$\boxed{\text{To: Will Potter}}$

Ziyan Li

B5 Problem Set 1 ~~Very~~ Excellent! $\frac{60}{62}$ 97%

1)

a) Transformation between differential 4-velocities

$$dU'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, dU^{\beta} \qquad \text{let } dU^{\bullet} = \begin{pmatrix} d(\gamma_v c) \\ d(\gamma_v v) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c\,d\gamma_v \\ dV \\ 0 \\ 0 \end{pmatrix}$$

with $V = \frac{v}{\sqrt{1-v^2}} = \gamma_v v$

~~then~~ and $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(here in Λ, $\gamma = \gamma_w$

w = relative velocity between frames).

Consider $dU'^{1} = \Lambda^{1}{}_{\beta}\, dU^{\beta} = \Lambda^{1}{}_{0}\, dU^{0} + \Lambda^{1}{}_{1}\, dU^{1}$

$\Rightarrow \quad dV' = -\gamma\beta\,(d\gamma_v) + \gamma\, dV$

(let $c=1$) $\quad = \gamma(dV - \beta\, d\gamma_v)$

in this case $\gamma = \gamma_w = \frac{1}{\sqrt{1-w^2}}$, $\beta = w$

~~$d\gamma_v$~~ $\gamma_v = \frac{1}{\sqrt{1-v^2}} = V^0$

$$\therefore \quad \underline{\underline{dV' = \frac{1}{\sqrt{1-w^2}}(dV - w\, dV^0)}}$$ 4/4

b) if $w = v$ $\quad dV' = \frac{1}{\sqrt{1-v^2}}(dV - v\, dV^0)$

$$\therefore\ d\left(\frac{v'}{\sqrt{1-v'^2}}\right) = \frac{1}{\sqrt{1-v^2}}\left(d\left(\frac{v}{\sqrt{1-v^2}}\right) - v\,d\left(\frac{1}{\sqrt{1-v^2}}\right)\right)$$

$$= \frac{1}{\sqrt{1-v^2}}\, d\left(\frac{1}{\sqrt{1-v^2}}\right) dv = \frac{dv}{1-v^2}$$

Also $d\left(\frac{v'}{\sqrt{1-v'^2}}\right) = v'\,d\left(\frac{1}{\sqrt{1-v'^2}}\right) + \frac{1}{\sqrt{1-v'^2}}\,dv'$

$$= dv' \quad (\because v'=0)$$

$$\therefore \Rightarrow \quad dv' = \frac{dv}{(1-v^2)}$$

$$\Rightarrow \quad \underline{\underline{dv = dv'(1-v^2)}} \checkmark$$

$$\therefore\ \frac{dv}{dt} = \frac{dv'}{dt}(1-v^2) = \frac{dv'}{dt'}\cdot\frac{dt'}{dt}(1-v^2)$$

$\because$ dt' is the time interval in rest frame

dt is the time interval in lab frame.

$$\therefore\ \frac{dt'}{dt} = \frac{d\tau}{dt} = \frac{1}{\gamma_w} = \frac{1}{\gamma_v} = (1-v^2)^{\frac{1}{2}}$$

$$\therefore\ \frac{dv}{dt} = \frac{dv'}{dt'}(1-v^2)^{3/2} \Rightarrow \underline{\underline{\frac{dv}{dt} = a'(1-v^2)^{3/2}}} \checkmark$$

5/5

c.) $\because a'$ is constant

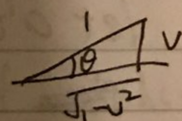
$$\therefore \frac{dv}{dt} = a'(1-v^2)^{3/2} \Rightarrow \int \frac{dv}{(1-v^2)^{3/2}} = \int a' dt = a't + c$$

compute $\int \frac{dv}{(1-v^2)^{3/2}}$, use $v = \sin\theta$

$dv = \cos\theta \, d\theta$

$$\therefore \int \frac{dv}{(1-v^2)^{3/2}} = \cancel{\int \sin\theta \, d\theta} \int \frac{\cos\theta \, d\theta}{(1-\sin^2\theta)^{3/2}} = \int \frac{\cos\theta \, d\theta}{\cos^3\theta}$$

$$= \int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta \, d\theta = \tan\theta$$



$\because \theta = \sin^{-1}(v) \quad \therefore \tan\theta = \frac{v}{\sqrt{1-v^2}}$

$$\therefore \int \frac{dv}{(1-v^2)^{3/2}} = \frac{v}{\sqrt{1-v^2}}$$

$$\therefore \frac{v}{\sqrt{1-v^2}} = a't + C$$

At $t=0$, $v=0 \Rightarrow C=0$

$$\therefore \frac{v}{\sqrt{1-v^2}} = a't \qquad \therefore \frac{v^2}{1-v^2} = a'^2 t^2$$

$$\therefore (1+a'^2t^2)v^2 = a'^2t^2 \Rightarrow \boxed{v = \frac{a't}{\sqrt{1+a'^2t^2}}}$$

c.) $\because a'$ is constant

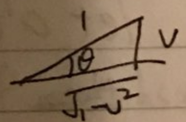
$$\therefore \frac{dv}{dt} = a'(1-v^2)^{3/2} \Rightarrow \int \frac{dv}{(1-v^2)^{3/2}} = \int a'dt = a't + C$$

Compute $\int \frac{dv}{(1-v^2)^{3/2}}$, use $v = \sin\theta$

$dv = \cos\theta \, d\theta$

$$\therefore \int \frac{dv}{(1-v^2)^{3/2}} = \cancel{\int \sin\theta \, d\theta} \int \frac{\cos\theta \, d\theta}{(1-\sin^2\theta)^{3/2}} = \int \frac{\cos\theta \, d\theta}{\cos^3\theta}$$

$$= \int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta \, d\theta = \tan\theta$$



$\because \theta = \sin^{-1}(v)$ $\therefore \tan\theta = \frac{v}{\sqrt{1-v^2}}$

$$\therefore \int \frac{dv}{(1-v^2)^{3/2}} = \frac{v}{\sqrt{1-v^2}}$$

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At $t=0$, $v=0 \Rightarrow C=0$

$$\therefore \frac{v}{\sqrt{1-v^2}} = a't \qquad \therefore \frac{v^2}{1-v^2} = a'^2t^2$$

$$\therefore (1+a'^2t^2)v^2 = a'^2t^2 \Rightarrow \boxed{v = \frac{a't}{\sqrt{1+a'^2t^2}}}$$

$\because$ $\cancel{dv = dv'(1-v^2)}$

$$\because \quad \frac{dt}{dt'} = \frac{dt}{d\tau} = \gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\frac{a'^2t^2}{1+a'^2t^2}}}$$

$$\therefore \quad = \frac{1}{\sqrt{\frac{1}{1+a'^2t^2}}} = \sqrt{1+a'^2t^2}$$

$$\therefore \quad \frac{d(a't)}{d(a't')} = \sqrt{1+(a't)^2}$$

$\therefore$ $\cancel{d(a't) = \sqrt{1+(a't)}}$

$$\int d(a't') = \int \frac{d(a't)}{\sqrt{1+(a't)^2}}$$

let $a't = u = \tan\theta$

~~then~~ $du = \sec^2\theta\, d\theta$

$$= \int \frac{du}{\sqrt{1+u^2}}$$

$$= \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta}{\sec\theta}\, d\theta = \int \sec\theta\, d\theta$$

$$= \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\, d\theta = \int \frac{d(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}$$

$$= \ln|\sec\theta+\tan\theta| + C$$

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→ Its not going

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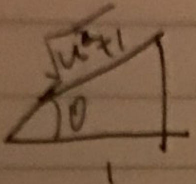

$\tan\theta = u \qquad \sec\theta = \sqrt{u^2+1}$

$$\Rightarrow \int d(a't') = \ln\left| u + \sqrt{1+u^2} \right| + C$$

$$\Rightarrow a't' = \ln\left| a't + \sqrt{1+(a't)^2} \right| + C$$

when $t=0$, $t'=0$

$$\therefore \quad 0 = \ln(1) + C = 0 + C \Rightarrow C = 0$$

$$\Rightarrow a't' = \ln\left| a't + \sqrt{1+(a't)^2} \right|$$

If $\sinh(x) = y$, then $y = \dfrac{e^x - e^{-x}}{2}$

$$\therefore \quad e^x - e^{-x} = 2y \Rightarrow (e^x)^2 - 2y(e^x) - 1 = 0$$

$$\therefore \quad e^x = \frac{2y + \sqrt{4y^2+4}}{2} \quad \text{(ignore negative solution)}$$

$$= y + \sqrt{1+y^2} \Rightarrow x = \ln(y + \sqrt{1+y^2})$$

$$\Rightarrow \ln\left(a't + \sqrt{1+(a't)^2} \right) = \sinh^{-1}(a't)$$

$$\Rightarrow a't' = \sinh^{-1}(a't) \Rightarrow a't = \sinh(a't')$$

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} + \frac{1}{2} - \frac{1}{4}e^{2x} - \frac{1}{4}e^{-2x} + \frac{1}{2} = \underline{\underline{1}}$$

$$\therefore \quad V = \frac{a't}{\underbrace{\sqrt{1+(a't)^2}}_{a't = \sinh(a't')}} = \frac{\sinh(a't')}{\sqrt{1+\sinh^2(a't')}}$$

$$= \frac{\sinh(a't')}{\cosh(a't')} = \underline{\underline{\tanh(a't')}}$$

2 $\quad V = \frac{dx}{dt} = \frac{a't}{\sqrt{1+(a't)^2}}$

$$\cancel{\therefore \frac{1}{a}dx =} \quad \int a'dx = \int \frac{a't}{\sqrt{1+(a't)^2}}\, d(a't)$$

$$\Rightarrow \quad a'x = \frac{1}{2}\int \left(1+(a't)^2\right)^{-\frac{1}{2}} d[(a't)^2]$$

$$= \frac{1}{2} \times 2\left(1+(a't)^2\right)^{\frac{1}{2}} + C = \sqrt{1+(a't)^2} + C$$

at $t=0=t'$, $x=0$ $\therefore$ $0 = \sqrt{1} + C$

$$\Rightarrow C = -1$$

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} + \frac{1}{2} - \frac{1}{4}e^{2x} - \frac{1}{4}e^{-2x} + \frac{1}{2} = \underline{\underline{1}}$$

$$\therefore \quad V = \frac{a't}{\underbrace{\sqrt{1+(a't)^2}}_{a't = \sinh(a't')}} = \frac{\sinh(a't')}{\sqrt{1+\sinh^2(a't')}}$$

$$= \frac{\sinh(a't')}{\cosh(a't')} = \underline{\underline{\tanh(a't')}}$$

$$\because \quad V = \frac{dx}{dt} = \frac{a't}{\sqrt{1+(a't)^2}}$$

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$$\Rightarrow \quad a'x = \frac{1}{2}\int \left(1+(a't)^2\right)^{-\frac{1}{2}} d[(a't)^2]$$

$$= \frac{1}{2} \times 2\left(1+(a't)^2\right)^{\frac{1}{2}} + c = \sqrt{1+(a't)^2} + c$$

at $t = 0 = t'$, $x = 0$ $\quad \therefore \quad 0 = \sqrt{1} + c$

$$\Rightarrow c = -1$$

$\therefore \quad a'x = \sqrt{1+(a't)^2} - 1$

$\because \quad a't = \sinh(a't')$

$\therefore \quad a'x = \sqrt{1+\sinh^2(a't')} - 1 = \cosh(a't') - 1$

$\therefore \quad x = \frac{1}{a'}[\cosh(a't') - 1]$ $\quad 11/11$

well done, it would have been a bit easier if you had used $v = \tanh u$ as a guess.

d)

Consider condition ii)

$\rightarrow$ t match t' at small x' and small t'

then $\quad t = A\sinh(a't') + B$

At $t'=0$, $t=0$ $\Rightarrow$ $B=0$ (for small x')

$B(x'=0)=0$

~~$\because$ B independent of~~

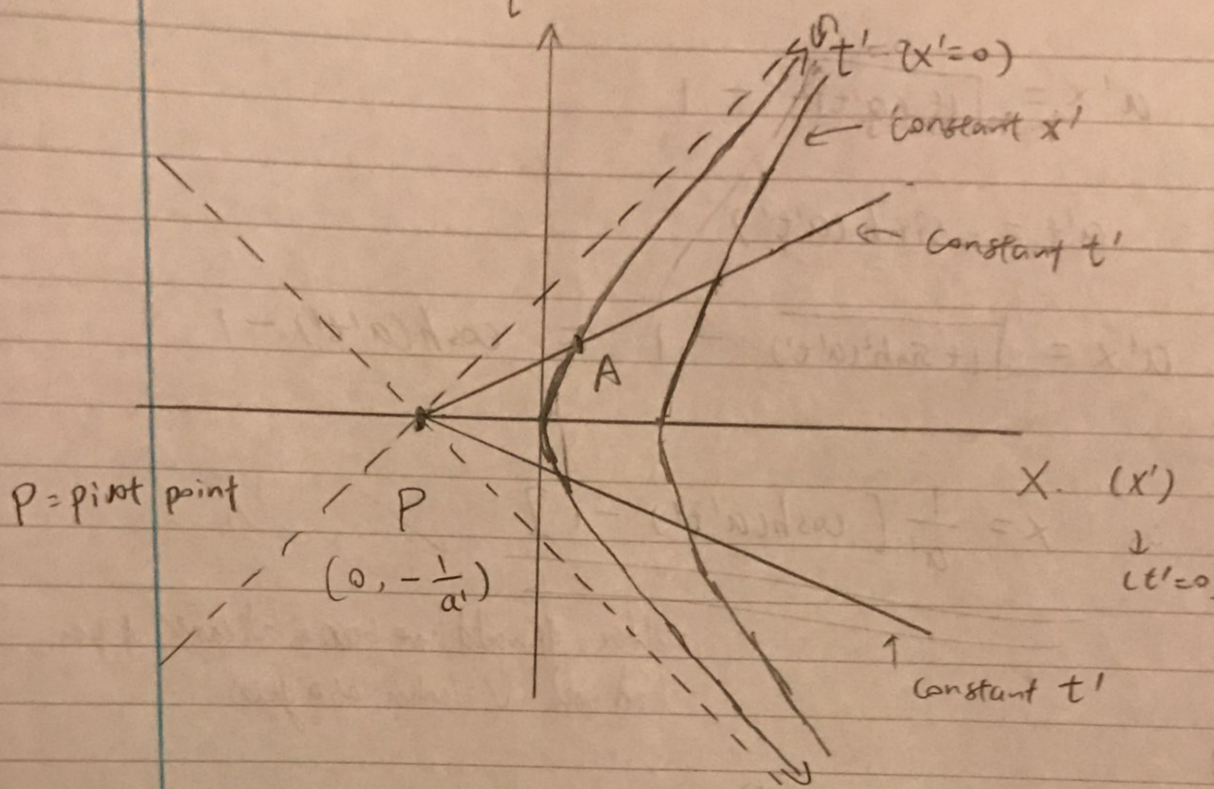
$\rightarrow$ x agrees with x' for small t'

$x = A\cosh(a't') + C$

$\therefore \quad x = x' = A + C$ at small t'

$\because$ A, C independent of t' $\therefore$ $x' = A + C$ for all x'

$A(x') + C(x') = x'$



Consider the pivot point in (t, x) ; $P\ (0, -\frac{1}{a'})$

and an event A on the world line of particle A (t_A, x_A)

$\because\ x = \frac{1}{a'}[\cosh(a't') - 1] = \frac{1}{a'}\sqrt{1+(a't)^2} - \frac{1}{a'}$

$\therefore\ (x+\frac{1}{a'})^2 - t^2 = (\frac{1}{a'})^2$ is the world line

$\therefore\ (x_A + \frac{1}{a'})^2 - t_A^2 = (\frac{1}{a'})^2$

In the instantaneous rest frame of A

A has coordinates $\tilde{t_A} = \cancel{\gamma(t_A - vt_A)}\ \gamma(t_A - vx_A)$

$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\tanh(a\tilde{t_A})^2}} = \frac{1}{\sqrt{\operatorname{sech}^2(a\tilde{t_A})}} = \cosh(a\tilde{t_A})$

$= \sqrt{1+(at_A)^2}$

$$\therefore \quad \tilde{t}_A = \underbrace{\sqrt{1+(a't_A)^2}}_{\gamma}\left(t_A - \underbrace{\frac{a't_A}{\sqrt{1+(a't_A)^2}}}_{v}\underbrace{\left(\frac{1}{a'}\right)\left(\sqrt{1+(a't_A)^2}-1\right)}_{x_A}\right)$$

$$= \left(t_A\sqrt{1+(a't_A)^2} - t_A\sqrt{1+(a't_A)^2} + t_A\right) = t_A$$

P has time coordinate

$$\tilde{t}_P = \gamma(t_P - vx_P) = \gamma(0 - v(-\frac{1}{a'})) = \gamma v \frac{1}{a'}$$

$$= \cancel{\sqrt{1+(a't_A)}}\,\frac{\cancel{a'}t_A}{\cancel{\sqrt{1+(a't_A)}}}\,\frac{1}{\cancel{a'}} = t_A$$

$$\therefore \quad \tilde{t}_P = \tilde{t}_A = t_A$$

$\because \tilde{t}_P = \tilde{t}_A$ in the ~~instata~~ instantaneous rest frame

$\therefore$ PA is the line of simultaneity in the instantaneous rest frame

By condition i) $\Rightarrow$ PA is the line of simultaneity in the accelerated frame (surface of constant t'), so are all lines starting from the pivot point P and intersects at the world line.

Consider the 4-displacement R from event P to event A

In lab frame $R = A - P = (t_A, x_A + \frac{1}{a'})$

$$R \cdot R = -t_A^2 + (x_A + \frac{1}{a'})^2 = (\frac{1}{a'})^2$$

In the instantaneous rest frame of A

$\because$ P and A are simultaneous

$\therefore$ $R \cdot R = (\frac{1}{a'})^2 = \Delta x^2$

where Δx is the distance in 3-space between P and A

clearly P is behind A $\therefore$ $\Delta \tilde{x} = \tilde{x_A} - \tilde{x_p} = \frac{1}{a'}$

By i) surfaces of constant t' in accelerated frame are surface of constant time $\tilde{t}$ in instantaneous rest frame

So accelerated frame and ~~instat~~ instantaneous rest frame have the same lines of constant time

$\rightarrow$ they have same x-axis (except a zero offset)

So in the accelerated frame, the distance between P and A ~~are~~ is

$$\Delta x' = \Delta \tilde{x} = \frac{1}{a'}$$

$\because$ in the accelerated frame, the ~~coor~~ spatial coordinate of A is 0 $(x_A' = 0)$

$\therefore$ $x_A' - x_p' = \Delta x' = \frac{1}{a'}$, becomes

$-x_p' = \frac{1}{a'}$ $\Rightarrow$ $\underline{\underline{x_p' = -\frac{1}{a'}}}$

This applies to all possible event A along the world line of the particle

$\rightarrow$ This means ~~it~~ that in the accelerated frame, at spatial coordinate $x' = -\frac{1}{a'}$, no matter what the time coordinate t' is, when this event is transformed back to the lab frame, we will always find that $t = 0$ and ~~x~~ $x = -\frac{1}{a'}$ (the event P in lab frame correspond to events in accelerated frame with $x' = -\frac{1}{a'}$ and all possible t')

$\therefore$ when $x' = -\frac{1}{a'}$, $t = 0$, $x = -\frac{1}{a'}$

$\because$ ~~$x = A\cosh$~~ $x = A(x')\cosh(a't') + C(x')$

and $A(x') + C(x') = x'$ $\Rightarrow$ $C(x') = x' - A(x')$

$\therefore$ $x = A(x')\cosh(a't') + x' - A(x')$

$\rightarrow$ $x = A(x')[\cosh(a't') - 1] + x'$

when $x' = -\frac{1}{a'}$, $x = -\frac{1}{a'}$ independent of t'

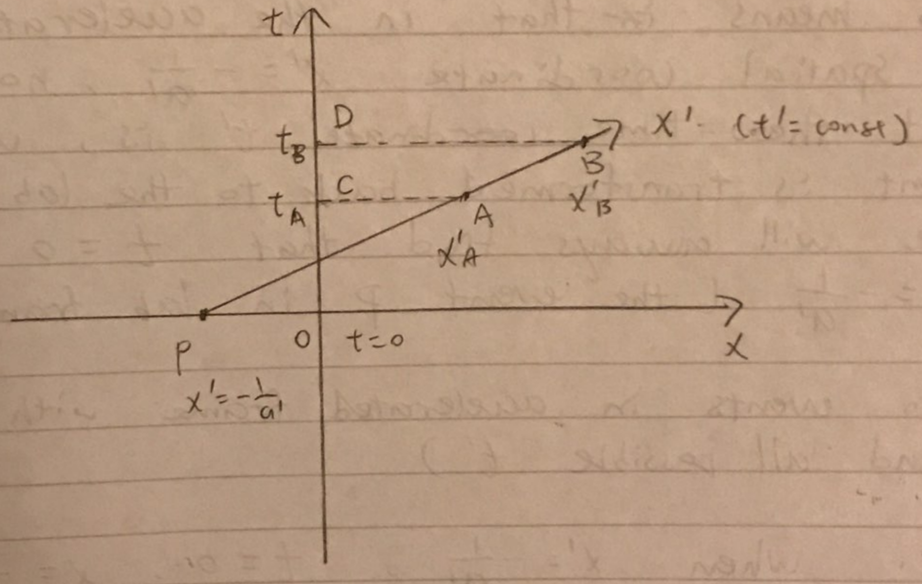
$\therefore A(x' = -\frac{1}{a'}) = 0$

$\therefore A(x')$ can be written as

$$A(x') = f(x')(x' + \frac{1}{a'})$$

$$C(x') = x' - f(x')(x' + \frac{1}{a'})$$

$$t = A\sinh(a't') + B \Rightarrow t = f(x')(x' + \frac{1}{a'})\sinh(a't') + B(x')$$



Consider PA to be one of the lines of simultaneity in accelerated frame

By similar triangles $\frac{PA}{PB} = \frac{OC}{OD}$

$\therefore \frac{x'_A + \frac{1}{a'}}{x'_B + \frac{1}{a'}} = \frac{t_A}{t_B} \Rightarrow$ when t' is constant, t is proportional to $x' + \frac{1}{a'}$

$\because \; t = f(x')(x'+\frac{1}{a'})\sinh(at') + B(x')$

$\therefore$ let $B(x') = g(x')(x'+\frac{1}{a'})$

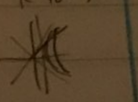
then $t = [f(x')\sinh(at') + g(x')](x'+\frac{1}{a'})$

and the coefficient

$f(x')\sinh(at') + g(x') = \text{constant}$

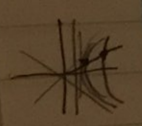
for all t'

const R for $x' \neq 0$



$\therefore \; f(x') = \text{constant} = f$

$g(x') = \text{constant} = g$



$\therefore \; B(x') = g(x'+\frac{1}{a'})$

evenly spaced

$\because \; B(x'=0) = 0 \quad \therefore \; g = 0 \quad \therefore \; \underline{\underline{B(x') = 0}}$

~~For x'~~ $\therefore \; t = f(x'+\frac{1}{a'})\sinh(a't')$

for $x' = 0$ (i.e. for events happen along the world line of the particle)

We've shown in (c.) that

$a't = \sinh(a't') \quad \therefore \; \frac{f}{a'} = \frac{1}{a'} \Rightarrow f = 1$

$\therefore \; \underline{\underline{A(x') = (x'+\frac{1}{a'})}}$

ⓓ $\therefore C(x') = x' - \left(x' + \frac{1}{a'}\right) = -\frac{1}{a'}$

i.e. ~~We've~~ Given the A, B, C we've found, we put ~~c back into~~ speed of light c back into the equations ~~to~~ to get

$$ct = \left(\frac{c^2}{a'} + x'\right) \sinh\left(\frac{a't'}{c}\right)$$

$$x = \left(\frac{c^2}{a'} + x'\right) \cosh\left(\frac{a't'}{c}\right) - \frac{c^2}{a}$$ 8/8

1e) $c\,dt = c\frac{\partial t}{\partial x'} dx' + c\frac{\partial t}{\partial t'} dt'$

$$dx = \frac{\partial x}{\partial x'} dx' + \frac{\partial x}{\partial t'} dt'$$

$\therefore c\,dt = \sinh(a't'/c)\, dx' + \left(\frac{c^2}{a'} + x'\right)\frac{a'}{c} \cosh\left(\frac{a't'}{c}\right) dt'$

$$dx = \cosh(a't'/c)\, dx' + \left(\frac{c^2}{a'} + x'\right)\frac{a'}{c} \sinh\left(\frac{a't'}{c}\right) dt'$$

$\therefore c^2 d\tau^2 = c^2 dt^2 - dx^2 = \sinh^2(a't'/c)\, d^2x'$

$\cancel{+ 2\sinh(a't'/c)\cosh(a't'/c)\left[c\left(1 + \frac{a'x'}{c^2}\right)\right] dx'dt'} + \cancel{\cosh^2}$

$c^2\left(1 + \frac{a'x'}{c^2}\right)^2 \cosh^2\left(\frac{a't'}{c}\right) d^2t' \quad - \cosh^2(a't'/c)\, d^2x'$

$\cancel{- 2\sinh\left(\frac{a't'}{c}\right)\cosh\left(\frac{a't'}{c}\right) c\left(1 + \frac{a'x'}{c^2}\right) dx'dt'} - \sinh^2\left(\frac{a't'}{c}\right) c^2\left(1 + \frac{a'x'}{c^2}\right)^2 d^2t'$

$\therefore C(x') = x' - (x' + \frac{1}{a'}) = -\frac{1}{a'}$

i.e. ~~We've~~ Given the A, B, C we've found,

We put ~~c back into~~ speed of light c back into the equations ~~to~~ to get

$$ct = \left(\frac{c^2}{a'} + x'\right)\sinh\left(\frac{a't'}{c}\right)$$

$$x = \left(\frac{c^2}{a'} + x'\right)\cosh\left(\frac{a't'}{c}\right) - \frac{c^2}{a}$$ 8/8

1e) $c\,dt = c\frac{\partial t}{\partial x'}dx' + c\frac{\partial t}{\partial t'}dt'$

$dx = \frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial t'}dt'$

$\therefore c\,dt = \sinh(a't'/c)\,dx' + \left(\frac{c^2}{a'} + x'\right)\frac{a'}{c}\cosh\left(\frac{a't'}{c}\right)dt'$

$dx = \cosh(a't'/c)\,dx' + \left(\frac{c^2}{a'} + x'\right)\frac{a'}{c}\sinh\left(\frac{a't'}{c}\right)dt'$

$\therefore c^2 d\tau^2 = c^2dt^2 - dx^2 = \sinh^2(a't'/c)\,d^2x'$

$\cancel{+2\sinh(a't'/c)\cosh(a't'/c)\left[c\left(1 + \frac{a'x'}{c^2}\right)\right]dx'dt'} + \cancel{\cosh^2}$

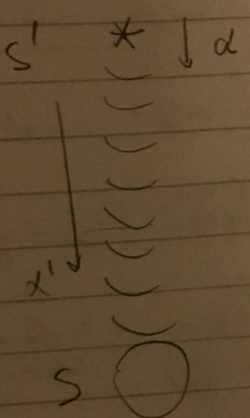
$c^2\left(1 + \frac{a'x'}{c^2}\right)^2\cosh^2\left(\frac{a't'}{c}\right)d^2t' - \cosh^2(a't'/c)\,d^2x'$

$\cancel{-2\sinh\left(\frac{a't'}{c}\right)\cosh\left(\frac{a't'}{c}\right)c\left(1 + \frac{a'x'}{c^2}\right)dx'dt'} - \sinh^2\left(\frac{a't'}{c}\right)c^2\left(1 + \frac{a'x'}{c^2}\right)^2 d^2t'$

$$= \left[\cosh^2\left(\frac{a't'}{c}\right) - \sinh^2\left(\frac{a't'}{c}\right)\right] c^2 \left(1+\frac{a'x'}{c^2}\right)^2 d^2t'$$

$$- \left[\cosh^2\left(\frac{a't'}{c}\right) - \sinh^2\left(\frac{a't'}{c}\right)\right] d^2x'$$

$$= \left(1+\frac{a'x'}{c^2}\right)^2 c^2 d^2t' - d^2x'$$ ✓ 4/4 32/32



$$c^2 d\tau^2 = c^2 d^2t - d^2x = \left(1+\frac{a'x'}{c^2}\right)^2 c^2 d^2t' - d^2x'$$

Consider 2 pulses passes through the position x' in the direction of acceleration a' and are separated by a time interval dt' in the source frame (accelerated frame) S' ✓

In S', $dx' = 0$

$$\therefore c^2 d\tau^2 = \left(1+\frac{a'x'}{c^2}\right) c^2 d^2t'$$

$\because a' > 0$, $x' > 0$ $\therefore$ $d\tau > dt$

The time interval in the inertial frame (the receiver frame) is longer

$\Rightarrow$ the frequency of pulse as seen by receiver is shorter $\Rightarrow$ red shifted /

1\. c) $\frac{dv}{dt} = a(1-v^2)^{3/2}$

$$\int \frac{dv}{(1-v^2)^{3/2}} = \int a' dt \qquad \text{let } V = \tanh(u)$$

$$\int \frac{dv}{(1-\tanh^2 u)^{3/2}} = a't \qquad dv = \frac{1}{\cosh^2 u} du \qquad \cosh^2 u - \sinh^2 u = 1$$

$$1 - \tanh^2 u = \frac{1}{\cosh^2 u}$$

$$\int \frac{\cosh^3 u \, du}{\cosh^2 u} = \sinh u = a't$$

$$V = \tanh u = \frac{a't}{\sqrt{1+(a't)^2}} \Big/ \frac{dt'}{dt} = (1-v^2)^{1/2} = (1-\tanh^2 u)^{1/2} = \frac{1}{\cosh u} = \frac{1}{\sqrt{1+(a't)^2}}$$

$$dt' = \int \frac{dt}{\sqrt{1+(a't)^2}} , \quad t' = \int_0^t \frac{dt}{\sqrt{1+(a't)^2}} \qquad \boxed{a't = \sinh u \quad a'dt = \cosh u \, du}$$

$$a't' = \int \frac{\cosh u \, du}{\sqrt{1+\sinh u}} = \int_0^u du \qquad \therefore \; u = a't' \quad \therefore V = \tanh(a't')$$

$$\therefore a't = \sinh(a't') \qquad x = \int_0^t \frac{a't}{\sqrt{1+a'^2t^2}} dt = \frac{1}{a'}\left[(1+a'^2t^2)^{\frac{1}{2}} - 1\right] = \frac{1}{a'}\left[\cosh(a't') - 1\right]$$

1\. d) $dt' = \gamma(dt - v\,dx)$, $\frac{dt'}{dx'} = \frac{dt}{dx'} - v\frac{dx}{dx'} = 0$ (surfaces of constant t')

$$0 = A'\sinh(a't') + B' - A'\sinh(a't') - C'\tanh(a't')$$

$$A' = \frac{dA}{dx'}, \text{ etc} \qquad B' = C'\tanh(a't') \Rightarrow B' = C' = 0$$

At early times $\quad t = A(x')a't' + B \qquad B = 0$

$$A = \frac{1}{a'} + f(x')$$

~~for~~ $x = A + C = x' \quad \therefore \quad A + C = x'$

for $x' \to 0 \quad A = \frac{1}{a'} \quad C = -\frac{1}{a'} \quad , \quad A = \frac{1}{a'} + x$

2) $$(T^{\mu\nu}V_\nu)' = T'^{\mu\nu}V'_\nu$$

$\because$ $T^{\mu\nu}V_\nu$ is a contravariant tensor

$$\therefore \quad (T^{\mu\nu}V_\nu)' = \frac{\partial x'^\mu}{\partial x^\lambda} T^{\lambda\nu}V_\nu = \underbrace{T^{\lambda\sigma}V_\sigma}_{\text{change dummy index}} \frac{\partial x'^\mu}{\partial x^\lambda}$$

$$= T^{\lambda\sigma}\frac{\partial x'^\mu}{\partial x^\lambda}\underbrace{\delta^\rho_\sigma V_\rho}_{V_\sigma} = T^{\lambda\sigma}\frac{\partial x'^\mu}{\partial x^\lambda}\left(\underbrace{\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\sigma}}_{\delta^\rho_\sigma}\right)V_\rho$$

$$= T^{\lambda\sigma}\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\sigma}\left(\frac{\partial x^\rho}{\partial x'^\nu}V_\rho\right)$$

$\because$ V_ρ is a covariant tensor

$$\therefore \quad V'_\nu = \frac{\partial x^\rho}{\partial x'^\nu}V_\rho$$

$$\therefore \quad (T^{\mu\nu}V_\nu)' = T^{\lambda\sigma}\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\sigma}V'_\nu$$

$$\Rightarrow \quad T'^{\mu\nu}V'_\nu = T^{\lambda\sigma}\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\sigma}V'_\nu$$

$$\Rightarrow \quad \left(T'^{\mu\nu} - T^{\lambda\sigma}\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\sigma}\right)V'_\nu = 0$$

3).

$\because$ V'_ν is an arbitrary covariant tensor,

$\therefore$ $T'^{\mu\nu} - T^{\lambda\sigma} \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\sigma} = 0$

$$\Rightarrow \quad T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\sigma} T^{\lambda\sigma}$$

This is precisely the way a contravariant tensor transforms.

$\Rightarrow$ $T^{\mu\nu}$ is a tensor

$\Rightarrow$ The proof does NOT depend on the rank of tensor involved.

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3). Starting from the contravariant geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \qquad ①$$

define $V^\mu \equiv \frac{dx^\mu}{d\tau}$ $\qquad V_\mu = \frac{dx_\mu}{d\tau}$

then ① becomes

$$\frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} V^\nu V^\lambda = 0 \qquad ②$$

multiply ② by V_μ gives

$$V_\mu \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} V_\mu V^\nu V^\lambda = 0$$

$$\because \quad V_\mu = g_{\mu\rho} V^\rho$$

$$\therefore \quad V_\mu \frac{dV^\mu}{d\tau} + g_{\mu\rho} \Gamma^\mu_{\nu\lambda} V^\rho V^\nu V^\lambda = 0$$

$\therefore$ $\cancel{V_\mu \frac{dV^\mu}{d\tau}}$ use the identity

$$g_{\mu\rho} \Gamma^\mu_{\nu\lambda} = \frac{1}{2}\left(\frac{\partial g_{\nu\rho}}{\partial x^\lambda} + \frac{\partial g_{\lambda\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\rho}\right)$$

then $V_\mu \frac{dV^\mu}{d\tau} + \frac{1}{2}\left(\frac{\partial g_{\nu\rho}}{\partial x^\lambda} + \frac{\partial g_{\lambda\rho}}{\partial x^\nu} - \frac{\partial g_{\lambda\nu}}{\partial x^\rho}\right) V^\rho V^\nu V^\lambda$

$\Rightarrow$ consider $\left(\frac{\partial g_{\lambda\rho}}{\partial x^\nu} - \frac{\partial g_{\lambda\nu}}{\partial x^\rho}\right) V^\rho V^\nu V^\lambda$

$$= \frac{\partial g_{\lambda\rho}}{\partial x^\nu} V^\rho V^\nu V^\lambda - \frac{\partial g_{\lambda\nu}}{x^\rho} V^\rho V^\nu V^\lambda$$

$$= \frac{\partial g_{\lambda\rho}}{\partial x^\nu} V^\rho V^\nu V^\lambda - \frac{\partial g_{\lambda\rho}}{\partial x^\nu} V^\rho V^\nu V^\lambda$$

renaming indices $\rho \leftrightarrow \nu$

$$= 0$$

$\therefore \quad V_\mu \frac{dV^\mu}{d\tau} + \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^\lambda} V^\rho V^\nu V^\lambda = 0$

Renaming $\lambda \rightarrow \mu$ $\Rightarrow$ $V_\mu \frac{dV^\mu}{d\tau} + \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^\mu} V^\mu V^\nu V^\rho = 0$

Consider $\frac{d}{d\tau}(V_\mu V^\mu) = V_\mu \frac{dV^\mu}{d\tau} + V^\mu \frac{dV_\mu}{d\tau}$.

In locally inertial coordinates $\frac{d}{d\tau}(V_\mu V^\mu)$

$$= \frac{d}{d\tau}(-c^2) = 0$$

$\because V_\mu V^\mu$ is a scalar $\therefore \frac{d}{d\tau}(V_\mu V^\mu) = 0$ always

$\therefore \quad V_\mu \frac{dV^\mu}{d\tau} = -V^\mu \frac{dV_\mu}{d\tau}$

Hence $-V^{\mu}\frac{dV_{\mu}}{d\tau} + \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^{\mu}} V^{\mu}V^{\nu}V^{\rho}$

$$\rightarrow \quad V^{\mu}\left(\frac{dV_{\mu}}{d\tau} - \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^{\mu}} V^{\nu}V^{\rho}\right) = 0$$

This is true for all V^{μ} and all $g_{\nu\rho}$

$$\therefore \quad \frac{dV_{\mu}}{d\tau} = \frac{1}{2}\frac{g_{\nu\rho}}{\partial x^{\mu}} V^{\nu}V^{\rho}$$

$$\Rightarrow \quad \underline{\underline{\frac{d^2 x_{\mu}}{d\tau^2} = \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^{\mu}}\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau}}}$$

$$\frac{dV_0}{d\tau} = \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^0} V^{\nu}V^{\rho} \checkmark$$

$\therefore$ For V_0 to be a constant of motion

$$\frac{dV_0}{d\tau} = 0 \quad \Rightarrow \quad \frac{\partial g_{\nu\rho}}{\partial x^0} = 0 \qquad \because x_0 = ct$$

$\therefore$ The condition is <u>$g_{\nu\rho}$ independent of time</u> ✓ 7/7

4).

The equation for hydrostatic equilibrium

$$\frac{\partial P}{\partial x^\nu} + (\rho c^2 + P)\frac{\partial \ln|g_{00}|^{1/2}}{\partial x^\nu} = 0$$

Assume spherically symmetric distribution

$$\frac{dP}{dr} + (\rho c^2 + P)\frac{d(\ln|g_{00}|^{1/2})}{dr} = 0$$

multiply by $\frac{dr}{(\rho c^2 + P)}$ gives

$$\frac{dP}{\rho c^2 + P} + d(\ln|g_{00}|^{1/2}) = 0$$

$$\rightarrow \int \frac{dP}{P + \rho c^2} + \ln|g_{00}|^{1/2} = \text{const}$$

The equation of state: $P = K\rho^\gamma$

$$\rightarrow dP = K\gamma\rho^{\gamma-1}d\rho$$

$$\therefore \int \frac{dP}{P+\rho c^2} = \int \frac{K\gamma\rho^{\gamma-1}}{P+\rho c^2}d\rho = \int \frac{K\gamma\rho^{\gamma-1}d\rho}{K\rho^\gamma + \rho c^2}$$

$$= \int \frac{K\gamma\rho^{\gamma-2}d\rho}{K\rho^{\gamma-1}+c^2}$$

∴ The hydrostatic equilibrium ~~eq~~ equation becomes

$$\int_{\rho_0}^{\rho} \frac{K\gamma \rho'^{\gamma-2}\, d\rho'}{K\rho'^{\gamma-1}+c^2} + \left[\ln|g_{00}|^{1/2}\right]_{r_0}^{r} = 0$$

$$\int_{\rho_0}^{\rho}\frac{K\gamma\rho^{\gamma-2}\,d\rho}{K\rho^{\gamma-1}+c^2} = \left[\cancel{K}\gamma \ln(K\rho^{\gamma-1}+c^2)\left(\frac{1}{\cancel{K}(\gamma-1)}\right)\right]_{\rho_0}^{\rho}$$

$$= \ln\left[\frac{1+K\rho^{\gamma-1}/c^2}{1+K\rho_0^{\gamma-1}/c^2}\right]^{\frac{\gamma}{\gamma-1}} = \ln|g_{00}|^{1/2}\Big|_{r}^{r_0}$$

$$\cancel{= \ln\left[|g_{00}|^{1/2}\right]\Big|^{r=r_0} - \left[|g_{00}|^{1/2}\right]\Big|^{r} - \ln\left[|g_{00}|^{1/2}\right]\Big|_{r=r_0}}$$

$$|g_{00}|(r) = 1-\frac{2GM}{rc^2} = 1-\frac{R_s}{r}$$

$$\therefore \ln\left[|g_{00}|^{1/2}\right]_{r}^{r_0} = \ln\left[\frac{1-\frac{R_s}{r_0}}{1-\frac{R_s}{r}}\right]^{1/2}$$

$$\therefore \ln\left[\frac{1+K\rho^{\gamma-1}/c^2}{1+K\rho_0^{\gamma-1}/c^2}\right]^{\frac{\gamma}{\gamma-1}} = \ln\left[\frac{1-\frac{R_s}{r_0}}{1-\frac{R_s}{r}}\right]^{1/2}$$

$$\therefore \quad \frac{1+K\rho^{\gamma-1}/c^2}{1+K\rho_0^{\gamma-1}/c^2} = \left(\frac{1-R_s/r_0}{1-R_s/r}\right)^{\frac{\gamma-1}{2\gamma}}$$

let $\quad 2\alpha\gamma \equiv \gamma-1 \quad \Rightarrow \quad \alpha = \frac{\gamma-1}{2\gamma}$

$$\therefore \quad \frac{1+K\rho^{\gamma-1}/c^2}{1+K\rho_0^{\gamma-1}/c^2} = \left(\frac{1-R_s/r_0}{1-R_s/r}\right)^{\alpha}$$

$\rightarrow$ Newtonian limit

$$a^2 = \frac{\gamma p}{\rho} = \frac{\gamma K\rho^\gamma}{\rho} = \gamma K\rho^{\gamma-1} \rightarrow K\rho^{\gamma-1} = \frac{a^2}{\gamma}$$

let $\underline{\frac{a_0^2}{\gamma} \equiv K\rho_0^{\gamma-1}}$ then

$$\frac{1+\frac{a^2}{\gamma c^2}}{1+\frac{a_0^2}{\gamma c^2}} = \left(1-\frac{R_s}{r_0}\right)^{\alpha}\left(1-\frac{R_s}{r}\right)^{-\alpha}$$

In the Newtonian limit $\quad R_s \ll r$

$$\therefore \quad \frac{1+\frac{a^2}{\gamma c^2}}{1+\frac{a_0^2}{\gamma c^2}} = \left(1-\frac{\alpha R_s}{r_0}\right)\left(1+\frac{\alpha R_s}{r}\right) = 1 + \alpha R_s\left(\frac{1}{r}-\frac{1}{r_0}\right) + O\left(\frac{R_s}{r}\right)^2$$

$$\Phi = -\frac{GM}{r} = -\frac{R_s c^2}{2r} \qquad \therefore \frac{R_s}{r} = \frac{-2\Phi}{c^2}$$

$$\therefore \quad \frac{1+\frac{a^2}{\gamma c^2}}{1+\frac{a_0^2}{\gamma c^2}} = 1 + \frac{2\alpha}{c^2}\left(\Phi(r_0) - \Phi(r)\right)$$

$$\Rightarrow \left(1+\frac{a^2}{\gamma c^2}\right) = \left(1+\frac{a_0^2}{\gamma c^2}\right)\left(1+\frac{2\alpha}{c^2}\left(\Phi(r_0) - \Phi(r)\right)\right]$$

if $\frac{a^2}{\gamma c^2}$ is small

$$\simeq 1 + \frac{a_0^2}{\gamma c^2} + \frac{2\alpha}{c^2}\left(\Phi(r_0) - \Phi(r)\right)$$

$$\therefore \quad a^2 - a_0^2 = 2\gamma\alpha\left[\Phi(r_0) - \Phi(r)\right]$$

$$\rightarrow \boxed{a^2 - a_0^2 = (\gamma - 1)\left[\Phi(r_0) - \Phi(r)\right].}$$

is the Newtonian limit $(a_0 = a(r=r_0))$

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5)

a) ∵ particle number conserved

∴ n, the number density in rest frame, is a scalar

∴ $J^\mu = n U^\mu$ is a vector

For this vector, ~~the~~ in the rest frame

the non-zero time component of J^μ is the ~~charge~~ number density, the space-components are zero, which is the particle flux in the rest frame. (flux in rest frame is obviously 0)

This combination is a 4-vector, and is true in the rest frame

→ It is the true flux vector

In Minkowski space with no gravitational field present,

we have $\partial_\alpha J^\alpha = 0$ for particle number conservation

Now for the ~~present~~ presence of gravitational field, replace ordinary derivatives to covariant derivatives

we get $J^\mu_{;\mu} = 0$ ✓

~~if nothing~~

$$J^{\mu}_{;\mu} = 0 \Rightarrow \frac{1}{\sqrt{|g|}} \frac{\partial(\sqrt{|g|} J^{\mu})}{\partial x^{\mu}} = 0$$

$$\therefore \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{\mu}} (\sqrt{|g|} J^{\mu}) = 0$$

$\because$ the time derivatives are 0

$$\cancel{\therefore \nabla \cdot (\sqrt{|g|}\,\underline{J}) = 0}$$

$$\cancel{\therefore \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)(\sqrt{|g|} J^{\mu}) = 0}$$

~~for~~ $\frac{\partial}{\partial x}(\sqrt{|g|} J^x) + \frac{\partial}{\partial y}(\sqrt{|g|} J^y) + \frac{\partial}{\partial z}(\sqrt{|g|} J^z) = 0$

For coordinates (ct, x, y, z), $|g| = 1$

$$\therefore \frac{\partial J^x}{\partial x} + \frac{\partial J^y}{\partial y} + \frac{\partial J^z}{\partial z} = 0$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J^r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, J^{\theta}) + \frac{1}{r\sin\theta}\frac{\partial J^{\phi}}{\partial \phi} = 0$$

$\because$ In Bondi Accretion, flow is spherically symmetrical $\therefore J^{\theta} = 0$, $J^{\phi} = 0$

$$\therefore \frac{\partial}{\partial r}(r^2 J^r) = 0 \rightarrow r^2 J^r = \text{const}$$

$$\therefore \frac{1}{\sqrt{|g|}}\left(\frac{\partial}{\partial r}(\sqrt{|g|}\,J^r) + \frac{\partial}{\partial \theta}(\sqrt{|g|}\,J^\theta) + \frac{\partial}{\partial \phi}(\sqrt{|g|}\,J^\phi)\right) = 0$$

$\because$ spherical symmetry

$\therefore \quad J^\theta = J^\phi = 0$

$$\therefore \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial r}(\sqrt{|g|}\,J^r) = 0$$

Also in spherical symmetry

$\therefore$ $g_{\theta\theta}$ and $g_{\phi\phi}$ are unaffected by gravity

$\therefore \quad g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2 \sin\theta$, the usual Minkowski spherical coordinates.

$$\therefore \quad |g| = |g_{tt}\, g_{rr}\, r^4 \sin^2\theta| = |g_{tt}\, g_{rr}|\, r^4 \sin^2\theta$$

$$\therefore |g'| = \frac{|g|}{\sin^2\theta} = |g_{tt}\, g_{rr}|\, r^4$$ ($|g'|$ only depends on r)

$$\therefore \sqrt{|g|} = |g'|^{1/2} \sin\theta$$

$$\therefore \frac{1}{|g'|^{1/2}\cancel{\sin\theta}}\frac{\partial}{\partial r}\left(|g'|^{1/2}\cancel{\sin\theta}\, J^r\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(|g'|^{1/2} J^r\right) = 0$$

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$\because$ $J^r = nU^r$, $|g'|$ only depends on r

$\therefore$ $\frac{d}{dr}\left(|g'|^{1/2} nU^r\right) = 0$

$\Rightarrow$ $|g'|^{1/2} nU^r = \text{const}$

∵ $J^r = n U^r$ ∴ $r^2 n U^r = \text{const}$

For spherical coordinates $\sqrt{|g|} = r^2 \sin\theta$

∴ $|g'|^{1/2} = \left|\frac{g}{\sin^2\theta}\right|^{1/2} = \frac{\sqrt{|g|}}{\sin\theta} = r^2$

$\Rightarrow n U^r |g'|^{1/2} = \text{const}$

5 b).

The energy equation $T^{t\nu}_{;\nu} = 0$

$\Rightarrow$ refer to Notes 4.6

$$\Rightarrow 0 = g^{t\nu}\frac{\partial P}{\partial x^\nu} + \frac{1}{|g|^{1/2}}\frac{\partial}{\partial x^\nu}\left[|g|^{1/2}\left(\rho + P/c^2\right)U^\nu U^t\right] + \Gamma^t_{\mu\lambda}\left(\rho + \frac{P}{c^2}\right)U^\mu U^\lambda$$

Spherical symmetry $\Rightarrow$ only $U^t, U^r \neq 0$

$U^\theta, U^\phi = 0$

∴ only need affine connection involving indices t and r, (Γ^r_{rr}, Γ^r_{tr} are useless because the upper free index needs to be t)

In general, affine connection

$$\Gamma^\sigma_{\mu\lambda} = \frac{g^{\nu\sigma}}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right)$$

For Γ^{a}_{ab} ($a=b$ permitted, No sum)

$$\Rightarrow \Gamma^{a}_{ab} = \frac{g^{\nu a}}{2}\left(\frac{\partial g_{a\nu}}{\partial x^{b}} + \frac{\partial g_{b\nu}}{\partial x^{a}} - \frac{\partial g_{ba}}{\partial x^{\nu}}\right)$$

For diagonal $g_{\mu\nu}$ and $g^{\mu\nu}$: (NO sums)

$g^{\nu a} \neq 0$ only if $\nu = a$

$$\therefore \Gamma^{a}_{ab} = \frac{g^{aa}}{2}\left(\frac{\partial g_{aa}}{\partial x^{b}} + \cancel{\frac{\partial g_{ab}}{\partial x^{a}}} - \cancel{\frac{\partial g_{ab}}{\partial x^{a}}}\right)$$

$$= \frac{1}{2} g^{aa} \frac{\partial g_{aa}}{\partial x^{b}} \quad \text{(NO sum)}$$

For diagonal $g^{\mu\nu}$, $g^{aa} = \frac{1}{g_{aa}}$

$$\therefore \Gamma^{a}_{ab} = \cancel{\tfrac{1}{2}g^{aa}} \frac{1}{2g_{aa}} \frac{\partial g_{aa}}{\partial x^{b}} \quad \text{(} a=b \text{ permitted No sum)}$$

Apply this to the problem :

$$\Gamma^{t}_{tt} = \frac{1}{2g_{tt}} \frac{\partial g_{tt}}{\partial (ct)} = 0 \quad \text{(} \because \text{ No time } \cancel{\text{depent}} \text{ dependence)}$$

$$\cancel{\Gamma^{t}_{tr} = \frac{1}{2g_{tt}} \partial g}$$

$$\Gamma^{t}_{rt} = \Gamma^{t}_{tr} = \frac{1}{2g_{tt}} \frac{\partial g_{tt}}{\partial r} = \frac{1}{2} \frac{\partial \ln |g_{tt}|}{\partial r}$$

affine connection
symmetric in lower indices

We now look at the energy equation term by term

→ $~~g^{tt}~~ \quad g^{t\nu} \frac{\partial P}{\partial x^\nu}$ $\quad g^{t\nu} \neq 0$ only for g^{tt}

then $\nu = t$ $\quad \frac{\partial P}{\partial x^t} = 0$

$\therefore$ this term is 0

→ $\frac{1}{|g|^{1/2}} \frac{\partial}{\partial x^\nu} \left[|g|^{1/2} \left(\rho + \frac{P}{c^2}\right) U^\nu U^t \right]$

this term $= 0$ if $\nu = t$ (no time dependence)

$\therefore$ this term is $\frac{1}{|g|^{1/2}} \frac{\partial}{\partial r} \left[|g|^{1/2} \left(\rho + \frac{P}{c^2}\right) U^r U^t \right]$

→ $\Gamma^t_{\nu\lambda} \left(\rho + \frac{P}{c^2}\right) U^\nu U^\lambda$

this term is

$$\Gamma^t_{tr} \left(\rho + \frac{P}{c^2}\right) U^t U^r + \Gamma^t_{rt} \left(\rho + \frac{P}{c^2}\right) U^r U^t$$

$$= 2 \Gamma^t_{tr} \left(\rho + P/c^2\right) U^r U^t$$

$$= \frac{\partial \ln |g_{tt}|}{\partial r} \left(\rho + P/c^2\right) U^r U^t$$

$$= \frac{1}{g_{tt}} \frac{\partial g_{tt}}{\partial r} \left(\rho + \frac{P}{c^2}\right) U^r U^t$$

$$\sqrt{|g|} = r^2 \sin\theta \sqrt{|g_{tt}|}\sqrt{|g_{rr}|} = \cancel{r^2}\sqrt{|g'|}\sin\theta$$

$$\therefore \frac{1}{\cancel{\sin\theta}\sqrt{|g'|}}\frac{\partial}{\partial r}\left[\cancel{\sin\theta}\sqrt{|g'|}\left(\rho + \frac{p}{c^2}\right)U^t U^r\right)$$

$$+ \frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial r}\left(\rho + \frac{p}{c^2}\right)U^t U^r = 0$$

multiply by $g_{tt}\sqrt{|g'|}$:

$$\Rightarrow g_{tt}\frac{\partial}{\partial r}\left(\sqrt{|g'|}\left(\rho + \frac{p}{c^2}\right)U^t U^r\right)$$

$$+ \sqrt{|g'|}\frac{\partial g_{tt}}{\partial r}\left(\rho + \frac{p}{c^2}\right)U^t U^r = 0$$

$$\cancel{\Rightarrow g_{tt}\frac{\partial}{\partial r}\left(\sqrt{|g'|}\left(\rho + \frac{p}{c^2}\right)U^t U^r\right)}$$

$$\cancel{+ \sqrt{|g'|}\frac{\partial}{\partial r}\left(g_{tt}\left(\rho + \frac{p}{c^2}\right)U^t U^r\right)}$$

$$\cancel{= \sqrt{|g'|}\, g_{tt}\frac{\partial}{\partial r}\left[\left(\rho + \frac{p}{c^2}\right)U^t U^r\right]}$$

$$\Rightarrow g_{tt}\frac{\partial}{\partial r}\left[\sqrt{|g'|}\left(\rho + \frac{p}{c^2}\right)U^t U^r\right]$$

$$+ \left[\sqrt{|g'|}\left(\rho + \frac{p}{c^2}\right)U^t U^r\right]\frac{\partial g_{tt}}{\partial r} = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left[g_{tt}\sqrt{|g'|}\left(\rho + \frac{p}{c^2}\right)U^t U^r\right] = 0$$

$$\Rightarrow \quad g_{tt}\sqrt{|g'|}\left(\rho + \frac{P}{c^2}\right) U^t U^r = \text{const}$$

$\because \quad U_t = g_{t\nu} U^\nu = g_{tt} U^t + g_{tr} U^r$

and $\quad g_{tr} = 0$

$\therefore \quad U_t = g_{tt} U^t$

substitute this in and multiply by c^2

$$\Rightarrow \quad \underline{(P + \rho c^2)\, U^r U_t\, |g'|^{1/2} = \text{const}}$$

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5.) $D_\mu J^\mu = 0$? $\qquad \frac{\partial n}{\partial t} + \nabla \cdot (n \underline{u}) = 0.$

$\partial_\alpha J^\alpha = 0 \quad \rightarrow \quad D_\mu J^\mu$

Always can go into local rest frame where curvature is not important and locally Minkowski

$$D_\mu J^\mu = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}\, J^\mu\right) = 0.$$

$$\frac{1}{|\sqrt{|g|}|} \frac{\partial}{\partial r}\left(\sqrt{|g|}\, J^r\right) = 0$$