To: Will Potter

50/51 98% Excellent work Ziyan, very thorough answers!

B5 Problem Set 4

Ziyan Li

1.

a) Dynamic field equation (FRW Form)

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = -kc^2$$

$k = 0, \pm 1$ for the metric to be maximally symmetric

For empty space $\rho = 0$

$\therefore \quad \dot{R}^2 = -kc^2 \leq 0$

If $k = 0$, $\dot{R} = 0 \rightarrow$ static universe

this is impossible

If $k < 0$, then $\because k = 0, \pm 1 \quad \therefore k = -1$

$\therefore \dot{R}^2 = c^2 \quad \because \dot{R} > 0$ at $t = 0$ (big bang)

$\therefore \dot{R} = c \Rightarrow R = ct$ (for $R(t=0) = 0$) ~~big bang~~

let $c = 1 \quad \therefore \underline{R = t}, \underline{k = -1}$ ✓

The FRW metric is given by ($c = 1$, $[R] = L$)

$$-d\tau^2 = -dt^2 + \frac{R^2 dr^2}{1 - kr^2} + R^2 r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

substitute in $R = t$, $k = -1$ gives

In empty space:

① $-d\tau^2 = -dt^2 + \frac{t^2 dr^2}{1+r^2} + r^2 t^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$

good 3/3

b)

$$-d\tau^2 = -dT^2 + ds^2 + s^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$ ②

$\because$ $S = S(r,t)$ $T = T(r,t)$

$\therefore$ $dr = 0$, $dt = 0$ $\Rightarrow$ $ds = 0$, $dT = 0$

$\therefore$ when $dr = 0$, $dt = 0$, we have

$$-d\tau^2 = r^2 t^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$= s^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

$\because$ s, r, t independent of θ, ϕ

$\therefore$ $s^2 = r^2 t^2$

Without loss of generality:

$$s = rt$$

✓ 2/2

c) We are left with

~~$-dt^2 + dr^2$~~ $-dt^2 + \frac{t^2 dr^2}{1+r^2} = -dT^2 + ds^2$

$\because S = rt \quad \therefore ds = r\,dt + t\,dr$

$\therefore$ ~~$-dt$~~ $ds^2 = r^2 dt^2 + t^2 dr^2 + 2rt\,dr\,dt$

$\therefore$ ~~$-dT^2 + r^2dt^2 + t^2dr^2 + 2rt\,dr\,dt$~~

$\Rightarrow dT = \frac{\partial T}{\partial r} dr + \frac{\partial T}{\partial t} dt, \quad dT^2 = \left(\frac{\partial T}{\partial r}\right)^2 dr^2 + \left(\frac{\partial T}{\partial t}\right)^2 dt^2 + 2\frac{\partial T}{\partial r}\cdot\frac{\partial T}{\partial t} dr\,dt$

$\therefore$ We have

$-dt^2 + \frac{t^2}{1+r^2} dr^2$

$= r^2 dt^2 + t^2 dr^2 + 2rt\,dr\,dt$

$- \left(\frac{\partial T}{\partial r}\right)^2 dr^2 - \left(\frac{\partial T}{\partial t}\right)^2 dt^2 - 2\frac{\partial T}{\partial r}\frac{\partial T}{\partial t} dr\,dt$

$\therefore \left(r^2 + 1 - \left(\frac{\partial T}{\partial t}\right)^2\right) dt^2 + \left(t^2 - \frac{t^2}{1+r^2} - \left(\frac{\partial T}{\partial r}\right)^2\right) dr^2$

$+ 2\left(-\frac{\partial T}{\partial r}\cdot\frac{\partial T}{\partial t} + rt\right) dr\,dt = 0$

~~So we have~~

$\because r, t$ are independent variables

$\therefore$

$$\therefore \begin{cases} r^2+1-\left(\frac{\partial T}{\partial t}\right)^2=0 & ③ \\ t^2-\frac{t^2}{1+r^2}\left(\frac{\partial T}{\partial r}\right)^2=0 & ④ \\ -\frac{\partial T}{\partial r}\cdot\frac{\partial T}{\partial t}+rt=0 & ⑤ \end{cases}$$

∴ ~~$\frac{\partial T}{\partial t}$ without loss of generality: Without loss of generality~~

assume $\frac{\partial T}{\partial t}>0$, $\frac{\partial T}{\partial r}>0$ for $r>0, t>0$

$$③ \Rightarrow \left(\frac{\partial T}{\partial t}\right)^2 = r^2+1 \rightarrow \frac{\partial T}{\partial t}=\sqrt{r^2+1}$$

$$\therefore \rightarrow T=t\sqrt{r^2+1}+F_1(r)$$

$$④ \Rightarrow \left(\frac{\partial T}{\partial r}\right)^2 = t^2\left(1-\frac{1}{1+r^2}\right)=\frac{t^2r^2}{1+r^2}$$

$$\therefore \frac{\partial T}{\partial r}=\frac{tr}{\sqrt{1+r^2}}$$

$$\therefore T=\int\frac{tr}{\sqrt{1+r^2}}dr = t\int\frac{1}{2}\frac{du}{u^{1/2}} = \frac{t}{2}(2u^{1/2})+F_2(t)$$

$u=1+r^2$

$rdr=\frac{1}{2}du$

$$= t\sqrt{r^2+1}+F_2(t)$$

$$⑤ \Rightarrow -\frac{\partial T}{\partial r}\cdot\frac{\partial T}{\partial t} = -\sqrt{r^2+1}\cdot\frac{tr}{\sqrt{r^2+1}} = -tr$$

$$\therefore -\frac{\partial T}{\partial r}\frac{\partial T}{\partial t}+rt=-rt+rt=0 \quad \text{(satisfied)}$$

∴ From ③, ④, ⑤

$$T(r,t) = t\sqrt{r^2+1} + F_1(r) = t\sqrt{r^2+1} + F_2(t)$$

this shows that $F_1(r) = 0$, $F_2(t) = 0$

$$\therefore T(r,t) = t\sqrt{1+r^2}$$

Now $s = rt$

$T = t\sqrt{1+r^2}$

$$\therefore T = \sqrt{t^2+t^2r^2} = \sqrt{t^2+s^2} \quad \therefore T^2 - s^2 = t^2$$

$$\therefore t = \sqrt{T^2-s^2} \quad \because s = rt \quad \therefore s = r\sqrt{T^2-s^2}$$

$$\therefore r = \frac{s}{\sqrt{T^2-s^2}}$$

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d) $$\frac{\partial t}{\partial s} = -\frac{s}{\sqrt{T^2-s^2}} = -r$$

$$\frac{\partial r}{\partial s} = \frac{\sqrt{T^2-s^2} - s\cdot(-\frac{1}{2})(T^2-s^2)^{-\frac{1}{2}}(2s)}{T^2-s^2}$$

$$= \frac{\sqrt{T^2-s^2} + s^2\frac{1}{\sqrt{T^2-s^2}}}{T^2-s^2}$$

$$= \frac{t + \frac{r^2t^2}{t}}{t^2} = \frac{1+r^2}{t}$$

$$\frac{\partial r}{\partial T} = \frac{\partial}{\partial T}\left(\frac{s}{\sqrt{T^2-s^2}}\right) = s\frac{\partial}{\partial T}(T^2-s^2)^{-\frac{1}{2}}$$

~~$= -\frac{1}{2}s(T^2-s^2)^{-\frac{3}{2}}(-2s)$~~

~~$= \frac{s^2}{(T^2-s^2)^{3/2}}$~~

~~$\frac{\partial t}{\partial T} =$~~

$$= -\frac{1}{2}s(T^2-s^2)^{-3/2}(2T)$$

$$= \frac{-sT}{(T^2-s^2)^{3/2}} = -\frac{s}{T^2-s^2}\cdot\frac{T}{(T^2-s^2)^{1/2}}$$

$$\frac{\partial t}{\partial T} = \frac{\partial}{\partial T}(T^2-s^2)^{1/2} = \frac{1}{2}(T^2-s^2)^{-\frac{1}{2}}(2T)$$

$$= \frac{T}{(T^2-s^2)^{1/2}}$$

$$\therefore \quad \frac{\partial r}{\partial T} = -\frac{s}{T^2-s^2}\frac{T}{(T^2-s^2)^{1/2}} = -\frac{r}{t}\frac{\partial t}{\partial T}$$

start with

$$-dT^2 = -dt^2 + \frac{t^2dr^2}{1+r^2} + t^2r^2(d\theta^2+\sin^2\theta d\phi^2)$$

transform

$$dt = \frac{\partial t}{\partial s}ds + \frac{\partial t}{\partial T}dT \quad , \quad dr = \frac{\partial r}{\partial s}ds + \frac{\partial r}{\partial T}dT$$

$$\therefore\ dt = -r\,ds + \frac{\partial t}{\partial T}dT$$

$$dr = \frac{1+r^2}{t}ds - \frac{r}{t}\frac{\partial t}{\partial T}dT$$

$$\therefore\ -dt^2 + \frac{t^2dr^2}{1+r^2} = -\left(-r\,ds + \frac{\partial t}{\partial T}dT\right)^2$$

$$+\frac{t^2}{1+r^2}\left(\frac{1+r^2}{t}ds - \frac{r}{t}\frac{\partial t}{\partial T}dT\right)^2$$

$$= -r^2ds^2 + 2r\frac{\partial t}{\partial T}ds\,dT - \left(\frac{\partial t}{\partial T}\right)^2dT^2$$

$$+\frac{t^2}{1+r^2}\frac{(1+r^2)^2}{t^2}ds^2 - \frac{t^2}{(1+r^2)}\frac{(1+r^2)}{t}\frac{2r}{t}\frac{\partial t}{\partial T}ds\,dT$$

$$+\frac{t^2}{1+r^2}\frac{r^2}{t^2}\left(\frac{\partial t}{\partial T}\right)^2dT^2$$

$$= (1+r^2-r^2)ds^2 + \frac{\partial t}{\partial T}(2r - 2r)ds\,dT$$

$$+\left(\frac{r^2}{1+r^2} - 1\right)\left(\frac{\partial t}{\partial T}\right)^2dT^2$$

$$= -\frac{1}{1+r^2}\left(\frac{\partial t}{\partial T}\right)^2dT^2 + ds^2$$

$$\because\ \frac{\partial t}{\partial T} = \frac{T}{(T^2-s^2)^{1/2}} = \sqrt{\frac{T^2}{T^2-s^2}} = \sqrt{\frac{T^2-s^2+s^2}{T^2-s^2}}$$

$$= \sqrt{1+\left(\frac{s}{\sqrt{T^2-s^2}}\right)^2} = \sqrt{1+r^2}$$

so,

$$\therefore -dT^2 = -dt^2 + \frac{t^2 dr^2}{1+r^2} + t^2r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -\frac{1}{1+r^2}\left(\frac{\partial t}{\partial T}\right)^2 dT^2 + ds^2 + S^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -\frac{1}{1+r^2}\left(\sqrt{1+r^2}\right)^2 dT^2 + ds^2 + S^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\rightarrow -dT^2 = -dT^2 + ds^2 + S^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

2/2

12/12

2. Dynamical equation of the universe with $E=0$

$$\dot{R}^2 - \frac{8\pi G \rho R^2}{3} = 0$$

$\because \rho = \rho_\gamma + \rho_m$, current value $R_0 = 1$

$\therefore$ For radiation : $\rho_\gamma R^4 = \rho_{\gamma 0}$

For matter : $\rho_m R^3 = \rho_{m0}$

$$\therefore \rho = \frac{\rho_{\gamma 0}}{R^4} + \frac{\rho_{m0}}{R^3}$$

$$\rightarrow \rho R^2 = \frac{\rho_{\gamma 0}}{R^2} + \frac{\rho_{m0}}{R}$$

$$\therefore \dot{R}^2 = \frac{8\pi G}{3}\left(\frac{\rho_{\gamma 0}}{R^2} + \frac{\rho_{m0}}{R}\right)$$

$$\therefore \frac{dR}{dt} = \sqrt{\frac{8\pi G}{3}}\sqrt{\frac{\rho_{\gamma 0}}{R^2} + \frac{\rho_{m0}}{R}}$$

$$\therefore \frac{dR}{\sqrt{\frac{\rho_{\gamma 0}}{R^2} + \frac{\rho_{m0}}{R}}} = \sqrt{\frac{8\pi G}{3}}\, dt$$

$$\therefore \int_0^R \frac{R\,dR}{\sqrt{\rho_{\gamma 0} + \rho_{m0} R}} = \int_0^t \sqrt{\frac{8\pi G}{3}}\, dt$$

$\because$ let $u = \rho_{\gamma 0} + \rho_{m0}(R)$ $\quad du = \rho_{m0}\, dR \rightarrow dR = \rho_{m0}^{-1}\, du$

$$R = \frac{u - \rho_{\gamma 0}}{\rho_{m0}}$$

$R=0 \Rightarrow u = \rho_{\gamma 0}$ $\quad R=R \quad u = \rho_{\gamma 0} + \rho_{m0} R$

$$\therefore \sqrt{\frac{8\pi G}{3}}\, t = \int_{\rho_{r0}}^{\rho_{r0}+\rho_{m0}R} \left(\frac{u}{\rho_{m0}} \frac{1}{\sqrt{u}} - \frac{\rho_{r0}}{\rho_{m0}} \frac{1}{\sqrt{u}} \right) \frac{du}{\rho_{m0}}$$

$$= \frac{1}{\rho_{m0}^2} \int_{\rho_{r0}}^{\rho_{r0}+\rho_{m0}R} u^{1/2}\, du - \rho_{r0} u^{-1/2}\, du$$

$$= \left[\frac{2}{3} u^{3/2} \Big|_{\rho_{r0}}^{\rho_{r0}+\rho_{m0}R} - \left(2u^{1/2} \Big|_{\rho_{r0}}^{\rho_{r0}+\rho_{m0}R} \right) \rho_{r0} \right] \frac{1}{\rho_{m0}^2}$$

$$\because \Omega_{m0} = \frac{8\pi G \rho_{m0}}{3H_0^2} \quad \therefore \Omega_{m0}^{1/2} H_0 = \sqrt{\frac{8\pi G \rho_{m0}}{3}}$$

$$\therefore \sqrt{\frac{8\pi G}{3}} = \frac{\Omega_{m0}^{1/2} H_0}{\sqrt{\rho_{m0}}}$$

$$\therefore \frac{3}{2} \frac{\Omega_{m0}^{1/2} H_0}{\sqrt{\rho_{m0}}}\, t = \frac{1}{\rho_{m0}^2} (\rho_{r0} + \rho_{m0}R)^{3/2} - \frac{1}{\rho_{m0}^2} \rho_{r0}^{3/2}$$

$$- 3\rho_{r0} (\rho_{r0} + \rho_{m0}R)^{1/2} \frac{1}{\rho_{m0}^2}$$

$$+ 3\rho_{r0} \rho_{r0}^{1/2} \frac{1}{\rho_{m0}^2}$$

~~$\therefore \frac{3}{2} \Omega_{m0}^{1/2} H_0 t$~~

$$\frac{3\Omega_{m0}^{1/2} H_0 t}{2} = \left(\left(\frac{\rho_{r0}}{\rho_{m0}} \right) + R \right)^{3/2} - 3\left(\frac{\rho_{r0}}{\rho_{m0}} + R \right) \frac{\rho_{r0}}{\rho_{m0}}$$

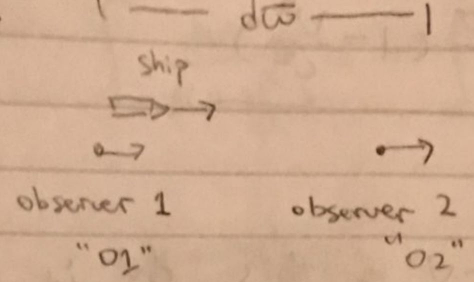
$$+ 2 \left(\frac{\rho_{r0}}{\rho_{m0}} \right)^{3/2}$$

$$\Rightarrow (R+I)^{3/2} - 3I(R+I)^{1/2} + 2I^{3/2} = \frac{3\Omega_{m0}^{1/2} H_0 t}{2}$$ ✓

$I = \frac{\rho_{r0}}{\rho_{m0}}$

6/6

3).



(/ ≡ relative to)

(⊕ ≡ add velocities relativistically)

$V = V(\text{ship}/O1)$

$V' = V(\text{ship}/O2)$, $-V' = V(O2/\text{ship})$

The relative velocity ~~separated for~~ between O2 and O2 separated by $d\varpi$ is

$$dV = V(O2/O1) = V(O2/\text{ship}) \oplus V(\text{ship}/O1)$$

$$= -V' \oplus V = V \ominus V'$$

$$= \frac{V-V'}{1-\frac{VV'}{c^2}} = \frac{V-V'}{1-\frac{V^2}{c^2}}$$

⇑ to first order in dV (thus in $d\varpi$)

By Hubble's Law: $V = \frac{\dot{R}}{R}\varpi$ ↳ proper distance

velocity of a fixed point expanding with space

$$\therefore dV = \frac{\dot{R}\,d\varpi}{R}$$

$$\Rightarrow \frac{V-V'}{1-\frac{V^2}{c^2}} = \frac{\dot{R}\,d\varpi}{R}$$

$$\Rightarrow \quad v' = v - \frac{\dot{R}\, d\varpi}{R}\left(1 - \frac{v^2}{c^2}\right)$$

Now define $dv = v' - v$, this dv is different from $\frac{\dot{R}}{R} d\varpi$, this $dv = v' - v$ denotes the change of ship's velocity as ~~mes~~ measured by comoving observers as the ship passes proper distance $d\varpi$

then $\quad v' - v = -\frac{\dot{R}}{R} d\varpi \left(1 - \frac{v^2}{c^2}\right) = dv$

$$\therefore \quad \frac{dv}{dt} = -\frac{\dot{R}}{R}\frac{d\varpi}{dt}\left(1 - \frac{v^2}{c^2}\right)$$

$$\therefore \quad \dot{v} = -\frac{\dot{R}}{R} v \left(1 - \frac{v^2}{c^2}\right)$$

$$\therefore \quad \frac{\dot{v}}{v\left(1 - \frac{v^2}{c^2}\right)} = -\frac{\dot{R}}{R}$$

$$\therefore \quad \frac{dv/dt}{v\left(1 - \frac{v^2}{c^2}\right)} = -\frac{dR/dt}{R}$$

$$\therefore \quad \frac{dv}{v\left(1 - \frac{v^2}{c^2}\right)} = -\frac{dR}{R}$$

$$\therefore \int \frac{dv}{v(1-\frac{v^2}{c^2})} = -\int \frac{dR}{R} = -\ln R + C$$

$$= \ln \frac{1}{R} + C$$

~~$d(1-\frac{v^2}{c^2}) =$~~

$$\int \frac{dv}{v(1-\frac{v^2}{c^2})} = \int \frac{dv}{v} + \frac{v/c^2}{1-\frac{v^2}{c^2}} dv$$

$$= \ln v + \frac{1}{2}\int \frac{d(v^2/c^2)}{1-v^2/c^2}$$

$$= \ln v - \frac{1}{2}\int \frac{d(1-v^2/c^2)}{1-v^2/c^2}$$

$$= \ln v - \frac{1}{2}\ln(1-v^2/c^2)$$

$$= \ln v - \ln[(1-v^2/c^2)^{1/2}]$$

$$= \ln\left[\frac{v}{\sqrt{1-v^2/c^2}}\right].$$

$$\therefore \ln\left[\frac{v}{\sqrt{1-v^2/c^2}}\right] = \ln\frac{1}{R} + C$$

$$\frac{v}{\sqrt{1-v^2/c^2}} = C' \frac{1}{R}$$

At $t=t_0$, $R=1$, $v=v_0$

$$\therefore c' = \frac{V_0}{\sqrt{1 - V_0^2/c^2}} = \gamma_{V_0} V_0 \equiv \upsilon_0$$

$$\therefore \frac{V}{\sqrt{1 - V^2/c^2}} = \frac{\upsilon_0}{R}$$ ✓ 6/6

~~b) The $\frac{h\nu}{k_B T}$ term appears~~

b) ∵ in thermal radiation the ν and T terms always appear as $\frac{h\nu}{k_B T}$ in the formula

∴ Temperature has the same evolutionary history of a photon with energy $h\nu$

→ Start from adiabatic expansion of a photon → $T \sim \nu$

$$TR \sim \text{constant}$$

$\because T \sim \nu$ $\quad \therefore \nu R \sim \text{constant}$

$$\rightarrow \frac{h\nu}{c} R \sim \text{constant}$$

$\because \frac{h\nu}{c}$ is momentum of photon

∴ $PR \sim$ constant, which is the same as ~~et~~ our ~~ta~~ equation for the Titanic.

✓

The equation for the Titanic is equivalent to an adiabatic expansion of photons.

→ (proof of classical $T\rho^{-2/3}$ ~ constant see after c)).

c) $$\frac{V}{\sqrt{1-V^2/c^2}} = \frac{U_0}{R} \qquad \frac{d\varpi}{dt} = V(R)$$

$$d\varpi = R\,dr \qquad \therefore \quad R\frac{dr}{dt} = V(R) \rightarrow \frac{dr}{dt} = \frac{V(R)}{R}$$

$$\therefore \quad \frac{V^2}{1-V^2/c^2} = \frac{U_0^2}{R^2}$$

$$\therefore \quad V^2 = \frac{U_0^2}{R^2} - \frac{U_0^2 V^2}{c^2 R^2}$$

$$\therefore \quad V^2\left(1+\frac{U_0^2}{c^2R^2}\right) = \frac{U_0^2}{R^2}$$

$$\therefore \quad V^2 = \frac{U_0^2/R^2}{1+U_0^2/c^2R^2} = \frac{U_0^2}{R^2+\frac{U_0^2}{c^2}}$$

$$\therefore \quad V(R) = \frac{U_0}{\sqrt{R^2+\frac{U_0^2}{c^2}}}$$

$$\therefore \quad \frac{dr}{dt} = \frac{V(R)}{R} = \frac{U_0}{R\sqrt{R^2+\frac{U_0^2}{c^2}}} \Rightarrow dr = \frac{U_0\,dt}{R\sqrt{R^2+\frac{U_0^2}{c^2}}}$$

For Einstein-de Sitter universe

$$R = \left(\frac{3H_0t}{2}\right)^{2/3} \quad \therefore \quad ~~dt~~ \; ~~dR = \frac{2}{3}\left(\frac{3H_0t}{2}\right)^{-1/3}dt~~$$

~~$\therefore dR = \frac{2}{3}R^{-1/2}dt \quad \therefore dt = \frac{3}{2}R^{1/2}dR$~~

$$\therefore dR = \frac{2}{3}\left(\frac{3}{2}H_0 t\right)^{-\frac{1}{3}} \frac{3}{2} H_0 \, dt. = R^{-1/2} H_0 \, dt$$

$$\underbrace{\left(\frac{3}{2}H_0 t\right)^{-\frac{1}{3}}}_{R^{-1/2}}$$

$$\therefore dt = \frac{R^{1/2} dR}{H_0}$$

$$\therefore dr = \frac{1}{H_0} \frac{v_0 R^{1/2} dR}{R(R^2 + \frac{v_0^2}{c^2})^{1/2}} = \frac{1}{H_0} \frac{v_0 \, dR}{R^{1/2}(R^2 + \frac{v_0^2}{c^2})^{1/2}}$$

$$= \frac{1}{H_0} \frac{v_0 \, dR}{(R^3 + R\frac{v_0^2}{c^2})^{1/2}} = \frac{c}{H_0} \frac{dR}{(R + \frac{c^2 R^3}{v_0^2})^{1/2}}$$

$$\therefore r(R) = \frac{c}{H_0}\int_0^R \frac{dx}{[x + \frac{c^2 x^3}{v_0^2}]^{1/2}}$$

As $R \to \infty$

$$r_{max} = r(\infty) = \frac{c}{H_0}\int_0^\infty \frac{dx}{(x + \frac{c^2}{v_0^2}x^3)^{1/2}}$$

$$= \frac{c}{H_0}\int_0^\infty \frac{dx}{x^{1/2}(1 + \frac{c^2}{v_0^2}x^2)^{1/2}}$$

$$= \frac{c}{H_0}\int_0^\infty \frac{v_0/c}{\sqrt{v_0/c}} \frac{dy}{y^{1/2}(1+y^2)^{1/2}}$$

$$= \frac{c}{H_0}\sqrt{v_0/c} \int_0^\infty \frac{dy}{(y + y^3)^{1/2}}$$

$$= \frac{\sqrt{v_0 c}}{H_0}\int_0^\infty \frac{dy}{(y+y^3)^{1/2}} = (3.708)\frac{\sqrt{v_0 c}}{H_0}$$ ✓ 8/8

$y = \frac{c}{v_0}x.$

$dx = \frac{v_0}{c}dy$

$\sqrt{x} = \frac{\sqrt{v_0}}{\sqrt{c}}\sqrt{y}$

~~$\frac{dx}{x} = \frac{dy}{y}$~~

~~As $v_0 \to c$ ... $v_0 \to \infty$ ... $r_{max} \to \infty$~~

$$\therefore dR = \frac{2}{3}\left(\frac{3}{2}H_0 t\right)^{-\frac{1}{3}}\frac{3}{2}H_0\, dt. = R^{-1/2} H_0\, dt$$

$$\underbrace{\left(\frac{3}{2}H_0 t\right)^{-\frac{1}{3}}}_{R^{-1/2}}$$

$$\therefore dt = \frac{R^{1/2}dR}{H_0}$$

$$\therefore dr = \frac{1}{H_0}\frac{v_0 R^{1/2} dR}{R(R^2 + \frac{v_0^2}{c^2})^{1/2}} = \frac{1}{H_0}\frac{v_0\, dR}{R^{1/2}(R^2 + \frac{v_0^2}{c^2})^{1/2}}$$

$$= \frac{1}{H_0}\frac{v_0\, dR}{(R^3 + R\frac{v_0^2}{c^2})^{1/2}} = \frac{c}{H_0}\frac{dR}{(R + \frac{c^2R^3}{v_0^2})^{1/2}}$$

$$\therefore r(R) = \frac{c}{H_0}\int_0^R \frac{dx}{[x + \frac{c^2x^3}{v_0^2}]^{1/2}}$$

As $R \to \infty$

$$\underline{\underline{r_{max}}} = r(\infty) = \frac{c}{H_0}\int_0^\infty \frac{dx}{(x + \frac{c^2}{v_0^2}x^3)^{1/2}}$$

$$= \frac{c}{H_0}\int_0^\infty \frac{dx}{x^{1/2}(1 + \frac{c^2}{v_0^2}x^2)^{1/2}}$$

$y = \frac{c}{v_0}x.$

$dx = \frac{v_0}{c}dy$

$\sqrt{x} = \frac{\sqrt{v_0}}{\sqrt{c}}\sqrt{y}$

~~$\frac{dx}{x} = \frac{dy}{y}$~~

$$= \frac{c}{H_0}\int_0^\infty \frac{v_0/c}{\sqrt{v_0/c}}\frac{dy}{y^{1/2}(1+y^2)^{1/2}}$$

$$= \frac{c}{H_0}\sqrt{v_0/c}\int_0^\infty \frac{dy}{(y+y^3)^{1/2}}$$

$$= \frac{\sqrt{v_0 c}}{H_0}\int_0^\infty \frac{dy}{(y+y^3)^{1/2}} = (3.708)\frac{\sqrt{v_0 c}}{H_0}$$ ✓ 8/8

~~As $v_0 \to c$ ... $v_0 \to \infty$ ... $r_{max} \to \infty$~~

As $V_0 \to c$ $\quad v_0 = \frac{v_0}{\sqrt{1 - v_0^2/c^2}} \to \infty$

$\therefore r_{max} \to \infty$

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b) $\rho R^3 = \text{constant}$

$\therefore \rho \sim \frac{1}{R^3} \quad R \sim \rho^{-3}$

momentum $P \sim v \sim \sqrt{v^2} \sim \sqrt{T} \sim T^{1/2}$

$\therefore PR \sim \text{const}$

$T^{1/2} \rho^{-3} \sim \text{const}$

$\to \quad T \rho^{-2/3} \sim \text{const.}$ ✓ 3/3

17/17

$T \sim \frac{1}{R}$ $\Rightarrow$ CMB always black body

$n \sim \frac{1}{R^3}$

CMB $\propto T^4$

$\lambda \sim R$

$E \sim T \sim \frac{1}{\lambda}$ $\quad \therefore T \sim \frac{1}{R}$

observer sees the particle pass by at velocity V, if it is at proper distance $d\varpi = R\,dr$ then it has a recession velocity v

$$v = H\,d\varpi = \frac{\dot{R}}{R}\,d\varpi$$

$$V' = \frac{V - v}{1 - \frac{Vv}{c^2}} \simeq (V - v)\left(1 + \frac{Vv}{c^2}\right) = V - v + \frac{vV^2}{c^2} + O(v^2)$$

$$= V - \left(\frac{\dot{R}}{R}\,d\varpi\right)\left(1 - \frac{V^2}{c^2}\right)$$

$$\int \frac{dV}{V\left(1 - \frac{V^2}{c^2}\right)} = \int -\frac{\dot{R}}{R}\,dt = -\int \frac{dR}{R}$$

$$\int \left[\frac{1}{V} + \frac{V/c^2}{1 - V^2/c^2}\right] dV = \int \frac{dR}{R}$$

4) From notes, the flux from an object at redshift z is

$$F(z) = \frac{L R^2(t)}{4\pi R_0^4 l^2(z)} = \frac{L}{4\pi (R_0 l(z))^2} \left(\frac{R^2(t)}{R_0^2}\right)$$

↓ proper radius of the sphere over which the photons from the distant source at z are now distributed

one factor of $\frac{R(t)}{R_0}$ comes from doppler shift of photon wavelength, the other comes from the time dilation of the emission ~~in~~ time interval between successive ~~phot~~ emitted photons.

let $R_0 = 1$, $r = R_0 l(z)$

$$\therefore F_z = \frac{L R^2(t)}{4\pi r^2}$$

conservation of number of galaxies:

$$nR^3 = n_0 R_0^3 = n_0 \quad \therefore n = \frac{n_0}{R^3}$$

→ The net flux from sources at distance r away is dF

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$n R^3 = n_0 R_0^3 = n_0$ $\therefore$ $n = \frac{n_0}{R^3}$ ✓

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~~$dF = \iint d\Omega$~~

The metric for E-dS universe

$$-c^2 d\tau^2 = -c^2 dt^2 + R^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$$

The 3d volume element is $dV = \sqrt{|g|}\, dr\, d\theta\, d\phi$
$= R^3(t)\, r^2 dr\, d\Omega$

$\because \int d\Omega = 4\pi$

$\therefore dF = F_z \times 4\pi R^3(t) r^2 dr \times n$

$$= \frac{L R^2(t)}{4\pi r^2} 4\pi r^2 R^3(t) \frac{n_0}{R^3(t)} dr$$

$$= n_0 L R^2(t) dr$$

radiation energy density

$$u = \frac{1}{c}\int dF = \frac{L n_0}{c}\int R^2(t) dr$$

photon geodesic $0 = -c^2 dt^2 + R^2 dr^2$

$$\Rightarrow dr = \frac{c}{R} dt$$

$$\therefore u = \frac{L n_0}{c}\int \frac{c}{R} R^2 dt = L n_0 \int R\, dt$$

we integrate from redshift $z = \infty$ to current value $z = 0$

$\therefore$ This is t from 0 to t_0

$\therefore \quad u = L n_0 \int_0^{t_0} R(t)\, dt$ ✓ good

$\because \quad R(t) = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{3H_0 t}{2}\right)^{2/3} \qquad \left(t_0 \equiv \frac{2}{3H_0}\right)$

$\therefore \quad u = L n_0 \int_0^{t_0} \left(\frac{t}{t_0}\right)^{2/3} dt = \frac{L n_0}{t_0^{2/3}} \int_0^{t_0} t^{2/3}\, dt$

~~$= L n_0 \frac{1}{t_0^{2/3}} (3)\, t^{1/3}$~~

$= \frac{L n_0}{t_0^{2/3}} \times \frac{3}{5} \times t^{5/3} \Big|_0^{t_0}$

$= \frac{3}{5} L n_0 t_0 = \frac{3}{5} \times L n_0 \times \frac{2}{3} \times \frac{1}{H_0}$

~~$= \frac{2}{5} L n_0$~~ $= \frac{2}{5} \frac{L n_0}{H_0}$ ✓

6/6

$\therefore \quad u = L n_0 \int_0^{t_0} R(t)\, dt$ ✓ good

$\because \quad R(t) = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{3H_0 t}{2}\right)^{2/3} \qquad \left(t_0 \equiv \frac{2}{3H_0}\right)$

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~~$= \frac{2}{5} L n_0$~~ $= \frac{2}{5} \frac{L n_0}{H_0}$ ✓

6/6

5. a) $\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -\frac{c^2}{a^2} = \text{const}$

At $t = t_0$ = current time $\dot{R}_0 = H_0$, $R_0 = 1$,

$\therefore \dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -\frac{c^2}{a^2} = \dot{R}_0^2 - \frac{8\pi G}{3}\rho_0 R_0^2$

$= H_0^2 - \frac{8\pi G \rho_0}{3} = H_0^2\left(1 - \frac{8\pi G\rho_0}{3H_0^2}\right)$ $\left(\Omega_{M0} \equiv \frac{8\pi G\rho_0}{3H_0^2}\right)$

$= H_0^2(1 - \Omega_{M0})$

$\therefore \frac{c^2}{a^2} = H_0^2(\Omega_{M0} - 1)$

b) The photon geodesic $d\tau = 0$

$\therefore 0 = -c^2dt^2 + \frac{R^2dr^2}{1 - r^2/a^2} + R^2r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

photon path along $r \Rightarrow d\theta = 0$, $d\phi = 0$

$\therefore c\,dt = \frac{R}{\sqrt{1 - r^2/a^2}}dr$

$\therefore c\int_0^t \frac{dt}{R} = \int_0^r \frac{1}{\sqrt{1 - r^2/a^2}}dr$

$r = a\sin\theta$
$dr = a\cos\theta\, d\theta$
$\theta = \sin^{-1}(\frac{r}{a})$

$= \int \frac{a\cos\theta}{\cos\theta}d\theta$

$= \int a\,d\theta = a\theta \Big|_{r=0}^{r=r}$

$= a[\sin^{-1}(\frac{r}{a}) - \underbrace{\sin^{-1}(0)}_{0}] = a\sin^{-1}(\frac{r}{a})$

$\therefore \quad a\sin^{-1}(\frac{r}{a}) = c\int_0^t \frac{dt}{R}$

$= c\int_{t=0}^{t=t} \frac{d\eta - \cos\eta\, d\eta}{2\frac{c}{a}(1-\Omega_{M0}^{-1})} \frac{2(1-\Omega_{M0}^{-1})}{1-\cos\eta}$

$= a\int_{t=0}^{t=t} \frac{(1-\cos\eta)}{(1-\cos\eta)} d\eta$

$= a\int_{\eta=0}^{\eta=\eta} d\eta$

$= a\eta$

$\therefore \quad a\sin^{-1}(\frac{r}{a}) = a\eta$

$\rightarrow \quad \eta = \sin^{-1}(\frac{r}{a})$

$\therefore \begin{cases} r = a\sin\eta \\ R = \frac{1}{2(1-\Omega_{M0}^{-1})}(1-\cos\eta) \end{cases}$

$\therefore$ As R goes from 0 to maximum then back to 0, η goes from 0 to 2π

~~$\therefore$ As η from $0 \rightarrow 2\pi$, the universe is created and distroyed~~

$\therefore$ The universe exists from $\eta = 0$ to 2π

~~During this time~~

~~$R = 1-\cos\eta$~~

$R = \frac{1-\cos\eta}{2(1-\Omega_{M0}^{-1})}$

$H_0 t = \frac{\eta - \sin\eta}{2\sqrt{\Omega_{M0}(1-\Omega_{M0}^{-1})^3}}$

$t = \frac{\eta - \sin\eta}{2\sqrt{\Omega_{M0} H_0^2 (1-\Omega_{M0}^{-1})}(1-\Omega_{M0}^{-1})}$

$dt = \frac{d\eta - \cos\eta\, d\eta}{2(H_0^2\Omega_{M0}-1)^{\frac{1}{2}}} \frac{1}{(1-\Omega_{M0}^{-1})}$

$(H_0^2\Omega_{M0}-1)^{\frac{1}{2}} = (\frac{c^2}{a^2})^{\frac{1}{2}} = \frac{c}{a}$

During this time r goes from 0 to a and back to 0

∴ a photon could only travel around such a universe once, ~~but~~ during the time when this universe exists. ✓

Could you see the back of your head?   7/8

9/10