To: Will Potter

BS Problem Set 2 Ziyan Li

1.

$$V^{\rho}_{;\mu;\nu} - V^{\rho}_{;\nu;\mu} = R^{\rho}_{\lambda\mu\nu} V^{\lambda}$$

where the curvature tensor

$$R^{\rho}_{\lambda\mu\nu} = \frac{\partial \Gamma^{\rho}_{\lambda\mu}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\rho}_{\lambda\nu}}{\partial x^{\mu}} + \Gamma^{\rho}_{\tau\nu}\Gamma^{\tau}_{\lambda\mu} - \Gamma^{\rho}_{\tau\mu}\Gamma^{\tau}_{\lambda\nu}$$

Go into locally free falling inertial coordinates:

$\cancel{\Gamma^{\rho}_{\tau\nu}}$ all affine connections are evaluated to be 0

$$\Rightarrow \Gamma^{a}_{bc} = 0$$

But the partial derivatives of affine connections do not vanish ✓

This is because the ~~free~~ frame is only locally inertial. ~~varying position will at~~ Although the affine connection components are evaluated to be 0 locally, its partial derivatives with respect to ~~position~~ coordinates will not be 0 generally

$\therefore$ $R^{\rho}_{\lambda\mu\nu} \neq 0$ at locally inertial frame

$\therefore$ $V^{\rho}_{;\mu;\nu} - V^{\rho}_{;\nu;\mu} \neq 0$ at locally inertial frame

$\rightarrow$ Its <u>not</u> going to be identically zero ✓

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2. The geodesic equation

$$\frac{d^2x^\lambda}{dp^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp} = 0 \qquad ①$$

(p is an arbitrary parameter)

The schwarzschild solution

$$-c^2d\tau^2 = -Bc^2dt^2 + Adr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

where $B = (1 - \frac{2GM}{rc^2})$, $AB = 1$

This is a diagonal metric

$\rightarrow$

$$\Gamma^a_{ab} = \Gamma^a_{ba} = \frac{1}{2g_{aa}}\frac{\partial g_{aa}}{\partial x^b}$$

$$\Gamma^a_{bb} = -\frac{1}{2g_{aa}}\frac{\partial g_{bb}}{\partial x^a}$$

$\therefore$ $\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{B'}{2B}$, $\Gamma^r_{tt} = \frac{B'}{2A}$

$\Gamma^r_{rr} = \frac{A'}{2A}$ $\quad \Gamma^r_{\theta\theta} = -\frac{r}{A}$ $\quad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{A}$

$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$ $\quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$

$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}$ $\quad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta$ (others are 0)

$\therefore$ the 4-components of equation ①

becomes

$$\frac{d^2(ct)}{dp^2} + \frac{B'}{B}\frac{dr}{dp}\frac{d(ct)}{dp}$$

$$\frac{d^2r}{dp^2} + \frac{B'}{2A}\left(\frac{cdt}{dp}\right)^2 + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dp}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{d\phi}{dp}\right)^2 = 0$$

$$\frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\theta}{dp} - \sin\theta\cos\theta\left(\frac{d\phi}{dp}\right)^2 = 0$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\phi}{dp} + 2\cot\theta\frac{d\theta}{dp}\frac{d\phi}{dp} = 0$$

$\theta = \frac{\pi}{2}$ is a solution for the above set of equations

$\rightarrow$ set $\theta = \pi/2$, we have

$$\frac{d^2(ct)}{dp^2} + \frac{B'}{B}\frac{dr}{dp}\frac{d(ct)}{dp} = 0 \quad ①$$

$$\frac{d^2r}{dp^2} + \frac{B'}{2A}\left(\frac{cdt}{dp}\right)^2 + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\phi}{dp}\right)^2 = 0 \quad ②$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\phi}{dp} = 0 \quad ③$$

$\because$ $A = A(r)$ $B = B(r)$

$\therefore$ ① $\Rightarrow$ $\frac{d}{dp}\left(B\frac{cdt}{dp}\right) = 0$

② $\Rightarrow$ $\frac{d}{dp}\left(r^2\frac{d\phi}{dp}\right) = 0$

① $\Rightarrow$ $B\frac{cdt}{dp} = \text{const}$

choose the parameter p such that const $= 0$

$\therefore$ $\frac{dt}{dp} = \frac{1}{B}$

③ $\rightarrow$ $r^2 \frac{d\phi}{dp} = J$ (J = constant)

multiply ② by $2A\frac{dr}{dp}$

$$\rightarrow \quad 2A\frac{dr}{dp}\frac{d^2r}{dp^2} + B'\left(\frac{c\,dt}{dp}\right)^2\frac{dr}{dp} + A'\left(\frac{dr}{dp}\right)^2\frac{dr}{dp}$$

$$- 2r\frac{dr}{dp}\left(\frac{d\phi}{dp}\right) = 0$$

$$\therefore \quad 2A\frac{dr}{dp}\frac{d^2r}{dp^2} + \cancel{\frac{B'}{B}\left(\frac{dr}{dp}\right)^2\frac{dr}{}} \; \frac{B'}{B^2}c\frac{dr}{dp}$$

$$+ A'\left(\frac{dr}{dp}\right)^2\frac{dr}{dp} - \frac{2}{r}\frac{dr}{dp}J = 0$$

$$\therefore \frac{d}{dp}\left[A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{c^2}{B}\right] = 0$$

$$\therefore \quad A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{c^2}{B} = -E \quad (\text{const})$$

$$-c^2\left(\frac{dT}{dp}\right)^2 = -Bc^2\left(\frac{dt}{dp}\right)^2 + A\left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\phi}{dp}\right)^2$$

for $\theta = \frac{\pi}{2}$

$$= -\frac{c^2}{B} + A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} = -E$$

$$\therefore \left(\frac{dT}{dp}\right)^2 = \frac{E}{c^2}$$

$$\therefore \frac{dr}{dT} = \frac{dr}{dp}\frac{dp}{dT} = \frac{c^2}{E}\frac{dr}{dp}$$

$$\therefore \left(\frac{dr}{dT}\right)^2 + \frac{c^2}{A} + \frac{c^2}{AE}\frac{J^2}{r^2} = \frac{c^4}{E}$$

$$\therefore \frac{1}{2}\left(\frac{dr}{dT}\right)^2 + \left(\frac{c^2J^2}{AE}\right)\frac{1}{2r^2} + \left(\frac{c^2}{2A}\right) = \frac{c^4}{2E}$$

~~For General relativity~~

$$\cancel{\Phi_{eff}(r) = \left(\frac{c^2J^3}{AE}\right)\left(\frac{c^2J^2}{E}\right)\left(1-\frac{2GM}{rc^2}\right)}\frac{1}{r}$$

$$\therefore \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{c^2J^2}{2E}\frac{1}{r^2}\left(1-\frac{2GM}{rc^2}\right)$$

$$+ \frac{c^2}{2}\left(1-\frac{2GM}{rc^2}\right) = \frac{c^4}{2E}$$

$$\therefore \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{c^2J^2}{E}\left(\frac{1}{2r^2}\right) + \left(-\frac{GM}{r} - \frac{GMJ^2}{Er^3}\right)$$

$$= \cancel{c^4}\ \frac{c^2}{2}\left(\frac{c^2}{E}-1\right) = \cancel{\frac{c^2}{E}}\ \frac{c^2}{2E}(c^2-E))$$

$$\therefore \cancel{\frac{1}{2}dr}\ \frac{1}{2}(v_s^2) + \frac{l_s^2}{2r^2} + \Phi_s(r) = \varepsilon_s$$

where $l_s = \frac{cJ}{\sqrt{E}}$, $\Phi_s(r) = -\frac{GM}{r} - \frac{GMJ^2}{Er^3}$

$$\varepsilon_s = \frac{c^2}{2E}(c^2-E)$$

For the Newtonian limit

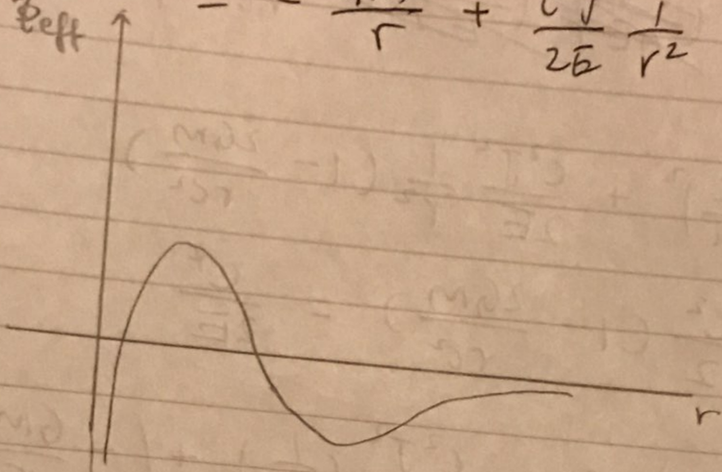
$E \to c^2$ & $c^2 - E =$ finite $c \to \infty$, $E \to \infty$

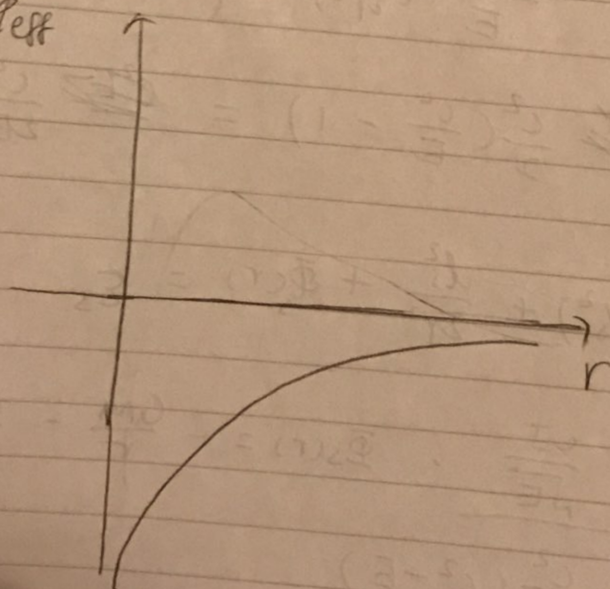
$$\therefore \frac{1}{2}v^2 + \frac{J}{2r^2} - \frac{GM}{r} = \frac{1}{2}(c^2-E) = \text{finite}$$

$$= \varepsilon$$

b) $\Phi_{eff} = \frac{l_s^2}{2r^2} + \Phi(r)$

$$= -\frac{GM}{r} + \frac{c^2J^2}{2E}\frac{1}{r^2} - \frac{GMJ^2}{E}\frac{1}{r^3}$$





or

For minimum in Φ_{eff}

$$\frac{d\Phi_{eff}}{dr} = 0 \Rightarrow 0 = \frac{GM}{r^2} - \frac{c^2J^2}{E}\frac{1}{r^3} + \frac{3GMJ^2}{E}\frac{1}{r^4}$$

$$\therefore (GM)r^2 - \left(\frac{c^2J^2}{E}\right)r + \left(\frac{3GMJ^2}{E}\right) = 0$$

For newtonian theory

$\frac{3GMJ^2}{E}$ term $\to 0$

$\therefore \quad GMr = \frac{c^2J^2}{E} = J^2$

$\therefore \quad r = \frac{J^2}{GM}$, always exist

For General relativity

$$\text{critical } r = \frac{\frac{c^2J^2}{E} \pm \sqrt{\left(\frac{c^2J^2}{E}\right)^2 - 4GM\left(\frac{3GMJ^2}{E}\right)}}{2GM}$$

minimum only exist if

$$\left(\frac{c^2J^2}{E}\right)^2 - \frac{12}{E}(GMJ)^2 > 0$$

$$\Rightarrow \quad \frac{c^2J^2}{E} > \sqrt{\frac{12}{E}}\, GMJ$$

$$\therefore \quad \frac{c^2J}{GM\sqrt{12E}} > 1$$

c)

$\because r^2\frac{d\phi}{dp} = J \;,\; \frac{dt}{dp} = B^{-1} \quad \therefore \frac{dp}{dt} = B$

$$\therefore \quad \frac{d\phi}{dt} = \frac{d\phi}{dp}\frac{dp}{dt} = \frac{J}{r^2}B = \frac{J}{r^2}\left(1 - \frac{2GM}{rc^2}\right)$$

If r = constant (circular orbit)

then for radial equation, $\frac{d^2r}{dp^2}=0$, $\frac{dr}{dp}=0$

$$\therefore \frac{B'c^2}{2A}\left(\frac{dt}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\phi}{dp}\right)^2 = 0$$

$$\cancel{\therefore \frac{dt}{dp}=B, \quad \frac{d\phi}{dp}=\frac{J}{r^2}, \quad AB=1}$$

$$\therefore \cancel{\frac{B'c^2}{2} - \frac{r}{A}\left(\frac{J}{r^2}\right)=0} \qquad \frac{d\phi}{dp} = \frac{d\phi}{dt}\frac{dt}{dp}$$

$$\therefore \cancel{\frac{B'c^2}{2B} = \frac{r}{A}\left(\frac{d\phi}{dt}\right)^2\left(\frac{dt}{dp}\right)^2}$$

$$\therefore \frac{B'c^2}{2A}\cancel{\left(\frac{dt}{dp}\right)^2} = \frac{r}{A}\left(\frac{d\phi}{dt}\right)^2\cancel{\left(\frac{dt}{dp}\right)^2}$$

$$\therefore \left(\frac{d\phi}{dt}\right)^2 = \frac{B'c^2}{2r} \qquad \because B = 1-\frac{2GM}{rc^2}$$

$$\therefore B' = \frac{2GM}{r^2c^2}$$

$$\therefore \left(\frac{d\phi}{dt}\right)^2 \equiv \Omega^2 = \frac{GM}{r^3}$$

$$\because \frac{d\phi}{dt} = \frac{J}{r^2}\left(1-\frac{2GM}{rc^2}\right) = \sqrt{\frac{GM}{r^3}}$$

$$\therefore J = \cancel{r^2 J}\left(\frac{\sqrt{GMr}}{1-\frac{2GM}{rc^2}}\right)$$

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$$\frac{dr}{d\varphi} = 0$$

$$\therefore \quad E = \frac{c^2}{B} - \frac{J^2}{r^2}$$

$$\therefore E = \frac{c^2}{\left(1-\frac{2GM}{rc^2}\right)} - \frac{1}{r^2}\left(\frac{\sqrt{GMr}}{1-\frac{2GM}{rc^2}}\right)^2$$

d.) For the lowest r to have local extreme need

$$\frac{d\Phi_{eff}}{dr} = 0 \quad \text{to have one real root}$$

(see b))

$$\therefore \quad \left(\frac{c^2J^2}{E}\right)^2 - 4GM\left(\frac{3GMJ^2}{E}\right) = 0 \quad ④$$

and $r = \dfrac{c^2J^2}{2GME}$ ⑤

$$④ \Rightarrow \quad \frac{J^2}{E}\left(\left(\frac{J^2}{E}\right)c^4 - 12(GM)^2\right) = 0$$

$$\therefore \quad \frac{J^2}{E} = \frac{12(GM)^2}{c^4}$$

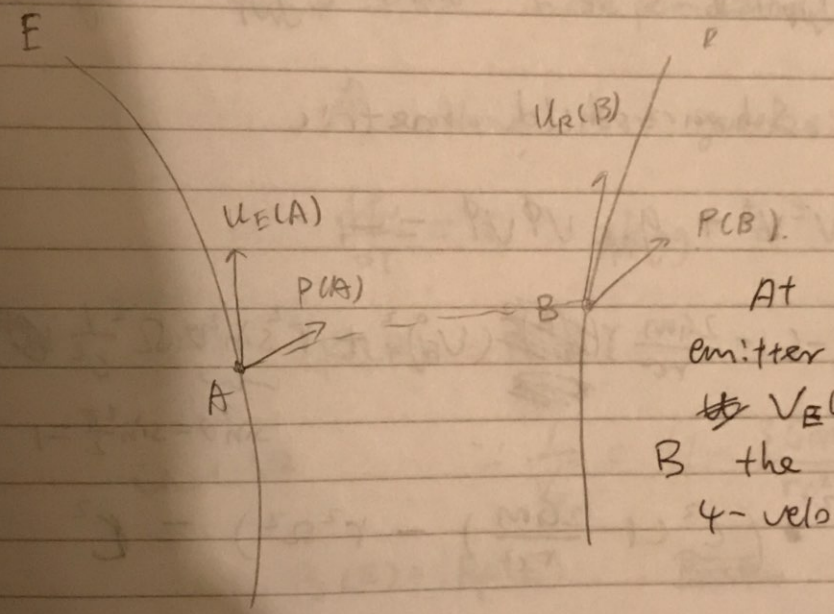
Substitute into ⑤ $\Rightarrow$ $r = \dfrac{c^2}{2GM}\left(\dfrac{12(GM)^2}{c^4}\right)$

$$\Rightarrow \quad r = \frac{6GM}{c^2}$$

4\. a) suppose emitter (E) and receiver (R) have worldlines $X^N_E(T_E)$ and $X^N_R(T_R)$ respectively

T_E and T_R are proper times. of E and R

at some event A, E emits a photon with 4-momentum $P(A)$ that is received by ~~B~~ R at event B



At event A the emitter has 4-velocity $V_E(A)$ and at event B the receiver ~~has~~ has 4-velocity $V_R(B)$

The energies of photon as observed by the emitter and receiver are respectively given by

$$E(A) = P(A) \cdot V_E(A) = P_N(A) V^N_E(A)$$

$$\cancel{= P_N(E) V_E} \quad = P_N(E) V^N_E(E)$$

$$E(B) = P(B) \cdot V_R(B) = P_N(B) V^N_R(B) = P_N(R) V^N_R(R)$$

$$\because \quad E(A) = h\nu_E \qquad E(B) = h\nu_R$$

$$\therefore \quad \underline{\frac{\nu_R}{\nu_E} = \frac{P_N(R) V^N(R)}{P_N(E) V^N(E)}}$$

✓ 3/3

assuming observer fixed at infinity / edge on

circular o.

b) $V^\mu(R) = (1,0,0,0)$, $V^\mu(E) = V_E^0 (1,0,0,\frac{d\phi}{cdt})$

$= (V_E^0, 0, 0, V_E^3)$

$V_E^0 = \frac{dt}{d\tau}$

$\because$ $\Omega = \frac{d\phi}{dt} = \left(\frac{GM}{r^3}\right)^{\frac{1}{2}}$

$\therefore$ $g_{\nu\rho} V^\nu V^\rho = -1$ and $g_{\nu\rho}$ is given by

$\because$ the Schwarzschild metric

$\therefore$ $g_{tt} V^t V^t + g_{\phi\phi} V^\phi V^\phi = -1$

$\therefore -1 = -\left(1 - \frac{2GM}{rc^2}\right) (V_E^0)^2 + \underbrace{r^2 \sin^2\theta}_{} \, \Omega^2 \frac{1}{c^2} (V_E^0)^2$

$\sin^2\theta = \sin^2\frac{\pi}{2} = 1$

$\therefore$ $(V_E^0)^2 \cdot \left(c^2\left(1 - \frac{2GM}{rc^2}\right) - r^2\Omega^2\right) = c^2$

$\therefore$ $(V_E^0)^2 = \left(c^2\left(1 - \frac{2GM}{rc^2}\right) - r^2\frac{GM}{r^3}\right)^{-1} \cdot c^2$

$\therefore$ $V_E^0 = \left(1 - \frac{2GM}{rc^2} - \frac{GM}{rc^2}\right)^{-1/2}$

$= \left(1 - \frac{3GM}{rc^2}\right)^{-1/2}$ ✓ 4/4

$\because$ $\frac{\nu_R}{\nu_E} = \frac{P_\mu(R)\, V_R^\mu(R)}{P_\mu(E)\, V_E^\mu(E)} = \frac{P_0(R)}{P_0(E) V_E^0 + P_3(E) V_E^3}$

$= \frac{1}{V_E^0} \frac{P_0(R)}{P_0(E) + P_3(E)\frac{1}{c}\frac{d\phi}{dt}}$

$$= \frac{1}{V_E^0} \frac{P_0(R)}{P_0(E)} \left(1 + \frac{P_3(E)}{P_0(E)} \frac{\Omega}{c}\right)^{-1}$$

$\because$ $\cancel{\frac{d^2 x}{}}$ $\frac{dV_\mu}{d\tau} = \cancel{\frac{1}{2}} \frac{1}{2} V^\nu V^\rho \frac{\partial g_{\nu\rho}}{\partial x^\mu}$ along a geodesic (PS 1, Q3)

and $g_{\nu\rho}$ is independent of time

$\therefore$ Along a geodesic $\frac{dV_0}{d\tau} = 0$

$\therefore$ $\frac{dP_0}{d\tau} = 0$ along a geodesic

$\Rightarrow$ $P_0(R) = P_0(E)$

in ~~uk~~ $\because \frac{1}{V_E^0} = \left(1 - \frac{3GM}{rc^2}\right)^{\frac{1}{2}}$

$P_3(E) = P_\phi(E)$ ~~$P_0(E)$~~

$$\therefore \frac{V_R}{V_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} \left(1 + \frac{\Omega}{c} \frac{P_\phi(E)}{P_0(E)}\right)^{-1}, \quad \Omega^2 = \frac{GM}{r^3}$$

Photon has null geodesic

$\therefore$ $g^{\nu\rho} P_\nu P_\rho = 0$

$$\therefore \frac{1}{c^2}\left(1 - \frac{2GM}{rc^2}\right)^{-1}(P_0)^2 - \left(1 - \frac{2GM}{rc^2}\right)(P_r)^2 - \frac{1}{r^2}(P_\phi)^2 = 0 \quad (1)$$

~~$\therefore$ disk material~~ moving ~~precisely along~~ the ~~line of sight~~

$\therefore$ ~~$P_r = 0$~~

- From disk materials moving at right angles across line of sight

$$P_\phi(E) = 0$$

$$\therefore \frac{V_R}{V_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2}$$

- From disk materials moving ~~prece~~ precisely along the line of sight $\Rightarrow P_r(E) = 0$

$$① \Rightarrow \frac{P_\phi(E)}{P_0(E)} = \frac{\pm r}{c\left(1 - \frac{2GM}{rc^2}\right)^{1/2}}$$

$$\therefore \frac{\Omega}{c}\frac{P_\phi(E)}{P_0(E)} = \pm\left(\frac{GM}{c^2 r^3}\right)^{\frac{1}{2}} \frac{r}{c}\left(1 - \frac{2GM}{rc^2}\right)^{-\frac{1}{2}}$$

$$= \pm\left(\frac{rc^2}{GM} - 2\right)^{-\frac{1}{2}}$$

$$\therefore \frac{V_R}{V_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} \Big/ \left(1 \pm (rc^2/GM - 2)^{-1/2}\right)$$

$\therefore$ + ~~sing~~ sign indicates source moving towards the observer

opposite

− sing indicates ~~sorce~~ Source moving away from the observer.

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5. a) ~~$a(1+2\epsilon\cos\phi+\epsilon^2\cos^2\phi)$~~

~~real part of $e^{2i\phi}$~~

$$e^{2i\phi} = \cos 2\phi + i\sin 2\phi$$

∴ real part of $e^{2i\phi} = Re(e^{2i\phi}) = \cos 2\phi$

$$\cos 2\phi = 2\cos^2\phi - 1$$

$$\therefore \quad Re\left(1+\frac{\epsilon^2}{2} + 2\epsilon e^{i\phi} + \epsilon^2\frac{1}{2}e^{2i\phi}\right)$$

$$= Re\left(1+\cancel{\frac{\epsilon^2}{2}} + 2\epsilon e^{i\phi} + \frac{\epsilon^2}{2}(2\cos^2\phi \cancel{-1})\right)$$

$$= Re(1+2\epsilon e^{i\phi} + \epsilon^2\cos^2\phi)$$

$$= \cancel{1+2\epsilon e^{i}} \quad 1+2\epsilon\cos\phi + \epsilon^2\cos^2\phi$$

⇒ we only need to solve the real part of

$$\frac{d^2\delta u}{d\phi^2} + \delta u = a(b+2\epsilon e^{i\phi} + \epsilon^2 e^{2i\phi}/2)$$

try $\delta u = A_0 + A_1\phi e^{i\phi} + A_2 e^{2i\phi}$

- We need a cumulative ~~solut~~ term in the solution ~~to~~ for the solution to be testable

⇒ we need a ϕ in front of $e^{i\phi}$

1/3

a=? b=?

b) try $\delta u = A_0 + A_1 e^{i\phi}\phi + A_2 e^{2i\phi}$

$$\frac{d\delta u}{d\phi} = A_1\phi(i)e^{i\phi} + A_1 e^{i\phi} + 2iA_2 e^{2i\phi}$$

$$= \cancel{(iA_1\phi +} A_1(i\phi+1)e^{i\phi} + 2iA_2 e^{2i\phi}$$

$$\frac{d^2\delta u}{d\phi^2} = iA_1(i\phi+1)e^{i\phi} + iA_1 e^{i\phi} - 4A_2 e^{2i\phi}$$

$$= -A_1\phi e^{i\phi} + 2iA_1 e^{i\phi} - 4A_2 e^{2i\phi}$$

$\therefore$ $-\cancel{A_1\phi e^{i\phi}} + 2iA_1 e^{i\phi} - 4A_2 e^{2i\phi}$

$+ A_0 + \cancel{A_1 e^{i\phi}(\phi)} + A_2 e^{2i\phi}$

$= \cancel{abt}\ ab + 2\epsilon a e^{i\phi} + \frac{1}{2}a\epsilon^2 e^{2i\phi}$

$\therefore$ $A_0 = ab$

$A_1 = -i\epsilon a$

$A_2 = -\frac{a\epsilon^2}{6}$

$\because$ $u_N = \frac{GM}{J^2}(1+\epsilon\cos\phi)$

~~$\delta u = Re(ab + 2\epsilon a e^{i\phi} + \frac{1}{2}a\epsilon^2$~~

$\delta u = Re(ab - i\epsilon a e^{i\phi}\phi - \frac{a\epsilon^2}{6}e^{2i\phi})$

$= ab + \epsilon a\phi\sin\phi - \frac{a\epsilon^2}{6}\cos(2\phi)$

b) try $\delta u = A_0 + A_1 e^{i\phi}\phi + A_2 e^{2i\phi}$

$$\frac{d\delta u}{d\phi} = A_1\phi(i)e^{i\phi}\phi + A_1 e^{i\phi} + 2iA_2 e^{2i\phi}$$

$$= \cancel{(iA_1\phi +} A_1(i\phi+1)e^{i\phi} + 2iA_2 e^{2i\phi}$$

$$\frac{d^2\delta u}{d\phi^2} = iA_1(i\phi+1)e^{i\phi} + iA_1 e^{i\phi} - 4A_2 e^{2i\phi}$$

$$= -A_1\phi e^{i\phi} + 2iA_1 e^{i\phi} - 4A_2 e^{2i\phi}$$

$\therefore$ $\cancel{-A_1\phi e^{i\phi}} + 2iA_1 e^{i\phi} - 4A_2 e^{2i\phi}$

$+ A_0 + \cancel{A_1 e^{i\phi}(\phi)} + A_2 e^{2i\phi}$

$= \cancel{abt}\ ab + 2\epsilon a e^{i\phi} + \frac{1}{2}a\epsilon^2 e^{2i\phi}$

$\therefore$ $A_0 = ab$ ✓

$A_1 = -i\epsilon a$ ✓

$A_2 = -\dfrac{a\epsilon^2}{6}$ ✓

$\therefore$ $u_N = \dfrac{GM}{J^2}(1+\epsilon\cos\phi)$

~~$\delta u = Re(ab + 2\epsilon a e^{i\phi} + \frac{1}{2}a\epsilon^2$~~

$\delta u = Re(ab - i\epsilon a e^{i\phi}\phi - \dfrac{a\epsilon^2}{6}e^{2i\phi})$

$= ab + \epsilon a\phi\sin\phi - \dfrac{a\epsilon^2}{6}\cos(2\phi)$ ✓

$\therefore \quad u = u_N + \delta u$

$$= \frac{GM}{J^2} + ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}\epsilon\cos\phi$$

$$+ \epsilon a\phi\sin\phi$$

~~If a is very small~~

$\cancel{\cos\phi + \epsilon}$

$$\Rightarrow u = ab - \frac{a\epsilon^2}{6}\cos(2\phi) + \frac{GM}{J^2}\left(1 + \epsilon\left(\cos\phi + \alpha\phi\sin\phi\right)\right)$$

where $\alpha = \frac{aJ^2}{GM} = 3\left(\frac{GM}{Jc}\right)^2$

$\therefore$ ~~ub=~~ for small a and thus small α

$\alpha\phi \simeq \sin\alpha\phi \quad , \quad 1 \simeq \cos(\alpha\phi)$

$$\therefore \cos\phi + \alpha\phi\sin\phi \simeq \cancel{\cos\phi\sin\epsilon}\ \cos\phi\cos(\alpha\phi) + \sin(\alpha\phi)\sin\phi$$

$$= \cos(\phi(1-\alpha))$$

$$\therefore u \simeq ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}\left[1 + \epsilon\cos((1-\alpha)\phi)\right]$$

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From the final term, we see that the orbit is periodic, but with a period $\frac{2\pi}{1-\alpha}$ (the r-values repeat ~~the~~ on a cycle that is larger than 2π). The result is that the orbit cannot close, and so ellipse precesses.

In one revolution, the ellipse will rotate about the focus by an amount

$$\Delta\phi = \frac{2\pi}{1-\alpha} - 2\pi \approx 2\pi(1-\alpha)^{-1} - 2\pi$$

$$\approx 2\pi(1+\alpha) - 2\pi = 2\pi\alpha$$

$$= 6\pi\left(\frac{GM}{cJ}\right)^2$$

✓ 2/2

8/10

To will potter

Ziyan Li

B5 Problem Set 2 Q3

3. a) Schwarzschild metric

$g_{tt} = -(1-\frac{2GM}{rc^2})$ $\quad g_{rr} = (1-\frac{2GM}{rc^2})^{-1}$

$g_{\theta\theta} = r^2$ $\quad g_{\phi\phi} = r^2\sin^2\theta$

$|g| = 1 \times r^2 \times r^2\sin^2\theta = r^4\sin^2\theta$

$|g'| = \frac{|g|}{\sin^2\theta} = \underline{\underline{r^4}}$

Bondi Accretion → spherical symmetry ~~$U^\theta = 0$~~ $(U^\theta, U^\phi = 0)$

$g_{\mu\nu}U^\mu U^\nu = -c^2 = g_{tt}U^tU^t + g_{rr}U^rU^r$

$U_t = g_{tt}U^t + \underbrace{g_{tr}U^r}_{=0 \because g \text{ is diagonal}} = g_{tt}U^t$

$\therefore \quad U^t = \frac{U_t}{g_{tt}}$

$\therefore -c^2 = \frac{(U_t)^2}{g_{tt}} + g_{rr}(U^r)^2 \qquad \because g_{rr} = -\frac{1}{g_{tt}}$

$\therefore \quad -c^2 = \frac{(U_t)^2}{g_{tt}} - \frac{(U^r)^2}{g_{tt}}$

$\therefore \quad (U_t)^2 = -g_{tt}c^2 + (U^r)^2$

$\Rightarrow \quad (U_t)^2 = (c^2(-)(1-\frac{2GM}{rc^2}) + (U^r)^2)$

$$\therefore \quad U_t = \left[c^2 - \frac{2GM}{r} + (U^r)^2 \right]^{1/2}$$

b) μ is rest mass per particle, so μ is a constant.

$$U = U^r \quad , \quad |g'| = r^4 \rightarrow |g'|^{1/2} = r^2$$

$$\rightarrow \quad n U r^2 = C_1 \quad ①$$

$$(P + \rho c^2) U U_t r^2 = C_2 \quad ②$$

$$\frac{②}{①} \Rightarrow \quad \frac{P + \rho c^2}{n} U_t = \text{const}$$

$$\because \quad \bar{\omega} = \mu n \quad , \quad \mu = \text{const}$$

$$\therefore \quad \frac{P + \rho c^2}{\bar{\omega}} U_t = \text{const}$$

$$\because \quad a^2 = \frac{\gamma P}{\bar{\omega}} \quad \therefore \quad P = \frac{a^2}{\gamma} \bar{\omega}$$

$$\rho = \bar{\omega} + \frac{P}{c^2(\gamma - 1)} \quad \therefore \quad \rho c^2 = c^2 \bar{\omega} + \frac{P}{\gamma - 1}$$

$$\rho c^2 = \bar{\omega} \left[c^2 + \frac{a^2}{\gamma(\gamma - 1)} \right]$$

$$\therefore \quad \frac{\rho c^2 + P}{\bar{\omega}} = c^2 + \frac{a^2}{\gamma(\gamma - 1)} + \frac{a^2}{\gamma}$$

$$= c^2 + a^2 \left(\frac{1}{\gamma} \right) \left(\frac{1}{\gamma - 1} + 1 \right)$$

$$= c^2 + a^2 \left(\frac{1}{\gamma}\right)\left(\frac{\cancel{1} + \gamma - \cancel{1}}{\gamma - 1}\right) = c^2 + \frac{a^2}{\gamma - 1}$$

$$\therefore \left(c^2 + \frac{a^2}{\gamma-1}\right) U_t = \text{const}$$

$$\Rightarrow \quad \underline{\underline{\left(c^2 + \frac{a^2}{\gamma-1}\right)^2 (U_t)^2 = \text{const}}}$$

$$\Rightarrow \quad \underline{\underline{\left(c^2 + \frac{a^2}{\gamma-1}\right)^2 \left(c^2 + U^2 - \frac{2GM}{r}\right) = \text{const}}}$$

The C_1 equation itself is

$$nUr^2 = C_1 = \text{const.} \qquad \mu = \text{const}$$

$\dot{m}$ ~~rate~~ = accretion rate $= \begin{pmatrix}\text{radial} \\ \text{mass flux}\end{pmatrix} \times (\text{area})$

$$= (4\pi r^2) \times (\varpi U) = 4\pi \varpi r^2 U$$

$$\because \quad \dot{m} = 4\pi \mu n r^2 U = 4\pi \mu C_1$$

$$\therefore \quad \dot{m} = \text{const}$$

$$\underline{\underline{\dot{m} = 4\pi \varpi r^2 U = \text{const}}}$$

3) c) i. At infinity $r = \infty$, $-\frac{2GM}{r} \to 0$

$U = U^r \to 0$ because no radial accretion at infinity (gas accreting from rest)

and $a \to a_\infty$

~~const =~~

$$\therefore \text{const} = c^2\left(c^2 + \frac{a_\infty^2}{\gamma - 1}\right)^2$$

ii) $$\left(c^2 + \frac{a^2}{\gamma-1}\right)^2\left(c^2 + U^2 - \frac{2GM}{r}\right) = c^2\left(c^2 + \frac{a_\infty^2}{\gamma-1}\right)^2$$

$\cancel{c^4 + a^2c^2}$ only keep terms with $O(c^4)$

$\therefore \cancel{c^2\left(c^2+\frac{a^2}{\gamma-1}\right)^2}$ ($O(c^6)$ cancells, $O(c^2)$, $O(1)$ are too small in Newtonian limit)

$$c^4\left(U^2 - \frac{2GM}{r}\right) + 2c^4\frac{a^2}{\gamma-1} = 2c^4\frac{a_\infty^2}{\gamma-1}$$

Newtonian limit $U = v$

$$\therefore v^2 - \frac{2GM}{r} + 2\frac{a^2}{\gamma-1} = \frac{2a_\infty^2}{\gamma-1}$$

$$\therefore \frac{v^2}{2} + \frac{a^2}{\gamma-1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma-1}$$

iii)

$$\left(c^2+\frac{a^2}{\gamma-1}\right)^2\left(c^2+v^2-\frac{2GM}{r}\right)=c^2\left(c^2+\frac{a_\infty^2}{\gamma-1}\right)^2$$

$a \ll c$

$$\therefore \quad \cancel{c^2}\left(c^2+v^2-\frac{2GM}{r}\right)=\cancel{c^2}\cdot c^2$$

$$\therefore \quad c^2+v^2-\frac{2GM}{r}=c^2$$

as $\quad r \to R_s = \dfrac{2GM}{c^2}$

$$\therefore \quad c^2+v^2-c^2=c^2$$

$$\therefore \quad v^2 \to c^2 \qquad \text{(photon-like)}$$

~~Null geodesic~~

$\because \quad v = v^r = \dfrac{dr}{d\tau} \qquad \therefore \left(\dfrac{dr}{d\tau}\right)^2 = c^2$

$$\therefore \quad -c^2 d\tau^2 = -dr^2$$

geodesic equation for spherical flow ($d\theta=0$, $d\phi=0$)

$$-c^2 d\tau^2 = c^2 g_{tt}\, dt^2 + g_{rr}\, dr^2 = -dr^2$$

$$g_{rr} = \frac{1}{1-\frac{2GM}{rc^2}} = \frac{1}{1-\frac{R_s}{r}}$$

As $r \to R_s$, $\quad g_{rr} \to \infty$

$\therefore g_{rr} \neq 1 \rightarrow g_{rr}$

$\Rightarrow$ ignore $-dr^2$ on the RHS

we have $-c^2 g_{tt} dt^2 + g_{rr} dr^2 = 0$

$g_{tt} = 1 - \frac{R_s}{r}$

$\therefore c^2 (1 - \frac{R_s}{r}) dt^2 = \frac{1}{1 - \frac{R_s}{r}} dr^2$

$\therefore \left(\frac{dr}{dt}\right)^2 = c^2 (1 - \frac{R_s}{r})^2$

$\because$ Inflow

$\therefore \frac{dr}{dt} = -c(1 - \frac{R_s}{r})$