Conformal Field Theory Problem Sheet 4

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$\boxed{1}$ consider a primary $|h\rangle$ and the following descendent at level ~~three~~ three

$$|\chi\rangle = (L_{-3} + \eta L_{-1}L_{-2} + \xi L_{-1}L_{-1}L_{-1})|h\rangle.$$

we want to choose η, ξ, h such that $|\chi\rangle$ is null, it is suffice to require $L_1|\chi\rangle = L_2|\chi\rangle = 0$ and for $n>2$ $L_n|\chi\rangle = 0$ then follows from the Virasoro algebra.

we have, using $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$ and $L_1|h\rangle = 0$ $L_2|h\rangle = 0$ $L_3|h\rangle = 0$, $L_0|h\rangle = h|h\rangle$

$$0 \overset{!}{=} L_1|\chi\rangle = L_1(L_{-3} + \eta L_{-1}L_{-2} + \xi L_{-1}L_{-1}L_{-1})|h\rangle$$

$$= [L_1, L_{-3}]|h\rangle + \eta L_{-1}L_1L_{-2}|h\rangle + \eta[L_1, L_{-1}]L_{-2}|h\rangle + \xi[L_1, L_{-1}]L_{-1}L_{-1}|h\rangle + \xi L_{-1}L_1L_{-1}L_{-1}|h\rangle$$

$$= 4L_{-2}|h\rangle + \eta L_{-1}L_1L_{-2}|h\rangle + 2\eta L_0L_{-2}|h\rangle + 2\xi L_0L_{-1}L_{-1}|h\rangle + \xi L_{-1}L_1L_{-1}L_{-1}|h\rangle$$

$$= 4L_{-2}|h\rangle + \eta L_{-1}[L_1, L_{-2}]|h\rangle + 2\eta[L_0, L_{-2}]|h\rangle + 2\eta L_{-2}L_0|h\rangle + 2\xi[L_0, L_{-1}]L_{-1}|h\rangle + 2\xi L_{-1}L_0L_{-1}|h\rangle + \xi L_{-1}[L_1, L_{-1}]L_{-1}|h\rangle + \xi L_{-1}L_{-1}L_1L_{-1}|h\rangle$$

$$= 4L_{-2}|h\rangle + 3\eta L_{-1}L_{-1}|h\rangle + 4\eta L_{-2}|h\rangle + 2\eta h L_{-2}|h\rangle + 2\xi L_{-1}L_{-1}|h\rangle + 2\xi L_{-1}[L_0, L_{-1}]|h\rangle + 2h\xi L_{-1}L_{-1}|h\rangle + 2\xi L_{-1}L_0L_{-1}|h\rangle + \xi L_{-1}L_{-1}[L_1, L_{-1}]|h\rangle$$

$$= 4L_{-2}|h\rangle + \eta(2(h+2)L_{-2} + 3L_{-1}L_{-1})|h\rangle + 2\xi L_{-1}L_{-1}|h\rangle + 2\xi L_{-1}L_{-1}|h\rangle + 2h\xi L_{-1}L_{-1}|h\rangle + 2h\xi L_{-1}L_{-1}|h\rangle + 2\xi L_{-1}L_{-1}|h\rangle + 2h\xi L_{-1}L_{-1}|h\rangle$$

$$= 4L_{-2}|h\rangle + \eta(2(h+2)L_{-2} + 3L_{-1}L_{-1})|h\rangle + 6\xi(h+1)L_{-1}L_{-1}|h\rangle$$

$$= (4 + 2\eta(h+2))L_{-2}|h\rangle + (3\eta + 6\xi(h+1))L_{-1}L_{-1}|h\rangle.$$

so we need $4 + 2\eta(h+2) = 0$, $3\eta + 6\xi(h+1)6 = 0$.

so $\underline{\eta = -\frac{2}{h+2}}$, $\underline{\xi = \frac{-3\eta}{6(h+1)} = \frac{-1}{(h+1)(h+2)}}$

and

$$0 \overset{!}{=} L_{+2}|\chi\rangle = L_{+2}(L_{-3} + \eta L_{-1}L_{-2} + \xi L_{-1}L_{-1}L_{-1})|h\rangle$$

$L_2|h\rangle = 0$

$$= [L_2, L_{-3}]|h\rangle + \eta[L_2, L_{-1}L_{-2}]|h\rangle + \xi[L_2, L_{-1}L_{-1}L_{-1}]|h\rangle$$

$$= 5L_{-1}|h\rangle + \eta[L_2, L_{-1}]L_{-2}|h\rangle + \eta L_{-1}[L_2, L_{-2}]|h\rangle$$

$$+ \xi[L_2, L_{-1}]L_{-1}L_{-1}|h\rangle + \xi L_{-1}[L_2, L_{-1}L_{-1}]|h\rangle$$

$$= 5L_{-1}|h\rangle + \eta 3L_1L_{-2}|h\rangle + \eta L_{-1}\left(4L_0 + \frac{c}{12}\cdot 2\cdot 3\right)|h\rangle.$$

$$+3\xi L_1L_{-1}L_{-1}|h\rangle + \xi L_{-1}[L_2, L_{-1}]L_{-1}|h\rangle + \cancel{\xi L_{-1}L_{-1}\underbrace{[L_2, L_{-1}]}_{=3L_1}|h\rangle}$$

$$= 5L_{-1}|h\rangle + \eta 3[L_1, L_{-2}]|h\rangle + 4\eta h L_{-1}|h\rangle + \frac{c}{2}\eta L_{-1}|h\rangle.$$
$$+3\xi(2L_0)L_{-1}|h\rangle + 3\xi L_{-1}L_1L_{-1}|h\rangle + 3\xi L_{-1}L_1L_{-1}|h\rangle$$

$$= 5L_{-1}|h\rangle + 9\eta L_{-1}|h\rangle + 4\eta h L_{-1}|h\rangle + \frac{c}{2}\eta L_{-1}|h\rangle.$$
$$+6\xi h L_{-1}|h\rangle + 6h\xi L_{-1}|h\rangle + 6h\xi L_{-1}|h\rangle$$

$[L_0, L_{-1}] = L_{-1}$. $+6\xi L_{-1}|h\rangle$

$$= 5L_{-1}|h\rangle + \eta\left(9 + 4h + \frac{c}{2}\right)L_{-1}|h\rangle$$
$$+ 6\xi(3h+1)L_{-1}|h\rangle$$

$$= \left(5 + \eta\left(9 + 4h + \frac{c}{2}\right) + 6\xi(3h+1)\right)L_{-1}|h\rangle$$

so $5 + \eta(9 + 4h + \frac{c}{2}) + 6\xi(3h+1) = 0 \cdot (*)$ for $L_2|\chi\rangle = 0$

and if ~~L_1, L_2~~ $L_1|\chi\rangle = 0$, $L_2|\chi\rangle = 0$ then

$[L_2, L_1]|\chi\rangle = 0 = L_3|\chi\rangle$

$\to [L_3, L_2]|\chi\rangle = 0 = L_4|\chi\rangle$ etc ...

so $|\chi\rangle$ is null.

plug in η, ξ into $(*)$ gives

$$0 = 5 + \underbrace{\left(-\frac{2}{h+2}\right)}_{\eta}\left(9 + 4h + \frac{c}{2}\right) + 6\underbrace{\left(-\frac{1}{(h+1)(h+2)}\right)}_{\xi}(3h+1)$$

$$\Rightarrow 5 - \frac{18 + 8h + c}{2+h} - \frac{18h + 6}{(h+1)(h+2)} = 0$$

$\therefore$ we need

$$c = \left(5 - \frac{18h+6}{(h+1)(h+2)}\right)(h+2) - 18 - 8h.$$

where c is the ~~contant~~ central charge.

null ~~state~~ descendent

$$|\chi\rangle = \left(L_{-3} + \frac{2}{h+2} L_{-1}L_{-2} - \frac{1}{(h+1)(h+2)} L_{-1}L_{-1}L_{-1}\right)|h\rangle.$$

[2]

The differential equation, for $\phi(z)$ a primary field with a level two descendant, is

$$\left(\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^{2}\right)\langle\phi(z)\,\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)\rangle = 0$$

OR equivalently

$$\left(\sum_{i=1}^{n}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right) - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right)\langle\phi(z)\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)\rangle$$

a) for 2 point functions $n=1$:

so $$\left(\frac{h_1}{(z-z_1)^2} + \frac{1}{z-z_1}\frac{\partial}{\partial z_1} - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right)\langle\phi(z)\phi_{h_1}(z_1)\rangle$$

$= 0$ ①

the two point function $\langle\phi(z)\,\phi_{h_1}(z_1)\rangle$ can only have the form

$$\langle\phi(z)\phi_{h_1}(z_1)\rangle = \frac{C_{h,h_1}}{(z-z_1)^{h_1+h}}$$ if $h=h_1$ and 0 if $h\neq h_1$.

so sub this into the LHS of ① gives

if $h=h_1$:

$$\left(\frac{h_1}{(z-z_1)^2} + \frac{1}{z-z_1}\frac{\partial}{\partial z_1} - \frac{3}{\cancel{2h}\,2(2h+1)}\frac{\partial^2}{\partial z^2}\right)\frac{C_{h,h_1}}{(z-z_1)^{h+h_1}}$$

$$\left(h_1 + (-h-h_1)(-1) - \frac{3}{2} \cdot \frac{1(-1)^2}{2h+1}(h+h_1)(h+h_1+1) \right) \frac{C_{h,h_1}}{(z-z_1)^{h+h_1}}$$

$$= \left(2h_1 + h - \frac{3}{2}\frac{(h+h_1)(h+h_1+1)}{2h+1}\right) \frac{C_{h,h_1}}{(z-z_1)^{h+h_1}}$$

if $h=h_1$

$$= \left(3h - \frac{3}{2}\frac{(2h)(\cancel{2h+1})}{(\cancel{2h+1})}\right) \frac{C_{h,h_1}}{(z-z_1)^{2h}}$$

$$= (3h-3h) \frac{C_{h,h_1}}{(z-z_1)^{2h}} = 0 = RHS.$$

if $h \neq h_1$, $LHS = 0 = RHS$ trivially.

so ① is automatically satisfied ~~if~~ for two point functions.

b) a three point function has ~~$\langle\phi(z)\phi_{h_1}\phi_{h_2}$~~ (in CFT) the form

$$\langle \phi(z)\phi_{h_1}(z_1)\phi_{h_2}(z_2)\rangle = \frac{C_{h h_1 h_2}}{(z-z_1)^{h+h_1-h_2}(z_1-z_2)^{h_1+h_2-h}(z-z_2)^{h+h_2-h_1}}$$

~~so~~ and obeys the equation. with $n=2$

$$\left(\frac{h_1}{(z-z_1)^2} + \frac{h_2}{(z-z_2)^2} + \frac{1}{z-z_1}\frac{\partial}{\partial z_1} + \frac{1}{z-z_2}\frac{\partial}{\partial z_2} - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right)$$

$$\cdot \langle \phi(z)\phi_{h_1}(z_1)\phi_{h_2}(z_2)\rangle = 0$$

Sub in the form of 3 point function we get

$$0 = \frac{1}{2(1+2h)} C_{hh_1h_2} \overbrace{\left(h^2 + h_1 - 3h_1^2 + h_2 + 6h_1h_2 - 3h_2^2 + h(-1+2h_1+2h_2)\right)}^{\equiv D}$$

$$\times (z-z_1)^{-2-h-h_1+h_2} \times (z-z_2)^{-2-h+h_1-h_2} \times (z_1-z_2)^{2+h-h_1-h_2}$$

(I used Mathematica, too tedious .....)

if $C_{hh_1h_2} \neq 0$, we need $D=0$, rewrite:

$$D = 2(2h+1)(h+2h_2-h_1) - 3(h-h_1+h_2)(h-h_1+h_2+1)$$

$\Rightarrow$ for $C_{hh_1h_2} \neq 0$ we need

$$2(2h+1)(h+2h_2-h_1) = 3(h-h_1+h_2)(h-h_1+h_2+1) \quad (*)$$

this is a quadratic equation for h_2

In a minimal model, central charge

$$c(m) = 1 - \frac{6}{m(m+1)} \qquad m = 2, 3, 4 \ldots$$

and weight can be

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$$

h_1 is general so choose $h_1 = h_{r,s}(m)$

h is the weight of $\phi(z)$ which has level 2 null descendant, so

$$h = h_{1,2}(m) \text{ or } h_{2,1}(m)$$

when $h = h_{1,2}(m)$, $h_1 = h_{r,s}(m)$, use Mathematica to solve (*) gives

$$h_2 = \frac{(-1+m+r+mr-ms)(1+m+r+mr-ms)}{4m(m+1)}$$

$$= \frac{((m+1)r - m(s-1))^2 - 1}{4m(m+1)}$$

$$= \underline{\underline{h_{r,s-1}}}$$

or

$$h_2 = \frac{(-1+m+r+mr-ms)(1-m+r+mr-ms)}{4m(m+1)}$$

$$= \frac{((m+1)r - m(s+1))^2 - 1}{4m(m+1)}$$

$$= \underline{\underline{h_{r,s+1}}}$$

Similarly we can solve for h_2 when $h = h_{2,1}(m)$

So overall we get the selection rules:

$h = h_{1,2}(m)$ $h_1 = h_{r,s}(m) \rightarrow h_2 = h_{r,s-1}(m)$ or $h_{r,s+1}(m)$

$h = h_{2,1}(m)$ $h_1 = h_{r,s}(m) \rightarrow h_2 = h_{r-1,s}(m)$ or $h_{r+1,s}(m)$

$\boxed{3}$ a) consider the holomorphic part first.

$\langle t(z_1) \cdots t(z_4) \rangle$

conformal invariance fixes ~~t~~

$t(z_1)\, t(z_2) \sim \dfrac{1}{(z_{12})^{2h}}$ and $t(z_3)\, t(z_4) \sim \dfrac{1}{(z_{34})^{2h}}$

so $\langle t(z_1) \cdots t(z_4) \rangle \sim \dfrac{1}{(z_{12})^{2h} (z_{34})^{2h}}$

and similarly anti-holomorphic part

$\langle \bar{t}(\bar{z}_1) \cdots \bar{t}(\bar{z}_4) \rangle \sim \dfrac{1}{(\bar{z}_{12})^{2h} (\bar{z}_{34})^{2h}}$ $\quad (\because h = \bar{h} = \frac{1}{2})$

but ~~we~~ when we form 4 point functions the cross ratios

$\eta = \dfrac{z_{12} z_{34}}{z_{13} z_{24}}$ is also

conformally invariant $\left(\text{and } \bar{\eta} = \dfrac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}} \right)$.

so overall 4 point ~~for~~ function of t has to ~~ta~~ take the form

$$\langle t(z_1, \bar{z}_1) \cdots t(z_4, \bar{z}_4) \rangle = \frac{g(\eta, \bar{\eta})}{|z_{12}|^{4h} |z_{34}|^{4h}}.$$

$\left(|z_{12}|^{4h} = z_{12}^{2h} \bar{z}_{12}^{2h},\ g \text{ an arbitrary function of } \eta, \bar{\eta} \right)$.

b) Focus on the ~~holo~~ holomorphic ~~shape~~ part only we have 4 point function

$$\langle \epsilon(z_1) \cdots \epsilon(z_4) \rangle$$

ϵ has a level 2 null descendant so.

use $\left(\sum_{i=1}^{n}\left(\frac{h_i}{(z-z_i)^2} - \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right) - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right)\langle \epsilon \cdots \epsilon \rangle = 0$

with $\underline{n=3}$ (4 point function). so.

$$\left(\sum_{i=2}^{4}\left(\frac{h}{(z-z_i)^2} - \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right) - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z_1^2}\right)\frac{g(\eta)}{(z_1-z_2)^{2h}(z_3-z_4)^{2h}}$$

$$= 0 \quad , \eta = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$$

this gives

$$g''(\eta) + \frac{-4\eta + 4\eta h - 8h + 2}{3\eta - 3\eta^2} g'(\eta) - \frac{2h(2h+1)}{3(1-\eta)^2} g(\eta) = 0$$

where for ϵ, $h = \frac{1}{2}$

c) equation

$$g''(\eta) + \frac{-2\eta - 2}{3\eta - 3\eta^2} g'(\eta) - \frac{2}{3(1-\eta)^2} g(\eta) = 0$$

Mathematica gives solution

$$\boxed{g_{\pm}(\eta) = \frac{1 \pm (\eta^2 - \eta)}{1-\eta}}$$

and ~~g same g~~

$$\boxed{g_{-} = \frac{(-\eta(-1+\eta))^{5/6}}{-1+\eta} \text{LegendreQ}\left[\tfrac{1}{3}, \tfrac{5}{3}, -1+2\eta\right]}$$

include its antiholomorphic part gives

$$g(\eta, \bar{\eta}) = K_{++}\, g_+(z) g_+(\bar{z}) + K_{+-}\, g_+(z) g_-(\bar{z}) + K_{-+}\, g_-(z) g_+(\bar{z}) + K_{--}\, g_-(z) g_-(\bar{z})$$

d) we require the $g(\eta, \bar{\eta})$ to be

- single valued: on the real section $\bar{\eta} = \eta^*$, g should be single valued as we move η around the complex plane around $\eta = 1$

So $g(\eta, \bar{\eta})$ should be invariant under

$$M_1(g(\eta, \bar{\eta})) = \lim_{t \to 1} g(1 + (\eta - 1)e^{2\pi i t}, 1 + (\bar{\eta} - 1)e^{-2\pi i t})$$

- Crossing relations:

identical operators $\Rightarrow$ should satisfy crossing relations when we exchange any 2 of them

$$1 \leftrightarrow 2 : \eta \leftrightarrow \frac{\eta}{\eta - 1} \rightarrow g(\eta, \bar{\eta}) = g\left(\frac{\eta}{\eta-1}, \frac{\bar{\eta}}{\bar{\eta}-1}\right)$$

$$1 \leftrightarrow 3 : \eta \leftrightarrow 1 - \eta \rightarrow \frac{g(\eta, \bar{\eta})}{|z_{12}|^{4h} |z_{34}|^{4h}} = \frac{g(1-\eta, 1-\bar{\eta})}{|z_{23}|^{4h} |z_{14}|^{4h}}$$

— consistent with

$$\epsilon(z_1,\bar{z}_1)\,\epsilon(z_2,\bar{z}_2) = \frac{1}{z_{12}^{1}\,\bar{z}_{12}^{1}} + \cdots$$

(power $1 = 2h = 2\times\frac{1}{2}$)

so the small $\eta,\bar{\eta}$ behaviour is

$$g(\eta,\bar{\eta}) = 1+\cdots$$

crossing ratio invariance, single valuedness under monodromy and small $\eta,\bar{\eta}$ behavior fixes $g(\eta,\bar{\eta})$ completely.

$\boxed{4}$

partial decomposition in terms of virasoro conformal blocks.

$$g(\eta,\bar{\eta}) = \sum_p C_p^2\,F(p|\eta)\,\bar{F}(p|\bar{\eta})$$

so explicit answer can ~~to~~ be ~~deco~~ decomposed as

$$g(\eta,\bar{\eta}) = F(0|\eta)\,\bar{F}(0|\bar{\eta}) + C_\epsilon^2\,F(\tfrac{1}{2}|\eta) \times \bar{F}(\tfrac{1}{2}|\bar{\eta})$$