Conformal Field Theory $\boxed{\text{Ziyan Li}}$

Problem sheet 3 TA: Diego Suarez.

$\boxed{1}$ a) $$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

Global conformal transformation is

$$z_i \to \frac{a z_i + b}{c z_i + d} \quad \text{where} \quad ad - bc = 1$$

so $\cancel{z}$ $$\eta \to \eta' = \frac{(z_1' - z_2')(z_3' - z_4')}{(z_1' - z_3')(z_2' - z_4')}$$

look at $z_i' - z_j'$:

$$z_i' - z_j' = \frac{a z_i + b}{c z_i + d} - \frac{a z_j + b}{c z_j + d} \cancel{= \frac{}{(c z_i + d)(c z_j + d)}}$$

$$= \frac{(a z_i + b)(c z_j + d) - (a z_j + b)(c z_i + d)}{(c z_i + d)(c z_j + d)}$$

$$= \frac{ac \cancel{z_i z_j} + ad z_i + bc z_j + \cancel{bd} - ac \cancel{z_i z_j} - ad z_j - bc z_i \cancel{-bd}}{c^2 z_i z_j + cd z_i + cd z_j + d^2}$$

$$= \frac{(ad - bc) z_i - (ad - bc) z_j}{\cancel{cz}(c z_i + d)(c z_j + d)}$$

$$= \frac{z_i - z_j}{(c z_i + d)(c z_j + d)}$$

$$\therefore \eta \to \eta' = \frac{\dfrac{z_1 - z_2}{\cancel{(c z_1 + d)}\cancel{(c z_2 + d)}} \dfrac{z_3 - z_4}{\cancel{(c z_3 + d)}\cancel{(c z_4 + d)}}}{\dfrac{z_1 - z_3}{\cancel{(c z_1 + d)}\cancel{(c z_3 + d)}} \dfrac{z_2 - z_4}{\cancel{(c z_2 + d)}\cancel{(c z_4 + d)}}} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

$= \eta \Rightarrow$ invariant.

b) Consider transformation $z \to z' = \dfrac{az+b}{cz+d}$

$(ad-bc) = 1$

that maps $(0, i, 2)$ to $(0, 1, \infty)$

$\Rightarrow$ ~~so~~ $\underline{0 \to 0} \Rightarrow 0 = \dfrac{0+b}{0+d} \Rightarrow \underline{b = 0}$

$$z \to z' = \frac{az}{cz+d} \Rightarrow ad - 0 = 1 \quad \therefore d = \frac{1}{a}.$$

$$\therefore \quad z \to z' = \frac{az}{cz+\frac{1}{a}}$$

$\Rightarrow$ $\underline{i \to 1} \Rightarrow 1 = \dfrac{ia}{ic+\frac{1}{a}} \Rightarrow ic + \dfrac{1}{a} = ia.$

$$\therefore \quad iac + 1 = ia^2 \cancel{\Rightarrow a(a+c)}$$

$$\to \quad a(a-c) = -i \Rightarrow \underline{a(c-a) = i}$$

$\Rightarrow$ $\underline{2 \to \infty} \Rightarrow \infty = \dfrac{2a}{2c+\frac{1}{a}} \Rightarrow 2c + \dfrac{1}{a} = 0.$

$$\therefore \quad c = -\frac{1}{2a}$$

$$\Rightarrow \quad a\left(-\frac{1}{2a} - a\right) = i \quad \therefore -\frac{1}{2} - a^2 = i$$

$$\therefore a^2 = -\frac{1}{2} - i$$

$$\Rightarrow \{a, b, c, d\} = \left\{ \sqrt{-\tfrac{1}{2} \mp i},\ 0,\ -\frac{1}{\sqrt{-2-4i}},\ \frac{1}{\sqrt{-\frac{1}{2} \mp i}} \right\}$$

and map $z \to z' = \dfrac{az+b}{cz+d}$

$\boxed{2}$ free scalar bosonic field $\varphi(z)$. $\quad O_\alpha = :e^{i\alpha\varphi(z)}:$

stress tensor $T(z) = -2\pi :\partial\varphi(z)\partial\varphi(z):$

$$\because \quad \varphi(z)\varphi(w) = -\frac{1}{4\pi}\log(z-w)$$

$$\therefore \quad \partial\varphi(z)\varphi(w) = \partial_z(\varphi(z)\varphi(w)) = -\frac{1}{4\pi}\frac{1}{z-w}.$$

$$T(z)O_\alpha(w) = T(z):e^{i\alpha\varphi(w)}:$$

$$= -2\pi :\partial\varphi(z)\partial\varphi(z): :e^{i\alpha\varphi(w)}:$$

$$= -2\pi\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}:\partial\varphi(z)\partial\varphi(z): \varphi^n(w)$$

$$\sim (-2\pi)\left(\frac{-1}{4\pi}\right)\left(-\frac{1}{4\pi}\right)\frac{1}{(z-w)^2}\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}n(n-1)\varphi(w)^{n-2}$$

$$+(-2\pi)\left(-\frac{1}{4\pi}\right)\frac{1}{z-w}\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}(2n)\varphi(w)^{n-1}\partial\varphi(z)$$

(where we had $2n$ single contractions and $n(n-1)$ double contractions)

$$\sim \frac{\alpha^2}{8\pi}\frac{1}{(z-w)^2}\underbrace{\sum_{n=2}^{\infty}\frac{(i\alpha)^{n-2}}{(n-2)!}\varphi(w)^{n-2}}_{:e^{i\alpha\varphi(w)}:}$$

$$+\frac{1}{z-w}(i\alpha)\partial\varphi(w)\underbrace{\sum_{n=1}^{\infty}\frac{(i\alpha)^{n-1}}{(n-1)!}\varphi(w)^{n-1}}_{:e^{i\alpha\varphi(w)}:}$$

(in 2nd term we replaced $\partial\varphi(z)$ by $\partial\varphi(w)$ because their different leads to a regular term)

$$\sim \frac{\alpha^2}{8\pi}\frac{:e^{i\alpha\varphi(w)}:}{(z-w)^2}+\frac{\partial_w :e^{i\alpha\varphi(w)}:}{z-w}$$

~~satis~~ satisfy the ~~for~~ of the OPE of T with a primary operator.

$:e^{i\alpha\varphi(w)}:$ ~~has weight $\frac{\alpha^2}{8\pi}$~~ is a primary operator. Compare with

$$T(z)\,O_\alpha(w) \sim \frac{h}{(z-w)^2}\,O_\alpha(w) + \frac{\partial_w O_\alpha(w)}{z-w} + \cdots$$

we find weight $\boxed{h = \frac{\alpha^2}{8\pi}}$

To get two point function of these operators we ~~have~~ use

$$:e^{A_1}::e^{A_2}: = :e^{A_1+A_2}:\, e^{\langle A_1 A_2\rangle}$$

$$\therefore \langle O_\alpha(z)\, O_\beta(w)\rangle = :e^{i\alpha\varphi(z)}::e^{i\beta\varphi(w)}:$$

$$= :e^{i(\alpha+\beta)\varphi(w)}:\, e^{-\alpha\beta \underbrace{\varphi_1(z)\varphi_2(w)}_{-\frac{1}{4\pi}\log(z-w)}}$$

$$= e^{\frac{\alpha\beta}{4\pi}\log|z-w|}\, :e^{i(\alpha+\beta)\varphi(w)}:$$

$$= |z-w|^{\frac{\alpha\beta}{4\pi}}\, :e^{i(\alpha+\beta)\varphi(w)}:$$

Conformal invariance requires that for non-zero $O_\alpha O_\beta$, the scale dimension of O_α, O_β should be the same $\Rightarrow \frac{\alpha^2}{8\pi} = \frac{\beta^2}{8\pi}$ $\therefore \alpha = \pm\beta$..

And for $|z-w|^{\alpha\beta}$ to not grow with ~~de~~ distance need $\beta = -\alpha$ so $:e^{i(\alpha+\beta)\varphi(w)}: = 1$ and

$$\langle O_\alpha(z)\, O_\beta(w)\rangle \sim \frac{1}{|z-w|^{\frac{\alpha^2}{4\pi}}} = \frac{1}{|z-w|^{2h}} \quad \text{where } h = \frac{\alpha^2}{8\pi}$$

consistent.

$\boxed{4}$ Schwartzian derivative

$$\{f(z), z\} = \frac{\partial_z f \, \partial_z^3 f - \frac{3}{2}(\partial_z^2 f)^2}{(\partial_z f)^2}$$

Global conformal transformations $f(z) = \frac{az+b}{cz+d}$

with ~~ad~~ $ad - bc = 1$.

$\therefore$ ~~$\partial_z f =$~~ $f(z) = \frac{az + \cancel{b}\frac{da}{c} + (b - \frac{da}{c})}{cz+d} = \frac{a}{c} + \frac{b - \frac{da}{c}}{cz+d}$

$$\partial_z f = -(b - \frac{da}{c})c \frac{1}{(cz+d)^2} = (ad-bc)\frac{1}{(cz+d)^2}$$

$$= \frac{1}{(cz+d)^2}$$

$$\partial_z^2 f = (-2)\frac{1}{(cz+d)^3} \cdot c = \frac{-2c}{(cz+d)^3}$$

$$\partial_z^3 f = \frac{6c^2}{(cz+d)^4}$$

$$\Rightarrow \{f(z), z\} = \frac{\frac{1}{(cz+d)^2} \cdot \frac{6c^2}{(cz+d)^4} - \frac{3}{2}\frac{4c^2}{(cz+d)^2}}{\frac{4c^2}{(cz+d)^2}}$$

$$= (\underbrace{6c^2 - \frac{3}{2} \times 4c^2}_{=0})\left(\quad \right)$$

$$= 0$$ vanishes.

$\boxed{5}$ All level two descendents : $L_{-2}|h\rangle$, $L_{-1}L_{-1}|h\rangle$

So, use $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$,

$\rightarrow \langle h|L_2L_{-2}|h\rangle = \langle h|[L_2, L_{-2}]|h\rangle + \langle h|L_{-2}\underbrace{L_2|h\rangle}_{=0}$

$= \langle h|4L_0|h\rangle + \frac{c}{12}\cdot 2\cdot 3\,\langle h|h\rangle$

$= \underline{\underline{4h + \frac{c}{2}}}$

$\rightarrow \langle h|L_1L_1L_{-1}L_{-1}|h\rangle \cancel{= \langle h|L_1[L_1, L_{-1}]|h\rangle}$

$= \langle h|L_1[L_1, L_{-1}]L_{-1}|h\rangle + \langle h|L_1L_{-1}L_1L_{-1}|h\rangle$

$= \langle h|L_1(2L_0)L_{-1}|h\rangle + \langle h|L_1L_{-1}[L_1, L_{-1}]|h\rangle$
$+ \langle h|L_1L_{-1}L_{-1}\underbrace{L_1|h\rangle}_{=0}$

$= 2\langle h|L_1L_0L_{-1}|h\rangle + 2\langle h|L_1L_{-1}\underbrace{L_0|h\rangle}_{h|h\rangle}$

$= 2\langle h|L_1[L_0, L_{-1}]|h\rangle + 2\langle h|L_1L_{-1}L_0|h\rangle + 2\langle h|L_1L_{-1}L_0|h\rangle$

$= 2\langle h|L_1L_{-1}|h\rangle + 4h\,\langle h|L_1L_{-1}|h\rangle$

$= (4h+2)\langle h|L_1L_{-1}|h\rangle = (4h+2)(\langle h|[L_1, L_{-1}]|h\rangle + \langle h|L_{-1}\underbrace{L_1|h\rangle}_{=0})$

$= (4h+2)\langle h|2L_0 + 0|h\rangle = \underline{\underline{(4h+2)(2h)}}$