To: Will Potter

55/55 100%

B5 Problem Set 3

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Excellent Ziyan!

1. a) $c=1$

$$-d\tau^2 = -B(r,t)\,dt^2 + A(r,t)\,dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$
$$\equiv g_{\mu\nu}\, dx^\mu dx^\nu$$

($g_{\mu\nu}$ only has diagonal terms)

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)$$

$g_{tt} = -B \quad g_{rr} = A \quad g_{\theta\theta} = r^2 \quad g_{\phi\phi} = r^2\sin^2\theta$

$$\therefore \Gamma_{rt}^{r} = \frac{1}{2g_{rr}}\left(\frac{\partial g_{rr}}{\partial x^t}\right)$$

$$= \frac{1}{2g_{rr}}\frac{\partial g_{rr}}{\partial t} = \frac{\dot{A}}{2A}$$

$$\Gamma_{rr}^{t} = \frac{1}{2g_{tt}}\left(-\frac{\partial g_{rr}}{\partial t}\right)$$

$$= -\frac{1}{2(-B)}\dot{A} = \frac{\dot{A}}{2B}$$

$$\Gamma_{tt}^{t} = \frac{1}{2g_{tt}}\left(\frac{\partial g_{tt}}{\partial t} + \cancel{\frac{\partial g_{tt}}{\partial t}} - \cancel{\frac{\partial g_{tt}}{\partial t}}\right)$$

$$= \frac{1}{2g_{tt}}\frac{\partial g_{tt}}{\partial t} \cancel{= \frac{1}{2g_{tt}}\frac{\partial g_{tt}}{\partial t}}$$

$$= \frac{1}{2(-B)}(-\dot{B}) = \frac{\dot{B}}{2B}$$

b)

$$R_{\mu\kappa} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^\kappa \partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda}$$

$$\frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^\kappa \partial x^\nu} = \cancel{\frac{1}{2}\frac{1}{|g|}} = \frac{1}{2}\frac{\partial^2}{\partial x^\kappa \partial x^\nu}\left(\ln|g_{tt}\, g_{rr}\, g_{\theta\theta}\, g_{\phi\phi}|\right)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x^\kappa \partial x^\nu}\left(\ln|g_{tt}| + \ln|g_{rr}| + \ln|g_{\theta\theta}| + \ln|g_{\phi\phi}|\right)$$

$$\frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} = \frac{\partial}{\partial t}\Gamma^t_{\mu\kappa} + \frac{\partial}{\partial r}\Gamma^r_{\mu\kappa} + \frac{\partial}{\partial \theta}\Gamma^\theta_{\mu\kappa} + \frac{\partial}{\partial \phi}\Gamma^\phi_{\mu\kappa}$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{g_{tt}}\left(\frac{\partial g_{t\nu}}{\partial x^\kappa} + \frac{\partial g_{t\kappa}}{\partial x^\nu} - \frac{\partial g_{\mu\kappa}}{\partial t}\right)\right]$$

$$+ \frac{1}{2}\frac{\partial}{\partial r}\left[\frac{1}{g_{rr}}\left(\frac{\partial g_{r\nu}}{\partial x^\kappa} + \frac{\partial g_{r\kappa}}{\partial x^\nu} - \frac{\partial g_{\mu\kappa}}{\partial r}\right)\right]$$

$$+ \frac{1}{2}\frac{\partial}{\partial \theta}\left[\frac{1}{g_{\theta\theta}}\left(\frac{\partial g_{\theta\nu}}{\partial x^\kappa} + \frac{\partial g_{\theta\kappa}}{\partial x^\nu} - \frac{\partial g_{\mu\kappa}}{\partial \theta}\right)\right]$$

$$+ \frac{1}{2}\frac{\partial}{\partial \phi}\left[\frac{1}{g_{\phi\phi}}\left(\frac{\partial g_{\phi\nu}}{\partial x^\kappa} + \frac{\partial g_{\phi\kappa}}{\partial x^\nu} - \frac{\partial g_{\mu\kappa}}{\partial \phi}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial t}\left(\frac{1}{g_{tt}}\frac{\partial g_{t\nu}}{\partial x^\kappa}\right) + \frac{\partial}{\partial r}\left(\frac{1}{g_{rr}}\frac{\partial g_{r\nu}}{\partial x^\kappa}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\nu}}{\partial x^\kappa}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{g_{\phi\phi}}\frac{\partial g_{\phi\nu}}{\partial x^\kappa}\right)\right)$$

$$+ \frac{1}{2}\left(\frac{\partial}{\partial t}\left(\frac{1}{g_{tt}}\frac{\partial g_{t\kappa}}{\partial x^\nu}\right) + \frac{\partial}{\partial r}\left(\frac{1}{g_{rr}}\frac{\partial g_{r\kappa}}{\partial x^\nu}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\kappa}}{\partial x^\nu}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{g_{\phi\phi}}\frac{\partial g_{\phi\kappa}}{\partial x^\nu}\right)\right)$$

$$- \frac{1}{2}\left(\frac{\partial}{\partial t}\left(\frac{1}{g_{tt}}\frac{\partial g_{\mu\kappa}}{\partial t}\right) + \frac{\partial}{\partial r}\left(\frac{1}{g_{rr}}\frac{\partial g_{\mu\kappa}}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\mu\kappa}}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{g_{\phi\phi}}\frac{\partial g_{\mu\kappa}}{\partial \phi}\right)\right)$$

$\kappa = -1$

If $\nu \neq \kappa$ $\because$ $g_{\mu\nu} \neq 0$ only if $\mu = \nu$

$$\therefore \frac{\partial \Gamma^{\lambda}_{\nu\kappa}}{\partial x^{\lambda}} = \frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left(\frac{1}{g_{\nu\nu}}\frac{\partial g_{\nu\nu}}{\partial x^{\kappa}}\right) + \frac{1}{2}\frac{\partial}{\partial x^{\kappa}}\left(\frac{1}{g_{\kappa\kappa}}\frac{\partial g_{\kappa\kappa}}{\partial x^{\nu}}\right)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x^{\nu}\partial x^{\kappa}}\ln|g_{\nu\nu}| + \frac{1}{2}\frac{\partial^2}{\partial x^{\kappa}\partial x^{\nu}}\ln|g_{\kappa\kappa}|$$

$$= \frac{1}{2}\frac{\partial^2 \ln|g_{\nu\nu}g_{\kappa\kappa}|}{\partial x^{\nu}\partial x^{\kappa}}$$

~~If $\nu = \kappa$~~ then ~~$g_{\nu\kappa} = g_{\nu\nu}$~~

$$\cancel{\therefore \frac{\partial \Gamma^{\lambda}_{\nu\kappa}}{\partial x^{\lambda}} = \frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left(\frac{1}{g_{\nu\nu}}\frac{\partial g_{\nu\nu}}{\partial x^{\nu}}\right) + \frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left(\frac{1}{g_{\nu\nu}}\frac{\partial g_{\nu\nu}}{\partial x^{\nu}}\right)}$$

$$\cancel{\frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left(\frac{1}{g_{\nu\nu}}\frac{\partial g_{\nu\nu}}{\partial x^{\nu}}\right)}$$

$\therefore$ If $\nu = t$ $\kappa = r$

$$\frac{\partial \Gamma^{\lambda}_{\nu\kappa}}{\partial x^{\lambda}} = \frac{\partial \Gamma^{\lambda}_{tr}}{\partial x^{\lambda}} = \frac{1}{2}\frac{\partial^2}{\partial r\partial t}\ln|g_{tt}g_{rr}|$$

$$= \frac{1}{2}\frac{\partial^2}{\partial r\partial t}\left(\ln|g_{tt}| + \ln|g_{rr}|\right)$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^{\nu}\partial x^{\kappa}}|g| = \frac{1}{2}\frac{\partial^2}{\partial r\partial t}\left(\ln|g_{tt}| + \ln|g_{rr}| + \ln|g_{\theta\theta}| + \ln|g_{\phi\phi}|\right)$$

$\because$ $g_{\theta\theta}$, $g_{\phi\phi}$ independent of t

$$\therefore \frac{1}{2}\frac{\partial^2}{\partial x^{\nu}\partial x^{\kappa}} = \frac{1}{2}\frac{\partial^2}{\partial r\partial t}\left(\ln|g_{tt}| + \ln|g_{rr}|\right)$$

$$\rightarrow \quad -\frac{\partial \Gamma^{\lambda}_{tr}}{\partial x^{\lambda}} + \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial r \partial t} = 0$$ 4/4

c) $\because$ b) $\therefore$ ~~$R_{tr} = \Gamma^{\eta}$~~ $(|g| = ABr^4\sin^2\theta)$

$$\therefore R_{tr} = \Gamma^{\eta}_{t\lambda}\Gamma^{\lambda}_{r\eta} - \frac{\Gamma^{\eta}_{tr}}{2}\frac{\partial \ln(g)}{\partial x^{\eta}}$$

$$= \Gamma^{t}_{tt}\Gamma^{t}_{rt} + \Gamma^{r}_{tt}\Gamma^{t}_{rr} + \Gamma^{t}_{tr}\Gamma^{r}_{rt} + \Gamma^{r}_{tr}\Gamma^{r}_{rr}$$

$$- \frac{\Gamma^{t}_{tr}}{2}\frac{\partial \ln|g|}{\partial t} - \frac{\Gamma^{r}_{tr}}{2}\frac{\partial \ln|g|}{\partial r}$$

$$= \frac{\dot{B}}{2B}\frac{B'}{2B} + \frac{B'}{2A}\frac{\dot{A}}{2B} + \frac{B'}{2B}\frac{\dot{A}}{2A} + \frac{\dot{A}}{2A}\frac{A'}{2A}$$

$$- \frac{1}{2}\frac{B'}{2B}\frac{1}{ABr^4\cancel{\sin^2\theta}}\cancel{\sin^2\theta}\frac{\partial}{\partial t}(ABr^4)$$

$$- \frac{1}{2}\frac{\dot{A}}{2A}\frac{1}{ABr^4\cancel{\sin^2\theta}}\cancel{\sin^2\theta}\frac{\partial}{\partial r}(ABr^4)$$

$$= \frac{\dot{B}}{2B}\frac{B'}{2B} + \frac{B'}{2A}\frac{\dot{A}}{2B} + \frac{B'}{2B}\frac{\dot{A}}{2A} + \frac{\dot{A}}{2A}\frac{A'}{2A}$$

$$- \frac{1}{2}\frac{B'}{2B}\frac{1}{AB\cancel{r^4}}(\dot{A}B + \dot{B}A)$$

$$- \frac{1}{2}\frac{\dot{A}}{2A}\frac{1}{ABr^4}(r^4A'B + r^4B'A + 4r^3AB)$$

$$= \cancel{\frac{\dot{B}}{2B}\frac{B'}{2B}} + \cancel{\frac{B'}{2A}\frac{\dot{A}}{2B}} + \frac{B'}{2B}\frac{\dot{A}}{2A} + \cancel{\frac{\dot{A}}{2A}\frac{A'}{2A}}$$

$$- \cancel{\frac{B'}{2B}\frac{\dot{A}}{2A}} - \cancel{\frac{B'}{2B}\frac{\dot{B}}{2B}} - \cancel{\frac{\dot{A}}{2A}\frac{A'}{2A}} - \cancel{\frac{\dot{A}}{2A}\frac{B'}{2B}} - \frac{\dot{A}}{2A}\cdot\frac{1}{2}\cdot\frac{4r^3}{r^4}$$

$$= -\frac{\dot{A}}{rA}$$

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2. a)

Gravitational field equation should take the form

$$G_{\mu\nu} = C T_{\mu\nu}$$

$T_{\mu\nu}$, the stress energy tensor, is the source of the gravitational ~~to~~ field

$G_{\mu\nu}$ must be linear in the second derivative of $g_{\mu\nu}$ ~~to~~ so that poisson equation is recovered

→ Now for a linear theory, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ~~##~~ $(|h_{\mu\nu}| \ll 1)$

then the LHS of field equation must be a linear combination of second order derivatives of $h_{\mu\nu}$

For scalar second order derivatives, we need to multiply by $\eta_{\mu\nu}$, the only available tensor except $h_{\mu\nu}$, to ~~term~~ turn the term into a 2nd rank tensor.

~~...~~ Also, ∵ $T_{\mu\nu}$ is symmetric ✓

∴ We need 2nd order derivatives in ~~#~~ $h_{\mu\nu}$ that are symmetric

→ only possible terms are

$\Box h_{\mu\nu}$, $\partial_\mu \partial_\nu h$, $(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu)$, $\eta_{\mu\nu} \Box h$, $\eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda}$

✓ 3/3

b)

Conservation of energy :

covariant derivative of $T_{\mu\nu}$ vanish

$$T_{\mu\nu;\mu} = 0$$

Now in ~~weak~~ weak field, we replace covariant derivative with ordinary partial derivative

$$\Rightarrow \quad \partial^\mu T_{\mu\nu} = 0$$

$$\therefore \quad \partial^\mu \Big(\Box h_{\mu\nu} + \alpha(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu) + \beta(\partial_\mu \partial_\nu h) + \eta_{\mu\nu}(\gamma \Box h + \delta \partial_\rho \partial_\lambda h^{\rho\lambda}) \Big) = 0$$

$$\therefore \quad \partial_\rho \partial^\rho \partial^\mu h_{\mu\nu} + \alpha \partial_\rho \partial_\mu \partial^\mu h^\rho_\nu + \alpha \partial_\rho \partial_\nu \partial^\mu h^\rho_\mu + \beta \partial_\mu \partial^\mu \partial_\nu h + \gamma \underset{\partial_\nu \partial^\mu}{\underbrace{\eta_{\mu\nu}}} \partial_\rho \partial^\rho h + \delta \underset{\partial_\nu \partial^\mu}{\underbrace{\eta_{\mu\nu}}} \partial_\rho \partial_\lambda h^{\rho\lambda} = 0$$

$\rightarrow$ consider $\partial_\rho \partial^\rho \partial^\mu h_{\mu\nu} + \alpha \partial_\rho \partial_\mu \partial^\mu h^\rho_\nu$

$$= \partial_\rho \partial^\rho \partial^\mu h_{\mu\nu} + \alpha \partial_\rho \partial^\rho \partial_\mu h^\mu_\nu \quad \rightarrow \text{switching } \rho, \mu$$

$$\cancel{= \partial_\rho \partial^\rho (\partial^\mu h_{\mu\nu} + \alpha \eta^{\mu\kappa} \partial_\mu h_{\kappa\nu})}$$

$$= \partial_\rho \partial^\rho (\partial^\mu h_{\mu\nu} + \alpha \eta^{\mu\kappa} \partial_\mu h_{\kappa\nu})$$

$$= \partial_\rho \partial^\rho (\partial^\mu h_{\mu\nu} + \alpha \partial^\kappa h_{\kappa\nu})$$

$$= \partial_\rho \partial^\rho (\partial^\mu h_{\mu\nu} + \alpha \underset{\kappa \to \mu}{\partial^\mu h_{\mu\nu}}) = (1+\alpha) \partial_\rho \partial^\rho (\partial^\mu h_{\mu\nu})$$

$$= (1+\alpha) \Box (\partial^\mu h_{\mu\nu})$$

→ consider

$$\alpha \partial_\rho \partial_\nu \partial^\mu h^\rho_\mu + \delta \underbrace{\eta_{\mu\nu} \partial^\mu}_{\partial_\nu} \partial_\rho \partial_\lambda h^{\rho\lambda}$$

$$= \alpha \partial_\rho \partial_\nu \underbrace{\partial^\mu \eta_{\lambda\mu}}_{\partial_\lambda} h^{\rho\lambda} + \cancel{\delta \partial_\nu \partial^\mu} \, \delta \partial_\nu \partial_\rho \partial_\lambda h^{\rho\lambda}$$

$$= \alpha \partial_\rho \partial_\nu \partial_\lambda h^{\rho\lambda} + \delta \partial_\nu \partial_\rho \partial_\lambda h^{\rho\lambda}$$

$$= (\alpha + \delta) \partial_\rho \partial_\nu \partial_\lambda h^{\rho\lambda}$$

→ consider

$$\beta \partial_\mu \partial^\mu \partial_\nu h + \cancel{\partial^\mu \Box} \, \underbrace{\partial^\mu \eta_{\mu\nu}}_{\partial_\nu} \gamma \Box h$$

$$= \beta \partial_\mu \partial^\mu \partial_\nu h + \gamma \partial_\nu \partial_\mu \partial^\mu h$$

$$= (\beta + \gamma) \partial_\mu \partial^\mu \partial_\nu h$$

$$\therefore \quad (1+\alpha) \Box (\partial^\mu h_{\mu\nu}) + (\beta+\gamma) \partial_\mu \partial^\mu \partial_\nu h + (\alpha+\delta) \partial_\rho \partial_\nu \partial_\lambda h^{\rho\lambda}$$

$= 0$ for all metric $h_{\mu\nu}$

$$\therefore \quad \begin{cases} 1+\alpha = 0 \\ \beta+\gamma = 0 \\ \alpha+\delta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = -1 \\ \delta = 1 \\ \gamma = -\beta \end{cases}$$

✓ 5/5

c)

$\cancel{\Box h_{\mu\nu} - \partial_\rho\partial_\mu \partial^\rho + \partial}$

$$\Box h_{\mu\nu} - (\partial_\rho\partial_\mu h^\rho{}_\nu + \partial_\rho\partial_\nu h^\rho_\mu) + \beta\partial_\mu\partial_\nu h - \eta_{\mu\nu}(\beta\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda})$$
$$= CT_{\mu\nu} \quad ①$$

Now raise ν and take trace ~~(set $\nu=\mu$)~~ :

$\cancel{\Box h = 2\partial_\rho\partial_\mu h^{\rho\mu} + \beta}$

multiply ① by $\eta^{\nu\kappa}$ $\Rightarrow$

$\therefore$ $\cancel{\Box h^\kappa_\nu = \partial_\rho\partial^\kappa h}$

$$CT^\kappa_\nu = \Box h^\kappa_\nu - 2\partial_\rho\partial_\nu h^{\rho\kappa} + \beta\partial_\nu\partial^\kappa h - \underbrace{\eta^{\nu\kappa}\eta_{\mu\nu}}_{\delta^\kappa_\nu}(\beta\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda})$$

take trace (set $\kappa = \nu = \mu$)

$$\Rightarrow \Box h - 2\partial_\rho\partial_\lambda h^{\rho\lambda} + \beta\partial_\mu\partial^\mu h - \delta^\mu_\mu(\beta\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda}) = CT$$

$$\Rightarrow \Box h - 2\partial_\rho\partial_\lambda h^{\rho\lambda} + \beta\Box h - 4\beta\Box h + 4\partial_\rho\partial_\lambda h^{\rho\lambda} = CT$$

$$\Rightarrow (1+\beta-4\beta)\Box h + (-2+4)\partial_\rho\partial_\lambda h^{\rho\lambda} = CT$$

$$\therefore 2\partial_\rho\partial_\lambda h^{\rho\lambda} = (3\beta - 1)\Box h + CT$$

$$\therefore \partial_\rho\partial_\lambda h^{\rho\lambda} = \frac{3\beta-1}{2}\Box h + \frac{C}{2}T$$

Now $T = T^\mu_\mu = T^0_0 + T^1_1 + T^2_2 + T^3_3$

$$= \eta^{00}T_{00} + \eta^{11}T_{11} + \eta^{22}T_{22} + \eta^{33}T_{33}$$

$$= -T_{00} + T_{11} + T_{22} + T_{33}$$

$\because$ $T_{\mu\nu} = -P g_{\mu\nu} + (\rho + \frac{P}{c^2}) U_\mu U_\nu$

$\because$ $U_0 = \frac{d(ct)}{d\tau}$, $U_i = \frac{dx_i}{d\tau}$

Newtonian limit , $d(ct) \gg dx_i$ $\therefore U_0 \gg U_i$

Also $P/c^2 \ll \rho$, $g_{\mu\nu} \approx \eta_{\mu\nu}$

$\therefore$ $\cancel{T_{\mu\nu} = -P\eta_{\mu\nu} + \rho U_\mu U_\nu \approx \rho U_\mu U_\nu}$

$\cancel{T_{00} = P + \rho U_0 U_0}$

$\cancel{T_{ii} = -P}$

$\because \frac{P}{c^2} \ll \rho$ $\therefore$ $T_{\mu\nu} \approx \rho U_\mu U_\nu$ (dust)

$\because U_0 \gg U_i$ $\therefore$ $T_{00} \gg T_{ii}$

$\therefore$ $T \approx -T_{00}$

$\therefore$ $\partial_\rho \partial_\lambda h^{\rho\lambda} = \frac{3\beta - 1}{2} \Box h - \frac{C T_{00}}{2}$ good

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$T^{\mu\nu} = \rho U^\mu U^\nu$ $T^{\mu\nu} = (\rho + P) U^\mu U^\nu + P g^{\mu\nu}$

$$= \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

$T^{\mu\nu} = \sum_i \rho_i U^\mu U^\nu$

$U^0 U^0$ always positive $\Rightarrow$ sum up

$\therefore$ to have $T_{ii} \approx T_{00}$

need $U^i \approx U^0 \Rightarrow$ relativistic gas.

gives

d)

$$\Box h_{\mu\nu} - (\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu) + \beta \partial_\mu \partial_\nu h$$

$$- \eta_{\mu\nu} (\beta \Box h - \partial_\rho \partial_\lambda h^{\rho\lambda}) = C T_{\mu\nu}$$

Set $\mu = 0$, $\nu = 0$

Newtonian static limit : $\frac{\partial}{\partial t} = 0$, $\eta_{00} = -1$

$\therefore \Box h_{00} + \beta \Box h - \partial_\rho \partial_\lambda h^{\rho\lambda} = C T_{00}$

$\because \partial_\rho \partial_\lambda h^{\rho\lambda} = \frac{3\beta - 1}{2} \Box h - \frac{C T_{00}}{2}$ (Newtonian limit)

$\therefore \Box h_{00} + \beta \Box h - (\frac{3\beta}{2} + \frac{1}{2}) \Box h + \frac{C T_{00}}{2} = C T_{00}$

$\therefore \Box h_{00} + \frac{1-\beta}{2} \Box h = \frac{C}{2} T_{00}$

$\because \frac{\partial}{\partial t} = 0$ $\therefore \Box \to \nabla^2$

$\therefore \nabla^2 h_{00} + \frac{1-\beta}{2} \nabla^2 h = \frac{C}{2} T_{00}$ (2)

In Newtonian limit, from Newtonian geodesic

we have $h_{00} = -\frac{2\Phi}{c^2}$, $\nabla^2 \Phi = 4\pi G \rho$, $T_{00} = \rho c^2$

$\therefore \nabla^2 \Phi = 4\pi G \rho$ becomes

$-\frac{c^2}{2} \nabla^2 h_{00} = \frac{4\pi G}{c^2} T_{00} \Rightarrow \nabla^2 h_{00} = -\frac{8\pi G}{c^4} T_{00}$

$\because c = 1$ $\therefore \nabla^2 h_{00} = -8\pi G T_{00}$ (3)

Compare ② with ③, we have that

$$\frac{1-\beta}{2} = 0 \quad , \quad \frac{C}{2} = \cancel{-8\pi G} \; -8\pi G$$

$$\Rightarrow \quad \beta = 1 \checkmark \, , \quad C = -16\pi G \checkmark \quad 7/7$$

e) ~~The general~~

In notes:

(339) → the linear wave

(340) → the source term consists of $T_{\mu\nu}$ and $\tau_{\mu\nu}$, the stress energy contribution of the gravitational radiation itself

(348) → the Gauge transformation that keeps the ~~equation~~ wave equation invariant.

e)

The general field equation needs to be sourced by $T_{\mu\nu}$, and ~~on~~ its LHS should be linear in 2nd order derivatives of $g_{\mu\nu}$ ~~to recover~~ in weak field to recover ~~the~~ Newton's Laws.

∵ $R_{\mu\nu}$ and $g_{\mu\nu} R$ ~~one~~ are only 2 available tensors

over

$\therefore$ need $aR_{\mu\nu} + \cancel{b}\, b g_{\mu\nu} R = C' T_{\mu\nu}$

raise both $\mu, \nu \Rightarrow a R^{\mu\nu} + b g^{\mu\nu} R = C T^{\mu\nu}$

Covariant derivative, $\because T^{\mu\nu}_{;\mu} = 0$

$$\therefore (aR^{\mu\nu} + b g^{\mu\nu} R)_{;\mu} = 0$$

Also from Bianchi identity

$$(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R)_{;\mu} = 0$$

$$\therefore \underbrace{(aR^{\mu\nu} - \tfrac{a}{2} g^{\mu\nu} R)_{;\mu}}_{=0} + (\tfrac{a}{2} g^{\mu\nu} R + b g^{\mu\nu} R)_{;\mu} = 0$$

$$\Rightarrow (\tfrac{a}{2} + b)(g^{\mu\nu} R)_{;\mu} = 0$$

In general $(g^{\mu\nu} R)_{;\mu} \neq 0$

$$\therefore \tfrac{a}{2} + b = 0 \Rightarrow b = -\tfrac{a}{2}$$

$$\therefore a(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R) = c' T_{\mu\nu} \qquad \text{let } c = \frac{c'}{a}$$

$$\therefore R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = C T_{\mu\nu}$$

Following Chapter 6 on the Notes we see that to recover Newtonian Gravity in weak curvature, $C = -8\pi G$

$$\therefore R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$ 3/3

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3. a)

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 h_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right)$$

$$h_{\mu\nu} = A_{\mu\nu} \exp(ik_\rho x^\rho)$$

$$\therefore R_{\lambda\mu\nu\kappa} = \frac{1}{2}\Big((ik_\kappa)(ik_\mu) h_{\lambda\nu} - (ik_\kappa)(ik_\lambda) h_{\mu\nu} - (ik_\mu)(ik_\nu) h_{\lambda\kappa}$$

$$+ (ik_\nu)(ik_\lambda) h_{\mu\kappa}\Big)$$

$$= \frac{1}{2}\left(-k_\kappa k_\mu h_{\lambda\nu} + k_\kappa k_\lambda h_{\mu\nu} + k_\mu k_\nu h_{\lambda\kappa} - k_\nu k_\lambda h_{\mu\kappa}\right)$$

linear vacuum field equation

$$R^{(1)}_{\mu\nu} = 0 \Rightarrow$$

$$\frac{1}{2}\left(\frac{\partial^2 h}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda_\mu}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h^\lambda_\nu}{\partial x^\mu \partial x^\lambda} + \Box h_{\mu\nu}\right) = 0$$

$$\therefore -k_\mu k_\nu h + k_\nu k_\lambda h^\lambda_\mu + k_\mu k_\lambda h^\lambda_\nu - k_\rho k^\rho h_{\mu\nu} = 0$$

$$\because k_\lambda h^\lambda_\mu = k^\lambda h_{\lambda\mu}, \quad k_\nu k^\lambda = \eta_{\nu\lambda} k^\nu k^\alpha = k^\mu k_\lambda$$

$$\Rightarrow \cancel{0 = k_\rho k_\lambda \eta^{\lambda\rho} h^\lambda_\nu}$$

$$\therefore \Rightarrow 0 = k_\nu k^\lambda h_{\lambda\mu} - k_\mu k_\nu \frac{h}{2} + k_\mu k^\lambda h_{\lambda\nu} - k_\mu k_\nu \frac{h}{2} - k^2 h_{\mu\nu}$$

$$\cancel{= k_\nu k^\lambda h_{\lambda\mu} - k_\nu \eta_{\nu\lambda} k^\lambda \frac{h}{2} + k_\mu k^\lambda h_{\lambda\nu} - k_\mu \eta_{\nu\lambda} k^\lambda \frac{h}{2} + k^2 h_{\mu\nu}}$$

$$\cancel{= k_\lambda k^\nu h_{\lambda\mu} - k_\lambda}$$

$$\cancel{= k_\nu k^\lambda h_{\lambda\nu} - k^\mu \eta}$$

$$= k_\nu k^\lambda h_{\lambda\mu} - \eta_{\lambda\mu} k^\lambda k_\nu \frac{h}{2} + k_\mu k^\lambda h_{\lambda\nu} - \eta_{\lambda\nu} k_\mu k^\nu \frac{h}{2}$$
$$- k^2 h_{\mu\nu}$$

$$= k_\nu k^\lambda \left(h_{\lambda\mu} - \eta_{\lambda\mu}\frac{h}{2}\right) + k_\mu k^\lambda \left(h_{\lambda\nu} - \eta_{\lambda\nu}\frac{h}{2}\right) - k^2 h_{\mu\nu}$$

$$= k_\nu k^\lambda \bar{h}_{\lambda\mu} + k_\mu k^\lambda \bar{h}_{\lambda\nu} - k^2 h_{\mu\nu}$$

$\lambda \to \rho$
$\nu \to \kappa$

$$\Rightarrow \underline{\underline{k_\kappa k^\rho \bar{h}_{\rho\mu} + k_\mu k^\rho \bar{h}_{\rho\kappa} - k^2 h_{\mu\kappa} = 0}}$$ ✓ 5/5

b) if $k^2 \neq 0$, $h_{\mu\kappa} = \frac{1}{k^2}\left(k_\kappa k^\rho \bar{h}_{\rho\mu} + k_\mu k^\rho \bar{h}_{\rho\kappa}\right)$ ✓

$$\therefore R_{\lambda\mu\nu\kappa} = \frac{1}{2k^2}\Big[-k_\kappa k_\mu \cancel{k_\lambda k^\rho \bar{h}_{\rho\nu}} - k_\kappa k_\mu \cancel{k_\nu k^\rho \bar{h}_{\rho\lambda}}$$

$$+ k_\kappa k_\lambda \cancel{k_\mu k^\rho \bar{h}_{\rho\nu}} + k_\kappa k_\lambda \cancel{k_\nu k^\rho \bar{h}_{\rho\mu}}$$
$$+ k_\mu k_\nu \cancel{k_\lambda k^\rho \bar{h}_{\rho\kappa}} + k_\mu k_\nu \cancel{k_\kappa k^\rho h_{\rho\lambda}}$$
$$- k_\nu k_\lambda \cancel{k_\mu k^\rho \bar{h}_{\rho\kappa}} - k_\nu k_\lambda \cancel{k_\kappa k^\rho \bar{h}_{\rho\mu}} \Big] = 0$$ ✓

wave No curvature

~~No~~ so No energy carried.

just wavy coordinates.

3/3

$$= k_\nu k^\lambda h_{\lambda\mu} - \eta_{\lambda\mu} k^\lambda k_\nu \frac{h}{2} + k_\mu k^\lambda h_{\lambda\nu} - \eta_{\lambda\nu} k_\mu k^\nu \frac{h}{2}$$
$$- k^2 h_{\mu\nu}$$

$$= k_\nu k^\lambda (h_{\lambda\mu} - \eta_{\lambda\mu} \tfrac{h}{2}) + k_\mu k^\lambda (h_{\lambda\nu} - \eta_{\lambda\nu} \tfrac{h}{2}) - k^2 h_{\mu\nu}$$

$$= k_\nu k^\lambda \bar{h}_{\lambda\mu} + k_\mu k^\lambda \bar{h}_{\lambda\nu} - k^2 h_{\mu\nu}$$

$\lambda \to \rho$
$\nu \to \kappa$

$$\Rightarrow \underline{\underline{k_\kappa k^\rho \bar{h}_{\rho\mu} + k_\mu k^\rho \bar{h}_{\rho\kappa} - k^2 h_{\mu\kappa} = 0}}$$

5/5

b) if $k^2 \neq 0$, $h_{\mu\kappa} = \frac{1}{k^2}(k_\kappa k^\rho \bar{h}_{\rho\mu} + k_\mu k^\rho \bar{h}_{\rho\kappa})$

$$\therefore R_{\lambda\mu\nu\kappa} = \frac{1}{2k^2}\Big[-k_\kappa k_\mu \cancel{k_\lambda} k^\rho \bar{h}_{\rho\nu} - k_\kappa k_\mu \cancel{k_\nu} k^\rho \bar{h}_{\rho\lambda}$$

$$+ k_\kappa k_\lambda \cancel{k_\mu} k^\rho \bar{h}_{\rho\nu} + k_\kappa k_\lambda \cancel{k_\nu} k^\rho \bar{h}_{\rho\mu}$$

$$+ k_\mu k_\nu \cancel{k_\lambda} k^\rho \bar{h}_{\rho\kappa} + k_\mu k_\nu \cancel{k_\kappa} k^\rho \bar{h}_{\rho\lambda}$$

$$- k_\nu k_\lambda \cancel{k_\mu} k^\rho \bar{h}_{\rho\kappa} - k_\nu k_\lambda \cancel{k_\kappa} k^\rho \bar{h}_{\rho\mu} \Big] \underline{\underline{= 0}}$$

3/3

wave No curvature

~~No~~ so no energy carried.

just wavy coordinates.

c) if wave propagating at speed of light, we have $k^2 = 0$

$\therefore \quad k_\kappa k^\rho \bar{h}_{\rho\nu} + k_\nu k^\rho \bar{h}_{\rho\kappa} = 0$

Now set $\kappa = \nu = \sigma$ (we look at the diagonal terms of the LHS tensor)

$\Rightarrow \quad 2k_\sigma k^\rho \bar{h}_{\rho\sigma} = 0$ (No sum over σ)

$\because$ k_σ is generally non-zero

$\therefore \quad \underline{\underline{k^\rho \bar{h}_{\rho\sigma} = 0}} \rightarrow$ Harmonic gauge

good 4/4

12/12

giver

4. $\langle L_{GW} \rangle = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 M}{a^5}\left[\frac{1+\frac{73}{24}\epsilon^2+\frac{37}{96}\epsilon^4}{(1-\epsilon^2)^{7/2}}\right]$

$E_{GW} = \langle L_{GW} \rangle T$ ← orbital period

~~t~~ As orbit approaches parabolic

$L = a(1-\epsilon^2)$, $r(1+\cos\phi) = L$, $\epsilon \to 1$

Distance of ~~clos~~ closest approach occurs at $\phi = 0$

$\to \quad 2b = L$, ~~$\epsilon \to 1$~~

$\langle L_{GW} \rangle = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 M}{a^{3/2}}\left[\frac{1+\frac{73}{24}+\frac{37}{96}}{L^{7/2}}\right]$

$= \left(\frac{32}{5}\right)\left(\frac{425}{96}\right)\frac{G^4}{c^5}\frac{m_1^2 m_2^2 M}{a^{3/2}}\frac{1}{(2b)^{7/2}}$

$= \frac{85}{3}\frac{G^4}{c^5}\frac{1}{2^{7/2}}\frac{m_1^2 m_2^2 M}{a^{3/2} b^{7/2}}$

$\because \quad T = 2\pi\sqrt{\frac{a^3}{GM}} = 2\pi\frac{a^{3/2}}{G^{1/2}M^{1/2}}$

$\therefore \quad E_{GW} = \langle L_{GW} \rangle T = \frac{85}{3}\frac{2}{2^{7/2}}\frac{G^{7/2}}{c^5}\frac{m_1^2 m_2^2 M^{1/2}}{b^{7/2}}$

$= \frac{85\pi\sqrt{2}}{24}\frac{G^{7/2}M^{1/2}m_1^2 m_2^2}{c^5 b^{7/2}}$ 5/5