Conformal Field Theory

Problem Sheet 2.

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1. Free Boson energy-momentum tensor (canonical)

$$T_c^{\mu\nu} = -\frac{1}{2}\eta^{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi + \partial^\mu\varphi\partial^\nu\varphi$$

its trace $T_c{}^\mu{}_\mu = -\frac{1}{2}D\partial_\rho\varphi\partial^\rho\varphi + \partial_\rho\varphi\partial^\rho\varphi$

$$= \frac{1}{2}(2-D)\partial_\rho\varphi\partial^\rho\varphi$$

consider $T^{\mu\nu} = T_c^{\mu\nu} + \tilde{T}^{\mu\nu}$ where we use

ansatz $\tilde{T}^{\mu\nu} = (\alpha\eta^{\mu\nu}\partial_\rho\partial^\rho + \beta\partial^\mu\partial^\nu)f(x)$ such that

$\partial_\nu T^{\mu\nu} = 0$, $T^\mu{}_\mu = 0$.

$$T_c{}^\mu{}_\mu = -\frac{1}{2}D\partial_\rho\varphi\partial^\rho\varphi + \partial_\rho\varphi\partial^\rho\varphi$$

$$= (1-\frac{D}{2})\partial_\rho\varphi\partial^\rho\varphi$$

so we need $\tilde{T}^\mu{}_\mu = (\frac{D}{2}-1)\partial_\rho\varphi\partial^\rho\varphi$.

$\therefore$ ~~$(\alpha D + \beta(\alpha D+\beta)\partial_\rho\varphi\partial$~~

$$(\alpha D+\beta)\partial_\rho\partial^\rho f = (\frac{D}{2}-1)\partial_\rho\varphi\partial^\rho\varphi \quad ①$$

$\because$ $\partial_\mu T_c^{\mu\nu} = 0$ by construction

so we also need $\partial_\nu\tilde{T}^{\mu\nu} = 0$, this gives.

$$(\alpha\partial^\nu\partial_\rho\partial^\rho + \beta\partial_\mu\partial^\mu\partial^\nu)f(x) = 0$$

$$\rightarrow \quad (\alpha+\beta)\partial^\nu\partial^2 f(x) = 0 \quad \Rightarrow \quad \alpha = -\beta. \quad ②$$

$\Rightarrow$ ① becomes $\alpha(D-1)\partial_\rho\partial^\rho f = (\frac{D}{2}-1)\partial_\rho\varphi\partial^\rho\varphi$.

Free boson ~~的~~ Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\rho\varphi\,\partial^\rho\varphi$.

$\therefore$ equation of motion $\Rightarrow$ $\partial_\rho\partial^\rho\varphi = 0$.

$\therefore$ $\partial^\rho\varphi\,\partial_\rho\varphi = \partial_\rho\varphi\,\partial^\rho\varphi + \varphi\underbrace{\partial_\rho\partial^\rho\varphi}_{0} = \partial_\rho(\varphi\,\partial^\rho\varphi)$

$\Rightarrow$ $\alpha(D-1)\,\partial_\rho\partial^\rho f = (\frac{D}{2}-1)\,\partial_\rho(\varphi\,\partial^\rho\varphi)$

we can have

$\alpha(D-1)\,\partial_\rho f = (\frac{D}{2}-1)\,\varphi\,\partial_\rho\varphi$.

choose $f = \frac{\varphi^2}{2} \Rightarrow \partial_\rho f = \varphi\,\partial_\rho\varphi$, then

$\alpha(D-1)\,\varphi\,\partial_\rho\varphi = (\frac{D}{2}-1)\,\varphi\,\partial_\rho\varphi$

$\therefore$ $\alpha = \frac{D-2}{2(D-1)}$

$\therefore$ improvement term

$$\tilde{T}^{\mu\nu} = \frac{D-2}{2(D-1)}\,(\eta^{\mu\nu}\partial_\rho\partial^\rho - \partial^\mu\partial^\nu)\,(\frac{\varphi^2}{2})$$

$\square$.

[2] $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2 e^\phi$

the equation of motion $\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi}$

$\Rightarrow$ ~~$\partial_\mu t$~~ $\partial_\nu\partial^\nu\phi = \frac{1}{2}m^2 e^\phi$ (e.o.m)

Canonical energy momentum tensor :

$$T_c^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi$$

$$= -\eta^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi + \frac{1}{2}m^2e^\phi\right) + \partial^\mu\phi\partial^\nu\phi$$

conservation :

$$\partial_\mu T_c^{\mu\nu} = -\frac{1}{2}\partial^\nu(\partial_\rho\phi\partial^\rho\phi) - \frac{1}{2}m^2\partial^\nu e^\phi + \partial_\mu(\partial^\mu\phi\partial^\nu\phi).$$

(e.o.m)

$$= -\frac{1}{2}\partial^\nu(\partial_\rho\phi\partial^\rho\phi) - \frac{1}{2}m^2e^\phi\partial^\nu\phi + \partial_\mu(\partial^\mu\phi\partial^\nu\phi)$$

$$= -(\partial_\nu\partial_\rho\phi)(\partial^\rho\phi) - (\partial_\mu\partial^\mu\phi)\partial^\nu\phi + (\partial_\mu\partial^\mu\phi)\partial^\nu\phi + \partial^\mu\phi(\partial_\mu\partial^\nu\phi).$$

$= 0 \Rightarrow T_c$ conserved.

use again the ansatz $\tilde{T}^{\mu\nu} = \alpha(\eta^{\mu\nu}\partial_\rho\partial^\rho - \partial^\mu\partial^\nu)f(x)$.

(as in [1], $\alpha = -\beta$ is required for conservation of.
$T^{\mu\nu} = T_c^{\mu\nu} + \tilde{T}^{\mu\nu}$. (so $\partial_\mu\tilde{T}^{\mu\nu} = 0$))

consider trace :

$$T_{c\,\mu}^{\mu} = \left(1 - \frac{D}{2}\right)\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}Dm^2e^\phi$$

$= (1-\frac{D}{2})\partial_\rho\phi\partial^\rho\phi - D\,\partial_\rho\partial^\rho\phi$ by (e.o.m).

so $\tilde{T}^\mu_{\ \mu} = -T_c{}^\mu{}_\mu = (\frac{D}{2}-1)\,\partial_\rho\phi\,\partial^\rho\phi + D\,\partial_\rho\partial^\rho\phi$

~~②~~ on the other hand ~~$\tilde{T}^\mu_{\ \mu} = \alpha(D-1)\partial_\rho\partial^\rho f$~~

$\tilde{T}^\mu_{\ \mu} = \alpha(D-1)\,\partial_\rho\partial^\rho f$ by ~~anst~~ ansatz.

So $\alpha(D-1)\,\partial_\rho\partial^\rho f = (\frac{D}{2}-1)\,\partial_\rho\phi\,\partial^\rho\phi + D\,\partial_\rho\partial^\rho\phi$

~~$\partial_\rho\phi\,\partial^\rho\phi = \partial_\rho(\phi\,\partial^\rho\phi) - \partial\phi\,\partial_\rho\partial^\rho\phi$~~
~~$= \partial_\rho(\phi\,\partial^\rho\phi) - \frac{1}{2}m^2\phi e^\phi$~~

$\because D=2 \quad \therefore \quad \alpha\,\partial_\rho\partial^\rho f =$ ~~$D\partial_\rho\partial^\rho\phi$~~ $2\,\partial_\rho\partial^\rho\phi$

we have $f=\phi \quad \alpha = 2$

So the added term $\tilde{T}^{\mu\nu} = 2(\eta^{\mu\nu}\partial_\rho\partial^\rho - \partial^\mu\partial^\nu)\phi$

$\rightarrow T^{\mu\nu} = -\frac{1}{2}\eta^{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\eta^{\mu\nu}e^\phi + \partial^\mu\phi\partial^\nu\phi$

$+ 2(\eta^{\mu\nu}\partial_\rho\partial^\rho - \partial^\mu\partial^\nu)\phi$.

3 under special conformal transformations (SCT)

$$x_i^\mu \to x_i^{\mu\prime} = \frac{x_i^\mu - b^\mu x_i^2}{1-2b\cdot x_i + b^2 x_i^2} = \frac{x_i^\mu - b^\mu x_i^2}{\gamma_i}$$

so $|x_i - x_j|^2 = (x_i - x_j)\cdot(x_i, x_j) = x_i^2 + x_j^2 - 2x_i\cdot x_j$

under ~~to~~ SCT,

$$|x_i' - x_j'|^2 = x_i'^2 + x_j'^2 - 2x_i'\cdot x_j'$$

$$= \frac{1}{\gamma_i^2}(x_i - bx_i^2)\cdot(x_j - bx_j^2) + \frac{1}{\gamma_j^2}(x_j - bx_j^2)\cdot(x_j - bx_j^2)$$

$$- \frac{2}{\gamma_i\gamma_j}(x_i - bx_i^2)\cdot(x_j - bx_j^2)$$

$$= \frac{1}{\gamma_i^2}\left(x_i^2 - 2(b\cdot x_i)x_i^2 + (b^2)(x_i^2)x_i^2\right)$$

$$+ \frac{1}{\gamma_j}\left(x_j^2 - 2(b\cdot x_j)x_j^2 + (b^2)(x_j^2)(x_j^2)\right)$$

$$- \frac{2}{\gamma_i\gamma_j}\left(x_i\cdot x_j - (b\cdot x_j)x_i^2 - (b\cdot x_i)x_j^2 + b^2x_i^2x_j^2\right)$$

$$= \frac{x_i^2}{\gamma_i^2}\underbrace{(1-2b\cdot x_i + b^2x_i^2)}_{\gamma_i} + \frac{x_j^2}{\gamma_j^2}\underbrace{(1-2b\cdot x_j + b^2x_j^2)}_{\gamma_j}.$$

$$- \frac{2}{\gamma_i\gamma_j}\left(x_i\cdot x_j - (b\cdot x_j)x_i^2 - (b\cdot x_i)x_j^2 + b^2x_i^2x_j^2\right)$$

$$= \frac{1}{\gamma_i\gamma_j}\Big(x_i^2(1-2b\cdot x_j + b^2x_j^2) + x_j^2(1-2b\cdot x_i + b^2x_i^2)$$

$$-2x_i\cdot x_j + 2(b\cdot x_j)x_i^2 + 2(b\cdot x_i)x_j^2 - 2b^2x_i^2x_j^2.$$

$$= \frac{1}{\gamma_i\gamma_j}\left(x_i^2 - 2x_i\cdot x_j + x_j^2\right) = \frac{|x_i - x_j|^2}{\gamma_i\gamma_j}$$

so $\rightarrow$ $|x_i' - x_j'| = \dfrac{|x_i - x_j|}{\gamma_i^{1/2}\,\gamma_j^{1/2}}$ ☐

[4] $I_{\mu\nu}(x) = \eta_{\mu\nu} - \dfrac{2x_\mu x_\nu}{x^2}$

$$\to I_{\mu\alpha}(x) I^{\alpha\beta}(x-y) I_{\beta\nu}(y)$$

$$= \left(\eta_{\mu\alpha} - \frac{2x_\mu x_\alpha}{x^2}\right)\left(\eta^{\alpha\beta} - \frac{2(x^\alpha-y^\alpha)(x^\beta-y^\beta)}{(x-y)^2}\right)\left(\eta_{\beta\nu} - \frac{2y_\beta y_\nu}{y^2}\right).$$

$$= \left(\delta^\beta_\mu - \frac{2x_\mu x^\beta}{x^2} - \frac{2(x_\mu-y_\mu)(x^\beta-y^\beta)}{(x-y)^2} + \frac{4(x_\mu x_\alpha)(x^\alpha-y^\alpha)(x^\beta-y^\beta)}{(x-y)^2 x^2}\right)$$

$$\times\left(\eta_{\beta\nu} - \frac{2y_\beta y_\nu}{y^2}\right).$$

$$= \left(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} - \frac{2(x_\mu-y_\mu)(x_\nu-y_\nu)}{(x-y)^2} + \frac{4(x_\mu x_\alpha)(x^\alpha-y^\alpha)(x_\nu-y_\nu)}{(x-y)^2 x^2}\right.$$

$$- \frac{2y_\mu y_\nu}{y^2} + \frac{4x_\mu y_\nu\, x\cdot y}{x^2y^2} + \frac{4y_\nu(x_\mu-y_\mu)(x\cdot y-y^2)}{(x-y)^2y^2}$$

$$\left. - \frac{8x_\mu(x^2-x\cdot y)(x\cdot y-y^2)y_\nu}{x^2y^2(x-y)^2}\right).$$

$$= \eta_{\mu\nu} - \frac{1}{x^2y^2(x-y)^2}\Big[2x_\mu x_\nu y^2(x-y)^2 + 2x^2y^2(x_\mu-y_\mu)(x_\nu-y_\nu)$$

$$- 4x_\mu(x_\nu-y_\nu)(x^2-x\cdot y)y^2 + 2y_\mu y_\nu x^2(x-y)^2$$

$$- 4x_\mu y_\nu(x\cdot y)(x-y)^2 - 4y_\nu(x_\mu-y_\mu)x^2(x\cdot y-y^2)$$

$$+ 8x_\mu y_\nu(x^2-x\cdot y)(x\cdot y-y^2)\Big]$$

$$= \eta_{\mu\nu} - \frac{2}{x^2y^2(x-y)^2}\Big[x_\mu x_\nu y^2x^2 - 2x_\mu x_\nu y^2(x\cdot y) + x_\mu x_\nu y^2y^2$$

$$+ x^2y^2x_\mu x_\nu - x^2y^2x_\mu y_\nu - x^2y^2y_\mu x_\nu + x^2y^2y_\mu y_\nu$$

$$- 2 x_\mu y_\nu x^2 y^2 + 2 x_\mu y_\nu x^2 y^2 + 2 x_\mu x_\nu (x\cdot y) y^2 - 2 x_\mu y_\nu (x\cdot y) y^2$$

$$+ y_\mu y_\nu x^2 x^2 - 2 y_\mu y_\nu x^2 (x\cdot y) + y_\mu y_\nu x^2 y^2 .$$

$$- 2 x_\mu y_\nu (x\cdot y) x^2 + 4 x_\mu y_\nu (x\cdot y)(x\cdot y) - 2 x_\mu y_\nu (x\cdot y) y^2$$

$$- 2 x_\mu y_\nu x^2 (x\cdot y) + 2 y_\mu y_\nu x^2 (x\cdot y) + 2 x_\mu y_\nu x^2 y^2$$

$$- 2 y_\mu y_\nu x^2 y^2$$

$$+ 4 x_\mu y_\nu x^2 (x\cdot y) - 4 x_\mu y_\nu x^2 y^2 - 4 x_\mu y_\nu (x\cdot y)(x\cdot y).$$

$$+ 4 x_\mu y_\nu (x\cdot y) y^2 \Big]$$

$$= \eta_{\mu\nu} - \frac{2}{x^2 y^2 (x-y)^2} \Big[x_\mu x_\nu y^2 y^2 + y_\mu y_\nu x^2 x^2 - x^2 y^2 (x_\mu y_\nu + y_\mu x_\nu) \Big].$$

$$\to \quad I_{\mu\nu}(x'-y') = \eta_{\mu\nu} - 2 \frac{(x'_\mu - y'_\mu)(x'_\nu - y'_\nu)}{(x'-y')^2}$$

$$= \eta_{\mu\nu} - 2 \frac{\left(\frac{x_\mu}{x^2} - \frac{y_\mu}{y^2}\right)\left(\frac{x_\nu}{x^2} - \frac{y_\nu}{y^2}\right)}{\left(\frac{x_\rho}{x^2} - \frac{y_\rho}{y^2}\right)\left(\frac{x^\rho}{x^2} - \frac{y^\rho}{y^2}\right)}$$

$$= \eta_{\mu\nu} - 2 \frac{1}{\frac{1}{x^2} + \frac{1}{y^2} - \frac{2 x\cdot y}{x^2 y^2}} \left(\frac{x_\mu x_\nu}{x^2 x^2} - \frac{(x_\mu y_\nu + y_\mu x_\nu)}{x^2 y^2} + \frac{y_\mu y_\nu}{y^2 y^2} \right)$$

~~$= \eta_{\mu\nu} - 2 \frac{1}{\ }$~~

$$= \eta_{\mu\nu} - \frac{2\,(x^2 x^2 y^2 y^2) \,/\, (x^2 x^2 y^2 y^2)}{x^2 y^2 y^2 + x^2 x^2 y^2 - 2(x \cdot y) x^2 y^2} \Big(x_\mu x_\nu y^2 y^2 + y_\mu y_\nu x^2 x^2 - (x_\mu y_\nu + y_\mu x_\nu) x^2 y^2 \Big).$$

$$= \eta_{\mu\nu} - \frac{2}{x^2 y^2 (x-y)^2} \Big(x_\mu x_\nu y^2 y^2 + y_\mu y_\nu x^2 x^2 - (x_\mu y_\nu + y_\mu x_\nu) x^2 y^2 \Big)$$

$$\Rightarrow \quad I_{\mu\alpha}(x)\, I^{\alpha\beta}(x-y)\, I_{\beta\nu}(y) = I_{\mu\nu}(x'-y')$$

$\boxtimes$

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$$\phi_1(x)\phi_2(0)|0\rangle = \frac{c}{|x|^k}(\phi_\Delta(0) + \alpha x^\mu \partial_\mu \phi_\Delta(0) + \beta x^\mu x^\nu \partial_\mu \partial_\nu \phi_\Delta(0) + \gamma x^2 \partial_\mu \partial^\mu \phi_\Delta(0) + \cdots)|0\rangle$$

use

$\partial_\mu \frac{1}{|x|^k} = -k \frac{x_\mu}{|x|^{k-2}}$,

$$[K_\mu, \phi_\Delta(x)] = 2ix_\mu \Delta \phi_\Delta(x) + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\nu)\phi_\Delta(x)$$

$$[K_\mu, \phi_\Delta(0)] = 0,$$

and we have fixed $k = \Delta_1 + \Delta_2 - \Delta$, $\alpha = \frac{\Delta_1 - \Delta_2 + \Delta}{2\Delta}$

we act K_μ from both LHS and RHS

Acting from L.H.S :

$$K_\mu \phi_1(x)\phi_2(0)|0\rangle = [K_\mu, \phi_1(x)]\phi_2(0)|0\rangle + \phi_1(x) K_\mu \phi_2(0)|0\rangle$$

$$= [K_\mu, \phi_1(x)]\phi_2(0)|0\rangle + \phi_1(x)\phi_2(0)\underbrace{K_\mu|0\rangle}_{=0}$$

$$= (2ix_\mu \Delta_1 + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu))\phi_1(x)\phi_2(0)|0\rangle$$

$$= (2ix_\mu \Delta_1 + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu))\left[\frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(\phi_\Delta(0) + \alpha x^\rho \partial_\rho \phi_\Delta(0) + \beta x^\rho x^\nu \partial_\rho \partial_\nu \phi_\Delta(0) + \gamma x^2 \partial_\rho \partial^\rho \phi_\Delta(0) + \cdots\right)\right]|0\rangle.$$

$$= ix_\mu(\Delta_1 - \Delta_2 + \Delta)\frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(\phi_\Delta(0) + \alpha x^\rho \partial_\rho \phi_\Delta(0) + \beta x_\rho x_\nu \partial^\rho \partial^\nu \phi_\Delta(0) + \gamma x^2 \partial_\rho \partial^\rho \phi_\Delta(0)\right).$$

$$+ \frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(\cdots\right)$$

where the $(\cdots)$ includes the following :

$\rightarrow$ $i(2x_\mu x^\nu \partial_\nu - x^2\partial_\mu)(\alpha x^\rho \partial_\rho \phi_\Delta(0))$

$= 2i\alpha x_\mu x^\nu \partial_\nu (x^\rho \partial_\rho \phi_\Delta(0)) - i\alpha x^2 \partial_\mu (x^\rho \partial_\rho \phi_\Delta(0))$.

$= 2i\alpha x_\mu x^\rho \partial_\rho \phi_\Delta(0) + 2i\alpha x_\mu x^\nu x^\rho \partial_\nu \partial_\rho \phi_\Delta(0)$

$- i\alpha x^2 \partial_\mu \phi_\Delta(0) - i\alpha x^2 x^\rho \partial_\mu \partial_\rho \phi_\Delta(0)$

$\rightarrow$ $i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)(\beta x^\rho x^a \partial_\rho \partial_a \phi_\Delta(0))$

$= 2i\beta x_\mu x^\nu \partial_\nu (x^\rho x^a \partial_\rho \partial_a \phi_\Delta(0))$

$- \beta i x^2 \partial_\mu (x^\rho x^a \partial_\rho \partial_a \phi_\Delta(0))$

$= 2i\beta x_\mu x^\nu x^a \partial^\rho \partial_\rho \partial_a \phi_\Delta(0)$ + other $\partial\partial\partial$ terms.

$\rightarrow$ some other $\partial\partial\partial$ terms.

$\therefore K_\mu \phi_1(x)\phi_2(0)|0\rangle$

~~$= i x_\mu (\Delta_1 - \Delta_2 + \Delta) \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}} \phi_\Delta(0)$~~

$= i \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}} \Big((\Delta_1 - \Delta_2 + \Delta) x_\mu \phi_\Delta(0)$

$+ \alpha(\Delta_1 - \Delta_2 + \Delta) x_\mu x^\rho \partial_\rho \phi_\Delta(0)$

$+ 2\alpha x_\mu x^\rho \partial_\rho \phi_\Delta(0) - \alpha x^2 \partial_\mu \phi_\Delta(0)$.

$+ \mathcal{O}(\partial\partial) \Big)|0\rangle$

From RHS

use $[K_\mu, x] = 0$

$[K_\mu, \phi_\Delta(0)] = 0$, $K_\mu|0\rangle = 0$

$[P_\mu, \phi_\Delta(0)]|0\rangle = i\partial_\mu \phi_\Delta(0)|0\rangle$

$K_\mu \phi_1(x) \phi_2(0)|0\rangle$

$$= K_\mu \left(\frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}} \left(\phi_\Delta(0) + \alpha x^\rho \partial_\rho \phi_\Delta(0) + \beta x^\rho x^\nu \partial_\rho \partial_\nu \phi_\Delta(0) + \gamma x^2 \partial_\rho \partial^\rho \phi_\Delta(0) \right) \right) |0\rangle$$

$$= \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}} K_\mu \left(\phi_\Delta(0) - i\alpha x^\rho P_\rho \phi_\Delta(0) - \beta x^\rho x^\nu P_\rho P_\nu \phi_\Delta(0) - \gamma x^2 P_\rho P^\rho \phi_\Delta(0) \right)$$

$$= \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}} \Big(-i\alpha x^\rho [K_\mu, P_\rho] \phi_\Delta(0)$$

$$- \beta x^\rho x^\nu [K_\mu, P_\rho] P_\nu \phi_\Delta(0) - \beta x^\rho x^\nu P_\rho [K_\mu, P_\nu] \phi_\Delta(0)$$

$$- \gamma x^2 [K_\mu, P_\rho] P^\rho \phi_\Delta(0) - \gamma x^2 P^\rho [K_\mu, P_\rho] \phi_\Delta(0) \Big) |0\rangle.$$

$L_{\mu\rho} \phi_\Delta(0) = 0$ for single field spin=0

$$= \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}} \Big(-i\alpha x^\rho (2i)(\eta_{\mu\rho} D - \cancel{L_{\mu\rho}}) \phi_\Delta(0)$$

$$- \beta x^\rho x^\nu (2i)(\eta_{\mu\rho} D - L_{\mu\rho}) P_\nu \phi_\Delta(0) - \beta x^\rho x^\nu P_\rho (2i)(\eta_{\mu\nu} D - \cancel{L_{\mu\nu}})$$

$$\phi_\Delta(0) - \gamma x^2 (2i)(\eta_{\mu\rho} D - L_{\mu\rho}) P^\rho \phi_\Delta(0) - \gamma x^2 P^\rho (2i)(\eta_{\mu\rho} D - \cancel{L_{\mu\rho}}) \phi_\Delta(0)$$

$$\Big) |0\rangle$$

$$= \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}} \left(2i\alpha \Delta x_\mu \phi_\Delta(0) + \cdots \right) |0\rangle$$

where the ... includes

$$\rightarrow (\eta_{\mu\rho} D - L_{\mu\rho}) P_\nu \phi_\Delta(0)|0\rangle$$

$$= \Big(\eta_{\mu\rho} \underbrace{[D, P_\nu]}_{iP_\nu} \phi_\Delta(0) + \eta_{\mu\rho} P_\nu \underbrace{D\phi_\Delta(0)}_{= i\Delta\phi_\Delta(0)} - \underbrace{[L_{\mu\rho}, P_\nu]}_{-i(\eta_{\nu\mu}P_\rho - \eta_{\rho\nu}P_\mu)} \phi_\Delta(0)$$

$$- P_\nu \underbrace{L_{\mu\rho}\phi_\Delta(0)}_{=0} \Big)|0\rangle$$

$$= \Big(i\eta_{\mu\rho} P_\nu \phi_\Delta(0) + i\Delta \eta_{\mu\rho} P_\nu \phi_\Delta(0) + i(\eta_{\mu\nu} P_\rho - \eta_{\rho\nu} P_\mu)\phi_\Delta(0) \Big)$$

$$\cdot |0\rangle$$

so $K_\mu \phi_1(x)\phi_2(0)|0\rangle$ (RHS)

$$= \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}} \Big(2i\Delta\alpha\, x_\mu \phi_\Delta(0) -$$

$$\beta x^\rho x^\nu/(2i) (-\eta_{\mu\rho} \partial_\nu \phi_\Delta(0) - \Delta \eta_{\mu\rho} \partial_\nu \phi_\Delta(0).$$

$$- \eta_{\mu\nu}\partial_\rho \phi_\Delta(0) + \eta_{\rho\nu}\partial_\mu \phi_\Delta(0)) - \Delta \eta_{\mu\nu}\partial_\rho\phi_\Delta(0)).$$

~~$- \gamma x^2 (2i) (-\eta_{\mu\rho}\partial_\nu \phi_\Delta(0))$~~

~~$= \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}} (2i\Delta\alpha\, x_\mu \phi_\Delta(0))$~~

$$- \gamma x^2 (2i) \Big(-\eta_{\mu\rho}\partial^\rho \phi_\Delta(0) - \Delta\eta_{\mu\rho}\partial^\rho \phi_\Delta(0)$$

$$- \eta_{\mu\rho}\partial^\rho \phi_\Delta(0) + \eta_\rho{}^\rho \partial_\mu \phi_\Delta(0) \Big).$$

$$= \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta_3}} \Big(2i\alpha \Delta x_\mu \phi_0(0)$$

$$+ 4i\beta(1+\Delta) x_\mu x^\nu \partial_\nu \phi_0(0) \; \cancel{- 2i\beta x^2 \partial_\mu \phi_0(0)}$$

$$+ 2i\gamma x^2 \big((1+\Delta+1)\partial_\mu \phi_0(0) - d\, \partial_\mu \phi_0(0)\big) \Big).$$

$d = \eta_\rho{}^\rho$ = dimensions.

$$= \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta_3}} \Big(2i\alpha\Delta x_\mu \phi_0(0)$$

$$+ 4i\beta(1+\Delta) x_\mu x^\nu \partial_\nu \phi_0(0)$$

$$~~+2i(\Delta+2-d)~~ + 2i\big((\Delta+2-d)\gamma - \beta\big) x^2 \partial_\mu \phi_0(0)$$

$$+ O(\partial\partial) \Big) |0\rangle$$

So compare LHS and RHS we have.

$$\Delta_1 - \Delta_2 + \Delta = 2\alpha\Delta$$

$$\alpha(\Delta_1 - \Delta_2 + \Delta + 2) = 4\beta(1+\Delta)$$

$$\alpha = 2[(\Delta+2-d)\gamma - \beta]$$

$$\Rightarrow \begin{cases} \alpha = \dfrac{\Delta_1 - \Delta_2 + \Delta}{2\Delta} \\[2ex] \beta = \dfrac{\alpha(\Delta_1 - \Delta_2 + \Delta + 2)}{4(\Delta+1)} \\[2ex] \gamma = \dfrac{\alpha + 2\beta}{\Delta + 2 - d} \end{cases}$$

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use $\langle \phi_\Delta(y) \phi_\Delta(z) \rangle = \frac{1}{|y-z|^{2\Delta}}$

$$\langle \phi_1(x) \phi_2(0) \phi_3(z) \rangle = \frac{C_{12\Delta}}{|x|^{\Delta_1+\Delta_2-\Delta} |z|^{\Delta_2+\Delta-\Delta_1} |x-z|^{\Delta_1+\Delta-\Delta_2}},$$

$$\langle \phi_1(x) \phi_2(0) \phi_\Delta(z) \rangle = \sum_{\text{primaries } \Delta'} C_{12\Delta'} C_{\Delta'}(x,\partial) \langle \phi_{\Delta'}(y) \phi_{\Delta'}(z) \rangle \Big|_{y=0}$$

implies $\langle \phi_1(x) \phi_2(0) \phi_\Delta(z) \rangle$

$$= C_{12\Delta} C_\Delta(x,\partial) \langle \phi_\Delta(y) \phi_\Delta(z) \rangle \Big|_{y=0}$$

$C_{12\Delta}$ cancels so

$$\frac{1}{|x|^{\Delta_1+\Delta_2-\Delta} |z|^{\Delta_2+\Delta-\Delta_1} |x-z|^{\Delta_1+\Delta-\Delta_2}} = C_\Delta(x,\partial) \langle \phi_\Delta(y) \phi_\Delta(z) \rangle \Big|_{y=0}.$$

$$\rightarrow \frac{1}{|x-z|^{\Delta_1+\Delta-\Delta_2}} = \frac{1}{|x-z|^k} = \left([x-z]^2 \right)^{-\frac{k}{2}}$$

$$= (x^2 - 2x\cdot z + z^2)^{-\frac{k}{2}} = \frac{1}{|z|^k} \left(1 - \frac{2x\cdot z}{z^2} + \frac{x^2}{z^2} \right)^{-\frac{k}{2}}$$

$$= \frac{1}{|z|^k} \left(1 + \frac{k}{2} \left(\frac{2x\cdot z}{|z|^2} - \frac{x^2}{|z|^2} \right) + \frac{1}{8} k(k+2) \left(\frac{2x\cdot z}{|z|^2} - \frac{x^2}{|z|^2} \right)^2 + \cdots \right)$$

$$\frac{1}{|z|^{\Delta_2+\Delta-\Delta_1}} \frac{1}{|z|^k} = \frac{1}{|z|^{\Delta_2+\Delta-\Delta_1+\Delta_1+\Delta-\Delta_2}} = \frac{1}{|z|^{2\Delta}}$$

$\therefore \langle \phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle / C_{12\Delta}$

$$= \frac{1}{|x|^{\Delta_1+\Delta_2-\Delta}} \frac{1}{|z|^{2\Delta}} \Big(1 + (\Delta_1+\Delta-\Delta_2)\frac{x\cdot z}{z^2}$$

$$- \frac{1}{2}(\Delta_1+\Delta-\Delta_2)\frac{x^2}{z^2}$$

$$+ \frac{1}{2}(\Delta_1+\Delta-\Delta_2)(\Delta_1+\Delta-\Delta_2+2)\frac{(x\cdot z)^2}{|z|^4} + \cdots \Big).$$

$C_\Delta(\alpha, \partial)\langle \phi_\Delta(y)\phi_\Delta(z)\rangle |_{y=0}$

$$= (1 + \alpha x^\rho \partial_\rho + \beta x^\rho x^\nu \partial_\rho \partial_\nu + \gamma x^2 \partial_\rho \partial^\rho + \cdots)$$

$$\cdot \frac{1}{|y-z|^{2\Delta}} \Big|_{y=0}.$$

$$\Rightarrow \quad \alpha x^\rho \partial_\rho \frac{1}{|y-z|^{2\Delta}} = \alpha x^\rho \frac{-1(2\Delta)}{z^{2\Delta+2}}(-z_\rho)$$

$$= \frac{2\Delta\alpha\, x\cdot z}{|z|^{2\Delta+2}} \quad \Rightarrow \quad 2\Delta\alpha = \Delta_1+\Delta-\Delta_2$$

$$\Rightarrow \alpha = \frac{\Delta_1+\Delta-\Delta_2}{2\Delta}$$

$$\Rightarrow \beta x^\rho x^\nu \partial_\rho \partial_\nu \frac{1}{|y-z|^{2\Delta}}\Big|_{y=0} = \beta x^\rho x^\nu \partial_\rho \left(\frac{2\Delta z_\nu}{|y-z|^{2(\Delta+1)}} \right)\Big|_{y=0}.$$

$$= \beta (x^\rho x^\nu)\, 4\Delta(\Delta+1) \frac{z_\rho z_\nu}{|y-z|^{2(\Delta+2)}} \Big|_{y=0}$$

$$= 4\Delta(\Delta+1)\beta \frac{(Z\cdot X)^2}{|Z|^4}$$

$$\Rightarrow \quad \beta(4\Delta(\Delta+1)) = \frac{1}{2}(\Delta_1+\Delta-\Delta_2)(\Delta_1+\Delta-\Delta_2+2)$$

$$\Rightarrow \quad \beta = \frac{(\Delta_1+\Delta-\Delta_2+2)}{4(\Delta+1)}\left(\frac{(\Delta_1+\Delta-\Delta_2)}{2\Delta}\right)$$

$$= \frac{\Delta_1+\Delta-\Delta_2+2}{4(\Delta+1)}\alpha$$

this agrees with [5].