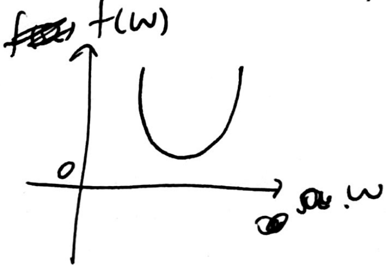
$$= \left[r^2+a^2 - \left(\frac{2mr}{\Sigma}\right)\left(r^2+a^2-a^2\sin^2\theta\right)\right]\sin^2\theta$$

$$= \left[r^2+a^2 - 2mr\,\cancel{\frac{1}{\Sigma}}\,\underbrace{(r^2+a^2\cos^2\theta)}_{\cancel{\Sigma}}\right]\sin^2\theta$$

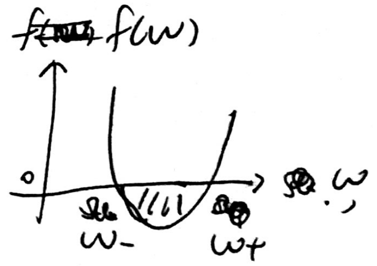
$$= \Delta\sin^2\theta.$$

For $r_- < r < r_+$, $\Delta < 0$, so $D < 0$

~~f(w)~~ $f(\omega)$ ~~hars~~ has no zero $\Rightarrow$ observer doesn't exist $\Rightarrow$



For $r > r_+$, we have



and $$\omega_\pm = \frac{1}{2g_{\phi\phi}}\left(-2g_{t\phi} \pm \sqrt{4g_{t\phi}^2 - 4g_{tt}g_{\phi\phi}}\right)$$

$$= -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

Define $\quad \Omega \equiv -\dfrac{g_{t\phi}}{g_{\phi\phi}}$, we have

$$\omega_\pm = \Omega \pm \sqrt{\Omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

inside ~~the~~ the ergosphere $\quad g_{tt} > 0$

$\because g_{\phi\phi} > 0$ always $\quad \therefore$ inside ergosphere we have $\dfrac{g_{tt}}{g_{\phi\phi}} > 0$

$$\Rightarrow \quad \omega_\pm = \Omega \pm \sqrt{\Omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} > 0$$

$\Rightarrow$ we must have $\dfrac{d\phi}{dt} > 0$

$\Rightarrow$ observer co-rotates with the ~~black~~ Black Hole.

$\square$.

$\boxed{3}$ $K^a \nabla_a K_b = \varkappa K_b$

$\because$ K a Killing vector and K a HSO on S.

$\therefore$ $\underbrace{\nabla_a K_b = \nabla_{[a} K_{b]}}_{\text{Killing}}$ and $\underbrace{K_{[a} \nabla_b K_{c]} = 0}_{\text{HSO}}$.

$\Rightarrow$ $K_a \nabla_b K_c = - K_b \nabla_c K_a - K_c \nabla_a K_b$

$\therefore$ ~~$K_c \nabla_a \nabla$~~

$K_c (\nabla_a K_b)(\nabla^a K^b) = -(K_a \nabla_b K_c + K_b \nabla_c K_a) \nabla^a K^b$

$= -(\underbrace{K_a \nabla^a K^b}_{\varkappa K^b})(\nabla_b K_c) - (K_b \underbrace{\nabla^a K^b}_{=-\nabla^b K^a})(\underbrace{\nabla_c K_a}_{=-\nabla_a K_c})$

$= -\varkappa \underbrace{K^b \nabla_b K_c}_{=\varkappa K_c} + K_b \nabla^b K^a (-\nabla_a K_c)$

$= -\varkappa^2 K_c - \varkappa K^a \nabla_a K_c$

$= -2\varkappa^2 K_c$ at S.

$\Rightarrow$ $\varkappa^2 = -\frac{1}{2} (\nabla_a K_b)(\nabla^a K^b) \Big|_S$

$\boxed{4}$ A stationary observer sees an ~~unchang~~ unchanging spacetime geometry, so the metric is constant for this observer.

$\Rightarrow$ if metric $g_{\mu\nu}$ depends on coordinate q

then $U^q = \frac{dq}{d\tau} = 0$ for this observer.

(U^q is the q-component of his/her 4-velocity)

$\Rightarrow$ $U^p \neq 0$ iff ~~coordin~~ metric is independent of coordinate p $\Rightarrow$ ∂_p is a Killing vector.

$\Rightarrow$ 4-velocity U must be proportional, up to a scalar function, to a Killing vector because the lineer ~~com~~ combination of all ∂_p such that $U^p \neq 0$ is a killing vector.

$$\Rightarrow \quad K^a = \alpha(x)\, U^a$$

$\because U^a$ is timelike $\therefore$ K^a must also be time-like.

~~Acceleration $A^a = U^b \nabla_b U^a = \alpha K^b \nabla_b (\alpha K^a)$~~

~~$= \alpha^2 K^b \nabla_b K^a + \alpha K^b K^a \nabla_b \alpha$~~

~~$= \alpha^2 \kappa K^a + \alpha (K^b \nabla_b \alpha) K^a$~~

$\because\ U^aU_a = -1 \quad \therefore\ K^aK_a = \alpha^2 U^aU_a = -\alpha^2$

$\Rightarrow \quad \alpha = \sqrt{-K^aK_a} \quad \Rightarrow \quad U^a = \dfrac{K^a}{\sqrt{-K^bK_b}} = \dfrac{K^a}{\sqrt{-K^2}}$

$$A^a = U^c\nabla_c U^a = \frac{K^c}{\sqrt{-K^2}}\nabla_c\left(\frac{K^a}{\sqrt{-K^2}}\right)$$

$$= \frac{K^c}{\sqrt{-K^2}}K^aK^b\nabla_c K_b + \frac{K^c}{-K^2}\nabla_c K^a \quad (*)$$

$\Downarrow -\nabla_c K_a = \nabla_a K_c.$

$$= \underbrace{\frac{K^a}{\sqrt{-K^2}}\underbrace{K^cK^b}_{\text{symmetric}}\underbrace{\nabla_c K_b}_{\text{anti symmetric}}}_{\Rightarrow\ =0} + \frac{-K^c(\nabla^a K_c)}{-K^2}$$

$$= \frac{1}{2}\frac{1}{-K^2}\nabla^a(-K^cK_c) = \frac{1}{2}\frac{1}{-K^2}\nabla^a(-K^2)$$

$$= \frac{1}{2}\nabla^a\log(-K^2) = \frac{1}{2}\nabla^a\log(\alpha^2)$$

$$= \frac{1}{2}\nabla^a(2\log\alpha) = \nabla^a\log\alpha$$

$$= \underline{g^{ab}\nabla_b\log\alpha} \qquad \square$$

$$A = \sqrt{A^aA_a} = \sqrt{\nabla^a\log\alpha\,\nabla_a\log\alpha} = \sqrt{\frac{1}{\alpha}\nabla^a\alpha\,\frac{1}{\alpha}\nabla_a\alpha}.$$

$$= \frac{1}{\alpha}\sqrt{\nabla^a\alpha\,\nabla_a\alpha}$$

$$\Rightarrow \quad \alpha A = \sqrt{\nabla^a\alpha\,\nabla_a\alpha}$$

At infinity $K^a K_a \to -1$ $\because U^a U_a = 1$

$\Rightarrow \quad \alpha \to 1$

$\therefore$ At ∞, $A_\infty = \frac{1}{\alpha}\int \sqrt{\nabla^a \alpha \nabla_a \alpha} \to \int \sqrt{\nabla^a \alpha \nabla_a \alpha}$

$= \alpha A$

D.

Consider

$$3 K^{[a} \nabla^b K^{c]} K_{[a} \nabla_b K_{c]}$$

$$= \frac{1}{3}\left(K^a \nabla^b K^c + K^b \nabla^c K^a + K^c \nabla^a K^b \right)$$

$$\times \left(K_a \nabla_b K_c + K_b \nabla_c K_a + K_c \nabla_a K_b \right)$$

$$= \frac{1}{3}\left[3 K^a K_a (\nabla^b K^c)(\nabla_b K_c) - 6 (K^a \nabla^b K^c)(K_b \nabla_a K_c) \right]$$

$$= K^a K_a (\nabla^b K^c)(\nabla_b K_c) - 2 (K^a \nabla^b K^c)(K_b \nabla_a K_c)$$

As $r \to r_H$, $K^{[a} \nabla^b K^{c]} = 0$ $\therefore$ LHS $= 0$.

And $\kappa^2 = -\frac{1}{2}(\nabla^b K^c)(\nabla_b K_c)$

$$\Rightarrow \quad 0 = -2\kappa^2 - 2\left[\frac{(K^a \nabla^b K^c)(K_b \nabla_a K_c)}{K^d K_d} \right]$$

$$\Rightarrow \quad \kappa^2 = \lim_{r \to r_H} \frac{-(K^b \nabla_b K^c)(K^a \nabla_a K_c)}{K^d K_d}$$

From previous calculations we know that

$$A^a = \frac{K^c \nabla_c K^a}{-K^d K_d}$$

(*)

$$\therefore \quad A^2 = A^c A_c = \lim_{r \to r_H} \frac{(K^a \nabla_a K^c)(K^b \nabla_b K_c)}{(K^d K_d)^2}$$

$$= \varkappa^2 \lim_{r \to r_H} \left(\frac{+1}{-K^d K_d} \right).$$

$$= \varkappa^2 \frac{1}{(\sqrt{-K^d K_d})^2} = \varkappa^2 \frac{1}{\alpha^2}$$

$$\Rightarrow \quad \varkappa^2 = \lim_{r \to r_H} \alpha^2 A^2$$

$$\Rightarrow \quad \varkappa = \lim_{r \to r_H} \alpha A$$

□

$\boxed{5}$

4+1 dimensional ~~met~~ Minkowski metric.

$$ds^2 = -dT^2 + dx^2 + dy^2 + dz^2 + dw^2$$

use

$$x = a\cosh(t/a)\sin\chi\sin\theta\cos\phi$$

$$y = a\cosh(t/a)\sin\chi\sin\theta\sin\phi$$

$$z = a\cosh(t/a)\sin\chi\cos\theta$$

$$w = a\cosh(t/a)\cos\chi$$

$$T = a\sinh(t/a)$$

$$g_{tt} = \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial t} g_{\mu\nu} = \left(\frac{\partial x}{\partial t}\right)^2 g_{xx} + \left(\frac{\partial y}{\partial t}\right)^2 g_{yy} + \left(\frac{\partial z}{\partial t}\right)^2 g_{zz}$$

$$+ \left(\frac{\partial w}{\partial t}\right)^2 g_{ww} + \left(\frac{\partial T}{\partial t}\right)^2 g_{TT}$$

$$= \sinh^2(t/a)\sin^2\chi\sin^2\theta\cos^2\phi$$

$$+ \sinh^2(t/a)\sin^2\chi\sin^2\theta\underbrace{\sin^2\phi}_{1}$$

$$+ \sinh^2(t/a)\sin^2\chi\underbrace{\cos^2\theta}_{1}$$

$$+ \sinh^2(t/a)\underbrace{\cos^2\chi}_{1}$$

$$+ \cosh^2(t/a)(-1)$$

$$= -(\cosh^2(t/a) - \sinh^2(t/a)) = \underline{\underline{-1}}$$

$$g_{\chi\chi} = \left(\frac{\partial x}{\partial \chi}\right)^2 g_{xx} + \left(\frac{\partial y}{\partial \chi}\right)^2 g_{yy} + \left(\frac{\partial z}{\partial \chi}\right)^2 g_{zz} + \left(\frac{\partial w}{\partial \chi}\right)^2 g_{ww} + 0.$$

$$= a^2\cosh^2(t/a)\cos^2\chi\sin^2\theta\cos^2\phi$$

$$+ a^2\cosh^2(t/a)\cos^2\chi\sin^2\theta\sin^2\phi$$

$$+ a^2\cosh^2(t/a)\cos^2\chi\cos^2\theta$$

$$+ a^2\cosh^2(t/a)\sin^2\chi = \underline{\underline{a^2\cosh^2(t/a)}}$$

$$g_{\theta\theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 g_{xx} + \left(\frac{\partial y}{\partial \theta}\right)^2 g_{yy} + \left(\frac{\partial z}{\partial \theta}\right)^2 g_{zz}$$

$$= a^2 \cosh^2(t/a) \sin^2\chi \,(\cos^2\theta \cos^2\phi + \cos^2\theta \sin^2\phi + \sin^2\theta)$$

$$= \underline{\underline{a^2 \cosh^2(t/a) \sin^2\chi}}$$

D.

$$g_{\phi\phi} = \left(\frac{\partial x}{\partial \phi}\right)^2 g_{xx} + \left(\frac{\partial y}{\partial \phi}\right)^2 g_{yy} = a^2 \cosh^2(t/a) \sin^2\chi \sin^2\theta \times (\sin^2\phi + \cos^2\phi)$$

$$= \underline{\underline{a^2 \cosh^2(t/a) \sin^2\chi \sin^2\theta}}$$

D.

$$\therefore ds^2 = -dt^2 + a^2 \cosh^2(t/a) (d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2))$$

$$\equiv \underline{\underline{-dt^2 + a^2 \cosh^2(t/a) \, d\Omega_3^2}}$$

D

(it can be explicitly computed that

$$g_{\mu\nu} = \frac{\partial x^a}{\partial x'^\mu} \frac{\partial x^b}{\partial x'^\nu} g_{ab} = 0 \text{ for } \mu \neq \nu \text{)}$$

define ~~cos~~ $\cos\lambda \cosh(t/a) = 1$

$$\Rightarrow \cancel{-\sin\lambda \, d\lambda + \frac{1}{a}\sinh(t/a)\, dt = 0}$$

$$\Rightarrow \cancel{a^2 \sin^2\lambda \, d\lambda^2 = \sinh^2(t/a)\, dt^2}$$

$$\Rightarrow \cancel{d^2\lambda = \frac{\sinh^2(t/a)}{a^2\sin^2\lambda} dt^2}$$

$\therefore\ a \sin\lambda \cosh(t/a)\, d\lambda = \sinh(t/a) \cos\lambda\, dt.$

$\Rightarrow\ a^2 \sin^2\lambda \cosh^2(t/a)\, d\lambda = \sinh^2(t/a) \cos^2\lambda\, dt.$

$$\therefore\ d^2\lambda = \frac{\sinh^2(t/a)\cos^2\lambda}{a^2\cosh^2(t/a)\sin^2\lambda}\, dt.$$

$$\cos^2\lambda = \frac{1}{\cosh^2(t/a)} \qquad \sin^2\lambda = 1 - \cos^2\lambda = 1 - \frac{1}{\cosh^2(t/a)}$$

$$= \frac{\cosh^2(t/a) - 1}{\cosh^2(t/a)} = \frac{\sinh^2(t/a)}{\cosh^2(t/a)}$$

$$\therefore\ d^2\lambda = \frac{\sinh^2(\cancel{t/a})}{a^2\cosh^2(t/a)\cosh^2(\cancel{t/a})\left(\frac{\cancel{\sinh^2(t/a)}}{\cancel{\cosh^2(t/a)}}\right)}\, dt^2$$

$$= \frac{1}{a^2\cosh^2(t/a)}\, dt^2 = \frac{\cos^2\lambda}{a^2}\, dt^2.$$

$$\therefore\ ds^2 = -dt^2 + a^2\cosh^2(t/a)\, d\Omega_3^2$$

$$= \frac{a^2}{\cos^2\lambda}\left(-dt^2 + d\Omega_3^2\right)$$

Penrose diagram :

$$ds^2 = \frac{a^2}{\cos^2\lambda}\left(-d\lambda^2 + d\chi^2 + \sin^2\chi \, d\Omega_2^2\right)$$

where $d\Omega_2 = d\theta^2 + \sin^2\theta \, d\phi^2$

$\sin^2\chi$, $\cos^2\lambda$ both have period π

$\therefore$ we can suppress $d\Omega_2^2$, and choose

$\lambda, \chi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ to get

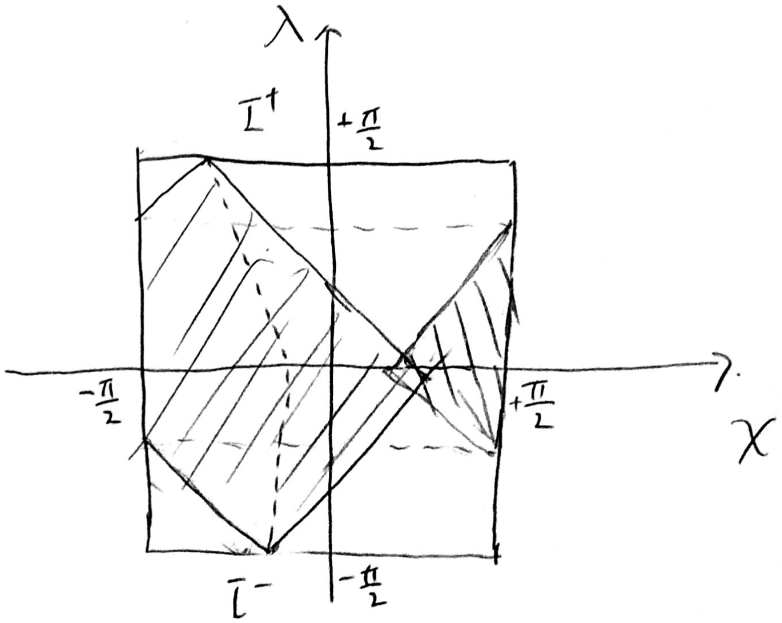
$$ds^2 = \frac{a^2}{\cos^2\lambda}\left(-d\lambda^2 + d\chi^2\right).$$

this blows up at $\lambda = \pm\frac{\pi}{2}$

$\because$ $\cos\lambda \cosh(t/a) = 1$ $\therefore$ At $\lambda = \pm\frac{\pi}{2}$

$\cosh(t/a) \to \infty$ this corresponds to

$t \to \pm\infty$ $\Rightarrow$ timelike infinity $\mathcal{I}^{\pm}$



penose diagram is a square,

the shaded areas ~~are~~ represents spacetime points that are causally connected (from the geodesic (time-like) represented by the dashed ~~line~~ line)

The ~~un~~ unshaded area areas the spacetime that this observer cannot observe.

$\Rightarrow$ cannot observe entire spacetime.