BS 2016

(1) the schwarzschild metric

$$-c^2 dT^2 = -\underbrace{c(1-\frac{2GM}{rc^2})}_{g_{tt}} dt^2 c^2 + \underbrace{(1-\frac{2GM}{rc^2})^{-1}}_{g_{rr}} dr^2$$

$$+ \cancel{r^2 \sin^2\theta d\theta} \underbrace{r^2}_{g_{\theta\theta}} d\theta^2 + \underbrace{r^2\sin^2\theta}_{g_{\phi\phi}} d\phi^2$$

$$-c^2 dT^2 = g_{\mu\nu} X^\mu X^\nu$$

Variational method :

Consider the Lagrangian $\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{dp}$ where p is an arbitrary parameter of time

then the Euler Lagrange equation gives

$$\frac{d}{dp}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\nu}\right) = \frac{\partial \mathcal{L}}{\partial x^\nu}$$

For $\nu = 2$, the θ equation

$$\frac{d}{dp}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\Rightarrow \frac{d}{dp}(2r^2\dot\theta) = 2r^2\dot\phi^2 \sin\theta\cos\theta$$

$$2r^2\ddot\theta + 4r\dot r\dot\theta = 2r^2\dot\phi^2\sin\theta\cos\theta$$

$$\therefore \ddot\theta + \frac{2}{r}\dot r\dot\theta - \sin\theta\cos\theta\,\dot\phi^2 = 0$$

→ We can fix $\theta = \frac{\pi}{2}$ and the above equation is satisfied

The t-equation :

$$\frac{d}{dp}\left(\frac{\partial L}{\partial \dot{(tc)}}\right) = \frac{\partial L}{\partial (tc)}$$

$$\therefore \frac{d}{dp}(-2 g_{tt} c\dot{t}) = 0$$

$$\therefore \frac{d}{dp}(g_{tt} c\dot{t}) = 0 \quad ①$$

$$\therefore \cancel{\frac{d^2(ct)}{dp^2} + \frac{dg_{tt}}{dp} = 0}$$

$$\therefore \cancel{c\ddot{t} + }$$

the r equation : with $\sin\theta = 1$, $d\theta = 0$

$$\frac{d}{dp}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\therefore \frac{d}{dp}(2 g_{rr}\dot{r}) = (c\dot{t})^2 \frac{dg_{tt}}{dr} + \dot{r}^2 \frac{dg_{rr}}{dr} + 2\dot{\theta}^2 r$$

$$\therefore 2 g_{rr}\ddot{r} + 2\dot{r}\frac{dg_{rr}}{dr}\dot{r} = (c\dot{t})^2 \frac{dg_{tt}}{dr} + \dot{r}^2\frac{dg_{rr}}{dr} + 2\dot{\phi}^2 r$$

let $\frac{dg_{rr}}{dr} = g_{rr}'$ $\quad \frac{dg_{tt}}{dr} = g_{tt}'$

then $$\ddot{r} + \frac{g_{rr}'}{2g_{rr}}\dot{r}^2 - \frac{g_{tt}'}{2g_{rr}}(c\dot{t})^2 - \frac{\dot{\phi}^2 r}{g_{rr}} = 0 \quad ②$$

the ϕ-equation :

$$\frac{d}{dp}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \quad \therefore \frac{d}{dp}(r^2\dot{\phi}) = 0 \quad ③$$

$\therefore$ Define constants $\quad r^2\dot\phi = J$

choose $\dot t = \frac{dt}{dp} = -g_{tt}^{-1}$ so that p and t are the same when relativistic effect is small

then multiply ② by $2g_{rr}\dot r$ gives

$$2g_{rr}\dot r\ddot r + g_{rr}'\dot r^3 - g_{tt}'\dot r(c\dot t)^2 - 2\dot\phi^2 r\dot r = 0$$

$\because \dot t = -g_{tt}^{-1}$, $\quad \dot\phi = \frac{J}{r^2}$

$$\therefore 2g_{rr}\dot r\ddot r + g_{rr}'\dot r^3 - g_{tt}'\dot r\frac{c^2}{g_{tt}^2} - \frac{2J\dot r}{r^3} = 0$$

$$= \frac{d}{dp}(g_{rr}\dot r^2) + \frac{d}{dp}\left(\frac{c^2}{g_{tt}}\right) + \frac{d}{dp}\left(\frac{J^2}{r^2}\right)$$

$$\therefore 0 = \frac{d}{dp}\left[g_{rr}\dot r^2 + \frac{J^2}{r^2} + \frac{c^2}{g_{tt}}\right]$$

$\therefore$ define constant $-E = g_{rr}\dot r^2 + \frac{J^2}{r^2} + \frac{c^2}{g_{tt}}$

The geodesic

$$-c^2\left(\frac{d\tau}{dp}\right)^2 = g_{tt}c^2\left(\frac{dt}{dp}\right)^2 + g_{rr}\left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\phi}{dp}\right)^2$$

$\frac{1}{g_{tt}^2}$ $\quad \frac{J^2}{r^4}$

$$= \frac{c^2}{g_{tt}} + g_{rr}\dot r^2 + \frac{J^2}{r^2}$$

$$= -E$$

$$\therefore \left(\frac{d\tau}{dp}\right)^2 = \frac{E}{c^2} \qquad \because \frac{dt}{dp} = -\frac{1}{g_{tt}}$$

$$\therefore \frac{d\tau}{dp} = \frac{\sqrt{|E|}}{c}$$

$$\therefore \frac{dt}{d\tau} = \frac{dt/dp}{d\tau/dp} = -\frac{c}{g_{tt}\sqrt{|E|}}$$

$$\frac{d\phi}{d\tau} = \frac{d\phi/dp}{\cancel{d\phi/d} \, d\tau/dp} = \frac{J/r^2}{\sqrt{|E|}/c} = \frac{cJ}{\sqrt{|E|}\, r^2}$$

$$-c^2 = g_{tt} c^2 \left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2$$

$$\therefore -c^2 = g_{tt} c^2 \cdot \frac{c^2}{g_{tt}^2 E} + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + r^2 \frac{c^2 J^2}{E r^4}$$

$$\therefore -c^2 = \frac{c^4}{g_{tt} E} + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + \cancel{r^2} \frac{c^2 J^2}{E r^2}$$

$\because g_{tt} g_{rr} = -1$

$$\therefore \left(\frac{dr}{d\tau}\right)^2 + \frac{c^2}{g_{rr}}\left(1 + \frac{J^2}{E r^2}\right) = \frac{c^4}{E}$$

$$\therefore \left(\frac{dr}{d\tau}\right)^2 = \frac{c^4}{E} - \frac{c^2}{g_{rr}}\left(1 + \frac{J^2}{E r^2}\right)$$

let $u = \frac{1}{r}$, $\frac{du}{d\phi} = \frac{d}{d\phi}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\phi}$

$$\cancel{\therefore \frac{du}{d\phi} \quad \frac{d^2u}{d\phi^2} = \frac{d}{d\phi}\left(-\frac{1}{r^2}\frac{dr}{d\phi}\right)}$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\tau}\right)^2\left(\frac{d\tau}{d\phi}\right)^2$$

$$= \frac{1}{r^4}\left(\frac{c^4}{E} - \frac{c^2}{g_{rr}}\left(1 + \frac{J^2}{E r^2}\right)\right)\left(\frac{r^4 E}{c^2 J^2}\right)$$

$$= \left[\frac{c^4}{r^4 E} - \frac{c^2}{r^4 g_{rr}}\left(1 + \frac{J^2}{E r^2}\right)\right]\left[\frac{r^4 E}{c^2 J^2}\right]$$

$$= \frac{c^2}{J^2} - \frac{E}{g_{rr}J^2}\left[1 + \frac{J^2}{Er^2}\right]$$

$$= \frac{c^2}{J^2} - \left(\frac{E}{J^2} + \frac{1}{r^2}\right)\left(1 - \frac{2GM}{rc^2}\right)$$

$$= \left(\frac{c^2}{J^2} - \frac{E}{J^2}\right) - \frac{1}{r^2} + \frac{2GME}{c^2J^2}\frac{1}{r} + \frac{2GM}{c^2}\frac{1}{r^3}$$

$$= \left(\frac{c^2 - E}{J^2}\right) - \cancel{1}u^2 + \frac{2GME}{c^2J^2}\cancel{\phi}u + \frac{2GM}{c^2}u^3$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 + u^2 = \left(\frac{c^2-E}{J^2}\right) + \left(\frac{2GME}{c^2J^2}\right)u + \left(\frac{2GM}{c^2}\right)u^3$$

Differentiate this. $\Rightarrow$

$$\cancel{2\left(\frac{du}{d\phi}\right)^2 + 2u\frac{du}{d\phi}}$$

$$2\left(\cancel{\frac{du}{d\phi}}\right)\left(\frac{d^2u}{d\phi^2}\right) + 2u\left(\cancel{\frac{du}{d\phi}}\right) = \left(\frac{2GME}{c^2J^2}\right)\cancel{\frac{du}{d\phi}} + \left(\frac{6GM}{c^2}\right)u^2\cancel{\frac{du}{d\phi}}$$

$$\therefore \boxed{\frac{d^2u}{d\phi^2} + u = \underbrace{\left(\frac{GME}{c^2J^2}\right)}_{A} + \underbrace{\left(\frac{3GM}{c^2}\right)}_{B}u^2}$$

~~For circular orbit $du = 0$ ($\because dr = 0$)~~

$$\cancel{\therefore u = \frac{GME}{c^2J^2} + \left(\frac{3GM}{c^2}\right)u^2}$$

$$\cancel{u = A + Bu^2} \qquad \cancel{\therefore u = \frac{1}{r} = \frac{1}{R} \quad (r = R)}$$

$$\cancel{\therefore \frac{1}{R} = A + \frac{B}{R^2}}$$

~~$V_{circ} = R\frac{d\phi}{d\tau}$~~

~~The~~ ~~$\frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{cJ}{\sqrt{B}r^2}\left(-\frac{g_{tt}\sqrt{B}}{c}\right)$~~

~~$= \frac{-g_{tt}J}{r^2}$~~

circular orbit : $dr = 0$ $\therefore \dot{r} = 0$ $\ddot{r} = 0$

$\therefore$ r-equation gives

$$-\frac{g_{tt}'}{2g_{rr}}(c\dot{t})^2 - \frac{\dot{\phi}^2 r}{g_{rr}} = 0$$

$$\therefore -\frac{g_{tt}'c^2}{2\cancel{g_{rr}}}\cancel{\left(\frac{dt}{dp}\right)^2} = \frac{r}{\cancel{g_{rr}}}\left(\frac{d\phi}{dt}\right)^2\cancel{\left(\frac{dt}{dp}\right)^2}$$

$$\therefore \left(\frac{d\phi}{dt}\right)^2 = \frac{-g_{tt}'c^2}{2r}$$

$$g_{tt} = -\left(1-\frac{2GM}{rc^2}\right)$$

$$-g_{tt}' = \frac{2GM}{r^2c^2}$$

$$\therefore \left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r^3} \quad \therefore \frac{d\phi}{dt} = \frac{\sqrt{GM}}{r^{3/2}}$$

$\because$ for $r = R = \text{const}$

the ~~speed~~ orbital speed as observed at infinity is $V_\infty = R\frac{d\phi}{dt}$

Observed by the astronaut, the time of astronaut t_a is given by

$$-c^2dt_a^2 = -c^2\left(1-\frac{2GM}{Rc^2}\right)dt^2$$

$\therefore \quad dt_a = \sqrt{\left(1 - \frac{2GM}{Rc^2}\right)} \, dt$

~~veloc~~ orbital speed of satellite is

$$V_{circ} = R \frac{d\phi}{dt_a} = \frac{R}{\sqrt{1 - \frac{2GM}{Rc^2}}} \frac{d\phi}{dt}$$

$$= \frac{R}{\sqrt{1 - \frac{2GM}{Rc^2}}} \frac{\sqrt{GM}}{R^{3/2}}$$

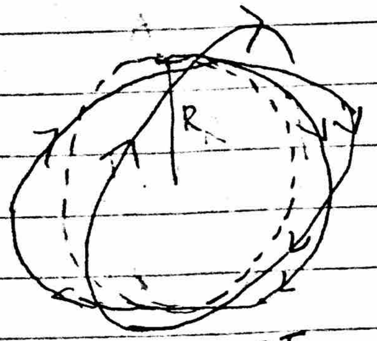
$$= \boxed{\sqrt{\frac{GM}{R - \frac{2GM}{c^2}}}} = \sqrt{\frac{GM}{R - r_s}}$$

The astronaut has to wait a full period for the collision

$$\therefore \quad T = \frac{2\pi R}{V_{circ}} = \boxed{2\pi R \sqrt{\frac{R - \frac{2GM}{c^2}}{GM}}}$$

If satellite ~~tea~~ launched with speed $V_{circ} + \Delta V$, then orbit becomes elliptical and ~~a~~ not closed

$\therefore$



let $\alpha = 3\left(\frac{GM}{Jc}\right)^2$

then the perihelion occurs with a period of phase of $\frac{2\pi}{1-\alpha}$, not 2π

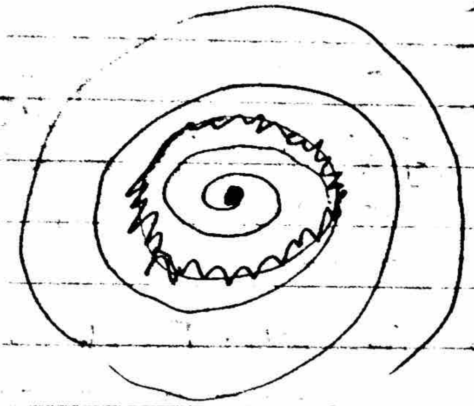
If $\frac{2\pi}{1-\alpha} \cdot m = 2\pi \cdot n$ where n, m are both integers, then collision will occur.

This occurs if $R > \frac{6Gm}{c^2}$ = max of stable orbit

If ~~RE~~ $R < \frac{6Gm}{c^2}$

A deviation from circular orbit will grow!

Since orbit is unstable



$\Rightarrow$ no collision

(2) $\because G^{ab} \propto T^{ab}$ and conseration of stress-energy tensor implies that

$$T^{ab}_{;a} = 0$$

$\therefore$ we need $G^{ab}_{;a} = 0$

~~Gab~~ G^{ab} is the property of spacetime, so it can only possibly depend on the ~~curv~~ ~~cur~~ tensors that decribe the space time curvature i.e. R^{ab} and R, and the metric tensor itself g^{ab}

Also from Bianchi identity, $(R^{ab} - \frac{1}{2} g^{ab} R)_{;a} = 0$

as required, so we ~~c~~ have $G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R$

$$T^{ab} = \left(\rho + \frac{p}{c^2}\right) u^a u^b + g^{ab} p$$

$\because$ $u^0 = \frac{c\,dt}{d\tau}$ is non-zero in any frame

~~(at local rest frame $u = (\;$ $u = \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}$, u^0 not 0)~~

because $u^0 = \frac{E}{m_0 c}$ if $u^0 = 0$, energy $= 0$, we are looking at 0 mass at rest, This doesn't exist.

Then $\because G^{ab}$ is diagonal $\therefore T^{ab}$ is diagonal

$\therefore$ ~~G^{ab} T^{ab}~~ and also g^{ab} is diagonal

$\therefore$ $T^{01} = 0 \quad T^{02} = 0 \quad T^{03} = 0$

$$T^{01} = \left(\rho + \frac{p}{c^2}\right) u^0 u^1 + \underbrace{g^{01}}_{=0} p = 0$$

$\therefore \quad u^0 u^1 = 0 \qquad \because u^0 \neq 0 \quad \therefore \underline{\underline{u^1 = 0}}$

Similarly $\quad T^{02} = 0 \Rightarrow u^2 = 0$

$\qquad\qquad\quad T^{03} = 0 \Rightarrow u^3 = 0$

$\therefore \underline{u^1 = u^2 = u^3 = 0}$ $\quad \left(\begin{array}{l} u_1 u_2 = 0 \\ u_2 u_3 = 0 \\ u_1 u_3 = 0 \end{array} \right.$

The geodesic $\quad -c^2 = g_{ab} u^a u^b$

$\because u^1 = u^2 = u^3 = 0 \quad \therefore -c^2 = g_{00} u^0 u^0 = g_{00} (u^0)^2$

$\cancel{= g_{00}(u_0)^2}$

$\because g_{ab}$ is diagonal $\quad \therefore g^{aa} = \frac{1}{g_{aa}}$ (No sum)

$\therefore g_{00} = \frac{1}{g^{00}} \quad \therefore -c^2 g^{00} = (u^0)^2$

$\therefore u^0 = (-c^2 g^{00})^{1/2} = (c^2 e^{-2\Phi})^{\frac{1}{2}}$

$= c e^{-\Phi}$

$\therefore \underline{u^0 = c e^{-\Phi}} \qquad \because u_0 = g_{0b} u^b$

$\therefore u_0 = g_{00} u^0 + \underbrace{g_{01} u^1}_{0} + \underbrace{g_{02} u^2}_{0} + \underbrace{g_{03} u^3}_{0}$

$= g_{00} u^0 = -e^{2\Phi} c e^{-\Phi} = \underline{\underline{-c e^{\Phi}}}$

$$G_{ab} = -\frac{8\pi G}{c^4} T_{ab}$$

$$G_{00} = \cancel{-}\frac{1}{r^2} \cancel{e^{2\Phi}} \frac{d}{dr}[r(1-e^{-2\Lambda})] = [(\rho + \frac{P}{c^2}) u_0 u_0 + g_{00} P]$$

$$\times -\frac{8\pi G}{c^4}$$

$$\cancel{= -\frac{8\pi G}{c^4}\left(\rho + \frac{P}{c^2}\right) c^2 e^{2\Phi}} \qquad (g_{00} = -e^{2\Phi})$$

$$= \cancel{-}\frac{8\pi G}{c^4}\left[\left(\rho + \frac{P}{c^2}\right) c^2 \cancel{e^{2\Phi}} - \cancel{e^{2\Phi}} P\right]$$

$$\therefore \frac{1}{r^2} \frac{d}{dr}[r(1-e^{-2\Lambda})] = \frac{8\pi G}{c^2 \cancel{c^4}}[\rho \cancel{c^2} \cancel{+P} \cancel{-P}]$$

$$\therefore \frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\Lambda})] = \frac{8\pi G \rho}{c^2}$$

$$\therefore \frac{d}{dr}[r(1-e^{-2\Lambda})] = 8\pi G \rho r^2/c^2$$

$$\therefore \int_0^r d[r'(1-e^{-2\Lambda})] = \frac{8\pi G}{c^2}\int_0^r \rho r'^2 dr'$$

$$\therefore r(1-e^{-2\Lambda}) = \frac{2G}{c^2}[\underbrace{4\pi\int_0^r \rho(r') r'^2 dr'}_{M}]$$

$$\therefore 1 - e^{-2\Lambda} = \frac{2GM}{c^2 r}$$

$$e^{-2\Lambda} = 1 - \frac{2GM}{rc^2}$$

$$\Rightarrow \boxed{\Lambda = -\frac{1}{2}\ln\left(1 - \frac{2GM}{rc^2}\right)}$$

the G_{11} component :

$$G_{11} = -\frac{8\pi G}{c^4} T_{11} = -\frac{8\pi G}{c^4} g_{11} P = -\frac{8\pi G}{c^4} e^{-2\Lambda} P$$

$$= \frac{1}{r^2} e^{2\Lambda} - \frac{1}{r^2} - \frac{2}{r}\Phi'$$

$$\therefore \quad \frac{2}{r}\Phi' = \frac{1}{r^2} e^{2\Lambda} - \frac{1}{r^2} + \frac{8\pi G P}{c^4} e^{+2\Lambda}$$

$$\therefore \Phi' = \frac{1}{2r} e^{2\Lambda} - \frac{1}{2r} + \frac{4\pi G P r}{c^4} e^{+2\Lambda}$$

$$e^{-2\Lambda} = 1 - \frac{2GM}{rc^2}, \quad e^{2\Lambda} = \frac{1}{1-\frac{2GM}{rc^2}}, \quad \Phi' = \frac{d\Phi}{dr}$$

$$\therefore \quad \frac{d\Phi}{dr} = \frac{1}{2r}\,\frac{1}{1-\frac{2GM}{rc^2}} - \frac{1}{2r} + \frac{4\pi G P r}{c^4}\,\frac{1}{1-\frac{2GM}{rc^2}}$$

$$= \frac{\frac{1}{2}rc^2 - \left(1-\frac{2GM}{rc^2}\right)\left(\frac{1}{2}rc^2\right) + \frac{4\pi G P r^3}{c^2}}{c^2 r^2\left(1-\frac{2GM}{rc^2}\right)}$$

$$\rightarrow \frac{d\Phi}{dr} = \frac{GM + \frac{4\pi G}{c^2} P r^3}{c^2 r^2\left(1-\frac{2GM}{c^2 r}\right)}$$

$$\because (\rho c^2 + P)\frac{d\Phi}{dr} = -\frac{dP}{dr}$$

$$\therefore \quad -\frac{dP}{dr} = (\rho c^2 + P)\,\frac{GM + \frac{4\pi G}{c^2} P r^3}{c^2 r^2\left(1-\frac{2GM}{c^2 r}\right)}$$

constant density ρ

$$\cancel{-\frac{dP}{dr} =} \quad M(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$$

$$= 4\pi \left[\frac{1}{3} r'^3\right]_0^r = \frac{4}{3}\pi r^3 \rho$$

$$\therefore \frac{d\Phi}{dr} = \frac{\frac{4}{3}\pi r^3 G\rho + \frac{4\pi G}{c^2} p r^3}{c^2 r^2 \left(1 - \frac{2G}{c^2 r}\frac{4}{3}\pi r^3\rho\right)}$$

$$= \frac{\left(\frac{4\pi G\rho}{3} + \frac{4\pi G P}{c^2}\right) r^3}{c^2 r^2 \left(1 - \frac{8\pi G\rho}{3c^2} r^2\right)}$$

$$= \frac{\left(\frac{4\pi G\rho}{3} + \frac{4\pi G P}{c^2}\right) r}{c^2 \left(1 - \frac{8\pi G\rho}{3c^2} r^2\right)}$$

$$\cancel{A = \frac{4\pi G\rho}{3c^2} + \frac{4\pi G P}{c^4}}$$

$$\cancel{B = \frac{8\pi G\rho}{3c^2}}$$

~~then~~ $\cancel{\frac{d\Phi}{dr} = \frac{Ar}{1 - Br^2}}$

$$\frac{dP}{dr} = -(\rho c^2 + P)\left(\frac{\frac{4\pi G\rho}{3c^2} r + \frac{4\pi G P}{c^4} r}{1 - \frac{8\pi G\rho}{3c^2} r^2}\right)$$

$$= -(\rho c^2 + P)(\rho c^2 + 3P)\left(\frac{4\pi G}{3c^4} r\right) \frac{1}{1 - \frac{8\pi G\rho}{3c^2} r^2}$$

$$\therefore \int_0^{P_0} \frac{dP}{(\rho c^2 + 3P)(\rho c^2 + P)} = \int_R^0 - \frac{\left(\frac{4\pi G}{3c^4}\right) r\, dr}{1 - \frac{8\pi G\rho}{3c^2} r^2}$$

Boundary conditions:

at the surface $r=R$ of the star, pressure is 0 because pressure in outer space is 0

at the centre of the star $r=0$, pressure is p_0

$$\therefore \int_0^{p_0} \frac{dp}{(\rho c^2+3p)(\rho c^2+p)} = \left[\frac{3}{2}\int_0^{p_0}\frac{dp}{\rho c^2+3p} - \frac{1}{2}\int_0^{p_0}\frac{dp}{\rho c^2+p}\right]\frac{1}{\rho c^2}$$

$$= \left[\frac{1}{2}\ln\left(\frac{\rho c^2+3p_0}{\rho c^2}\right) - \frac{1}{2}\ln\left(\frac{\rho c^2+p_0}{\rho c^2}\right)\right]\frac{1}{\rho c^2}$$

$$= \frac{1}{\rho c^2}\frac{1}{2}\ln\left(\frac{\rho c^2+3p_0}{\rho c^2+p_0}\right)$$

$$-\int_R^0 \frac{\frac{4\pi G}{3c^4} r\,dr}{1-\frac{8\pi G\rho}{3c^2}r^2} = \int_0^R \frac{\frac{4\pi G}{3c^4} r\,dr}{1-\frac{8\pi G\rho}{3c^2}r^2}$$

$$= -\frac{4\pi G}{3c^{4\,2}}\,\frac{3\cancel{c^2}}{\underset{4}{\cancel{16\pi}}G\rho}\ln\left(1-\frac{8\pi G\rho}{3c^2}R^2\right)$$

$$= \frac{-1}{4\rho c^2}\ln\left(1-\frac{8\pi G}{3c^2}R^2\right) \qquad \text{let } M \equiv \frac{4\pi R^3}{3}$$

$$= \frac{-1}{4\rho c^2}\ln\left(1-\frac{2GM}{c^2R}\right)$$

$$\therefore \ln\left(\frac{\rho c^2+p_0}{\rho c^2+3p_0}\right) = \ln\left[\left(1-\frac{2GM}{c^2R}\right)^{1/2}\right]$$

$$\therefore \frac{\rho c^2+p_0}{\rho c^2+3p_0} = \left(1-\frac{2GM}{c^2R}\right)^{1/2}$$

$$\therefore \quad \rho c^2 + p_0 = (1-\frac{2GM}{c^2R})^{1/2}\rho c^2 + (1-\frac{2GM}{c^2R})^{1/2}\cdot 3p_0$$

$$\therefore \quad p_0\left(3(1-\frac{2GM}{c^2R})^{1/2} - 1\right) = \rho c^2\left[1-(1-\frac{2GM}{c^2R})^{1/2}\right]$$

$$\Rightarrow \quad p_0 = \rho c^2 \frac{1-(1-\frac{2GM}{c^2R})^{1/2}}{3(1-\frac{2GM}{c^2R})^{1/2} - 1}$$

For stability $p_0 > 0$

As mass gets larger $(1-\frac{2GM}{c^2R})^{1/2}$ gets smaller so p_0 gets larger and larger (positive) until M reaches

$$3(1-\frac{2GM}{c^2R})^{1/2} - 1 = 0$$

in which case if we increase M further p_0 becomes negative and everything breaks down.

So upper limit for M is M_{max} given by

$$3(1-\frac{2GM_{max}}{c^2R})^{1/2} - 1 = 0$$

$$1 - \frac{2GM_{max}}{c^2R} = \frac{1}{9}$$

$$\therefore \frac{2GM_{max}}{c^2R} = \frac{8}{9}$$

$$\therefore \boxed{M_{max} = \frac{4c^2R}{9G}}$$

Redshift (Notes (1127)) is given by

observation

$$\frac{\nu_2}{\nu_1} = \cancel{\left(\frac{g_{00}(X_2)}{g_{00}(X_1)}\right)^{\frac{1}{2}}} \left(\frac{g_{00}(X_1)}{g_{00}(X_2)}\right)^{1/2}$$

emission

$$\cancel{\because g_{00} = -e^{2\Phi} = -\left(1-\frac{2GM}{rc^2}\right)}$$

Outside the star, space-time metric is the Schwardschild metric

$$\therefore \quad g_{00}(R) = -\left(1-\frac{2GM}{Rc^2}\right)$$

$$\therefore \quad g_{00}(\infty) = -1$$

$$1+z = \frac{\lambda_o}{\lambda_e} = \frac{\nu_e}{\nu_o} = \left(\frac{g_{00}(\infty)}{g_{00}(R)}\right)^{1/2}$$

$$= \frac{1}{\sqrt{1-\frac{2GM}{Rc^2}}}$$

$$\therefore z = \frac{1}{\sqrt{1-\frac{2GM}{Rc^2}}} - 1$$

z increases with M $\quad \because M_{max} = \dfrac{4c^2R}{9G}$

$$\therefore z_{max} = \frac{1}{\sqrt{1-\frac{8}{9}}} - 1 = 3-1 = 2$$

$\therefore$ redshift $z \leq 2$

$\therefore$ measurement of 3.1 is likely to be wrong.

(3) $\because$ $w^2+x^2+y^2+z^2=A^2$

$\therefore$ let
$$w=A\cos\chi \qquad , \quad \bar{r}=A\sin\chi$$
$$x=A\sin\chi\sin\theta\cos\phi \qquad =(x^2+y^2+z^2)^{\frac{1}{2}}$$
$$y=A\sin\chi\sin\theta\sin\phi$$
$$z=A\sin\chi\cos\theta$$

~~then $d^2s=d$ $ds^2=dw^2+dx^2+dy^2+dz^2$~~

~~$=A^2\sin^2\chi d\chi^2$~~ Hence $x=\bar{r}\sin\theta\cos\phi$
$$y=\bar{r}\sin\theta\sin\phi$$
$$z=\bar{r}\cos\theta$$

$\therefore$ $dx^2+dy^2+dz^2$
$$=d\bar{r}^2+\bar{r}^2d\theta^2+\bar{r}^2\sin^2\theta\,d\phi^2$$

$$dw^2=A^2\sin^2\chi\,d\chi^2=\bar{r}^2d\chi^2$$

$\because$ $\sin\chi=\frac{\bar{r}}{A}$ $\therefore$ $\cos\chi\,d\chi=\frac{d\bar{r}}{A}$
$$(\cos\chi)^{-1}=\frac{1}{\sqrt{1-\frac{\bar{r}^2}{A^2}}} \quad \therefore\ d\chi=\frac{d\bar{r}}{\sqrt{A^2-\bar{r}^2}}$$

$$\therefore\ dw^2=\frac{\bar{r}^2}{A^2-\bar{r}^2}d\bar{r}^2$$

$\therefore$ $ds^2=dw^2+dx^2+dy^2+dz^2$
$$=\frac{\bar{r}^2}{A^2-\bar{r}^2}d\bar{r}^2+d\bar{r}^2+\bar{r}^2d\theta^2+\bar{r}^2\sin^2\theta\,d\phi^2$$
$$=\frac{A^2}{A^2-\bar{r}^2}d\bar{r}^2+\bar{r}^2d\theta^2+\bar{r}^2\sin^2\theta\,d\phi^2 \quad \text{(spherical)}$$

flat : $ds^2 = d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2$

hyperbolic : $ds^2 = \frac{A^2}{A^2 + \bar{r}^2} d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2$

Surface area

flat : $S = 4\pi A^2 \chi^2$, $\bar{r} = A\chi = \bar{R}$

$$\underline{S = 4\pi \bar{R}^2}$$

hyperbolic : $S = 4\pi A^2 \sinh^2 \chi$ $\bar{r} = A \sinh \chi = \bar{R}$

$$\underline{S = 4\pi \bar{R}^2}$$

(see : HEL pp. 363-365)

The FRW metric : $ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right]$

where : $k = 0, 1, -1$.

→ $g_{tt} = -1$ since time parameter t is taken to be the proper time along the worldline of any fundamental observer (who moves with the cosmic fluid). So ~~for~~ along the worldline $dx^i = 0$, we have $d\tau = dt$ satisfied.

→ For any hypersurface $t = const$ ($dt = 0$) The terms inside the brackets [ ] represent maximally symmetric 3D hypersurface and thus) is homogeneous and isotropic the scale factor $a(t)$ depends only on time and thus is constant in a hypersurface $t = \text{const}$.

$\therefore$ ~~The~~ This common scale factor doesn't alter the symmetry. And metric is still homogeneous and maximal isotropic

For a photon $ds = 0$, we observe from ~~~~ line-of-sight with $d\theta = 0$, $d\phi = 0$

$\therefore$ ~~$c^2 dt^2 = a^2(t)$~~

$$\Rightarrow \quad c^2 dt^2 = a^2(t) \frac{dr^2}{1-kr^2}$$

Consider a light wave leaves a typical galaxy with fixed comoving coordinates (R, θ_E, ϕ_E) at time t_E, then it reaches an ~~obs~~ comoving observer at time t_0 at position ~~~~ $r = 0$

We have $$\int_{t_E}^{t_0} \frac{c\,dt}{a(t)} = f(R) = \int_0^R \frac{dr}{\sqrt{1-kr^2}}$$

$f(R)$ is time-independent since the emitter is comoving with the Universe.

Next wave crest leaves ~~~~ R at time $t_E + \delta t_E$, arrive at $r=0$ at $t_0 + \delta t_0$, then

$$\int_{t_E+\delta t_E}^{t_0+\delta t_0} \frac{c\,dt}{a(t)} = f(R)$$

Thus for small δt_1 and δt_0

$$f(R) = \int_{t_E}^{t_0} \frac{c\,dt}{a(t)} = \int_{t_E+\delta t_E}^{t_0+\delta t_0} \frac{c\,dt}{a(t)} = \int_{t_E}^{t_0} \frac{c\,dt}{a(t)} \cancel{+ \frac{\delta t_0}{R(t_0)} - \frac{\delta t_E}{R(t_E)}}$$

$$+ \frac{\delta t_0}{a(t_0)} - \frac{\delta t_E}{a(t_E)}$$

$$\therefore \quad \frac{\delta t_E}{\delta t_0} = \frac{a(t_E)}{a(t_0)}$$

let ν_E, ν_0 be the emitted and observed frequencies.

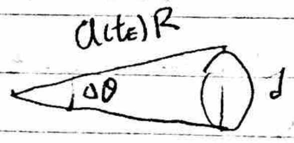
$$\frac{\nu_0}{\nu_E} = \frac{\delta(t_E)}{\delta(t_0)} = \frac{a(t_E)}{a(t_0)}$$

$\therefore$ ~~λ~~ wavelength

$$\frac{\lambda_0}{\lambda_E} = \frac{a(t_0)}{a(t_E)}$$

redshift : $z + 1 \equiv \frac{\lambda_0}{\lambda_E}$

$$\therefore \quad z = \frac{a(t_0)}{a(t_E)} - 1$$



light emitted at time t_E. The angle is invariant from t_E to t_0

$$\therefore \quad \Delta\theta = \frac{d}{a(t_E)R}$$

The Flux - luminosity relation:

$$F = \frac{L\, a^2(t_E)}{4\pi a^4(t_0) R^2} = \frac{L}{4\pi (1+z)^2 a^2(t_0) R^2}$$

the two factors of $1+z$

1 comes from that the frequency (energy) of photons ~~&~~ is doppler shifted.

1 comes from that the arrival rate of photons is dilated.

Flux F is energy per time per surface area

$\therefore$ surface brightness b is Flux per solid angle

$$b = \frac{F}{\Omega}$$

the solid angle Ω is calculated at the time of emission

$$\Omega = \frac{\pi d^2/4}{(a(t_E) R)^2}$$

proper area of galaxy

angular diameter distance

$$\therefore b = \frac{L\, a^2(t_E) \cancel{R^2} \times 4}{\cancel{4}\pi (1+z)^2 a^2(t_0) \cancel{R^2} (\pi d^2)} = \frac{L}{(\pi d)^2} \frac{1}{(1+z)^4}$$

$\because b \propto (1+z)^{-4}$ $\therefore$ high-redshift galaxies appears to be very dim. and are difficult to detect

$\therefore$ We need to increase telescope diameter to collect more light.

$$ds = \iint \sqrt{g_{xx} g_{yy}}\, dx\,dy \quad \text{invariant?}$$

(4)

→ Nucleosythesis is the process that creates atomic nuclei from pre-existing nucleons, primary protons and neutrons. It is responsible for the presence of Helium in the universe.

As temperature falls below $10^9 K$, a series of nuclear reactions occurs

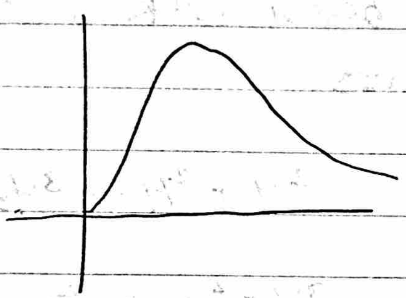
$$n + p \to {}^2H + \gamma \qquad {}^2H + {}^2H \to {}^3He + n$$

$$ {}^3He + n \to {}^3H + p \qquad {}^3H + {}^2H \to {}^4He + n$$

Nuclearsythesis changes the chemical makeup of the universe from entirely hydrogen (protons) into a more complex mixture of protons, deuterons, isotopes of Helium (${}^3He$, ${}^4He$) and small amounts of Li and Be. Eventually the density of neutrons gets two low (neutrons decay into protons), and the time between protons become longer, with the fusion processes freezing out. The stable ${}^4He$ nucleus produced in this period is around today.

→ Decoupling occurs when the universe is expanded and cooled further. The photons of light lost energy and become less and less able to ionise any atoms that form. Eventually all the electrons found their way into the ground state and the photons were no longer able to interact at all. Over a short interval of time the universe suddenly switched from being opaque to being completely transparent. Photons ~~deca~~ ~~have~~ have decoupled from the electrons as the electrons form atoms with the nuclei.

→ Spectral shape of CMB is a black body radiation shap



current temperature $T_0 \simeq 2.73K$

energy density (present) is ~~n_0~~ ~~$n_{\gamma 0} = 4 \times 10^{-14}\ J/m^3$~~

$$u_{\gamma 0} = 4 \times 10^{-14}\ J/m^3$$

mean energy of a ~~photon~~ typical photon in Black body radiation is

$$\epsilon \simeq 3k_BT \quad \text{(see Liddle, page 77)}$$

∴ number density of photon at present is

$$n_{\gamma 0} = \frac{u_{\gamma 0}}{\epsilon_0} = \frac{u_{\gamma 0}}{3k_BT_0} = \underline{\underline{3.6 \times 10^8\ m^{-3}}}$$

Assume the mass density of the universe is roughly the critical density $\rho_0 = \rho_c \simeq 10^{-26}\ kg/m^3$

Out of ρ_0, 4% is made of Baryons.

∴ # density of baryons is

$$n_{B0} = 0.04 \frac{\rho_0}{m_p} \simeq \underline{\underline{0.25\ m^{-3}}}$$

The photon - Baryon ratio is

$$\eta = \frac{n_{\gamma 0}}{n_{B0}} = \underline{\underline{1.4 \times 10^9}}$$

Decoupling occurs after recombination process

$$p + e^- \rightarrow \cancel{H} \ H$$

$\therefore$ Consider the saha equation - describes a ~~plasa~~ plasma of hydrogen

$$\frac{n_e n_p}{n_H} = \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} \exp\left(-\frac{R_y}{k_B T}\right) = f(T)$$

where $R_y = 13.6\,eV$ = Raydberg constant

For pure hydrogen $n_p = n_e$, Baryon density $n_B = n_p + n_H$ $\therefore$

$$\therefore f(T) = \frac{n_p^2}{n_H} = \frac{n_p^2}{n_B - n_p} = n_b \frac{\left(\frac{n_p}{n_b}\right)^2}{\frac{n_b}{n_b} - \left(\frac{n_p}{n_b}\right)}$$

$$\left(\text{let } \frac{n_p}{n_b} \equiv x\right) \quad = n_b \frac{x^2}{1-x}$$

$$\therefore \frac{x^2}{1-x} = \frac{1}{n_b} f(T)$$

$$\frac{1}{n_B} f(T) = \left(\frac{2.415 \times 10^{21} T^{3/2}}{n_B}\right) \exp\left(-\frac{1.578 \times 10^5}{T}\right)$$

in standard S.I units. $\because$ # of Baryons is conserved

$$n_B \cdot a^3 = n_{B0} \cdot (1)^3 \quad \therefore n_B \sim \frac{1}{a^3}$$

scaling factor $\because T \sim \frac{1}{a} \quad \therefore n_B \sim T^3$

$$\therefore \quad n_B = \left(\frac{T}{T_0}\right)^3 n_{B0}$$

$$\therefore \frac{x^2}{1-x} = \frac{(2.73)^3}{0.25} \left(\frac{2.415 \times 10^{21}}{T^{3/2}}\right) \exp\left(-\frac{1.578 \times 10^5}{T}\right)$$

Recombination occurs at $x \simeq 0.1$

The decoupling follows at $x \simeq 0.01$, when the majorities of electrons are combined to form atoms.

$$\therefore \frac{x^2}{1-x} \simeq 0.0001$$

By calculator ~~trial test~~ trial and error, we have the temperature at decoupling

$$T_{dec} \simeq 3100\,K$$

At $T = 3100$ K

$k_B T \simeq 4 \times 10^{-20}$ J $\ll m_e c^2 \simeq 8 \times 10^{-14}$ J

$\therefore$ Electrons are non-relativistic.
(the universe is matter dominated at the time of decoupling)

→ Our universe is flat, so for matter dominated case.

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

($t_0 \simeq$ age of universe $\simeq 13.8 \times 10^9$ years)

$$\because \quad T \sim \frac{1}{a} \sim t^{-2/3}$$

$$\therefore \quad \left(\frac{T}{T_0}\right) = \left(\frac{t_0}{t}\right)^{2/3}$$

$$\therefore \quad \left(\frac{T}{2.73K}\right) = \left(\frac{13.8 \times 10^9 \text{ years}}{t}\right)^{2/3}$$

temperature at decoupling is $T_{dec} = 3100$ K

age of universe for this T_{dec} is

$$t_{dec} = \left(\frac{3100}{2.73}\right)^{-\frac{3}{2}} \times 13.8 \times 10^9 = 360000 \text{ years}$$