16B2 Q1

proper time T is the time experienced by an observer in its own rest frame.

4-velocity $U = \frac{dX}{dT}$, where $X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$ is
$= \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$

the position 4-vector

4-acceleration $A = \frac{dU}{dT}$

$\frac{dt}{dT} = \gamma$, $\frac{d\gamma}{dt} = \gamma^3 (\underline{v} \cdot \underline{a}) \frac{1}{c^2}$

$$\therefore \quad A = \frac{dU}{dT} = \frac{d}{dT}\begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix} = \frac{dt}{dT}\frac{d}{dt}\begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$$

$$= \gamma \begin{pmatrix} \frac{d}{dt}(\gamma c) \\ \frac{d}{dt}(\gamma \underline{v}) \end{pmatrix}$$

$$= \gamma \begin{pmatrix} c\frac{d\gamma}{dt} \\ \underline{v}\frac{d\gamma}{dt} + \gamma \frac{d\underline{v}}{dt} \end{pmatrix}$$

$$= \gamma \begin{pmatrix} c\frac{\gamma^3}{c^2}(\underline{v} \cdot \underline{a}) \\ \underline{v}\frac{\gamma^3}{c^2}(\underline{v} \cdot \underline{a}) + \gamma \underline{a} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\gamma^4}{c}(\underline{v} \cdot \underline{a}) \\ \frac{\gamma^4}{c^2}(\underline{v} \cdot \underline{a})\underline{v} + \gamma^2 \underline{a} \end{pmatrix}.$$

- invariant

U ; $P = m_0 U$

4 velocity

4 momentum

$$P \cdot U = m_0 U \cdot U = \underline{\underline{-m_0 c^2}}$$

$$\left(U \cdot U = \begin{pmatrix} c \\ 0 \end{pmatrix} \cdot \begin{pmatrix} c \\ 0 \end{pmatrix} = \underline{\underline{-c^2}} \right)$$

in rest frame

$$P \cdot P = \underline{\underline{-m_0^2 c^2}}$$

$$U \cdot A = \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix} \cdot \begin{pmatrix} \frac{\gamma^4}{c}(\underline{v} \cdot \underline{a}) \\ \frac{\gamma^4}{c^2}(\underline{v} \cdot \underline{a})\underline{v} + \gamma^2 \underline{a} \end{pmatrix}$$

$$= -\gamma^5 (\underline{v} \cdot \underline{a}) + \frac{\gamma^5}{c^2}(\underline{v} \cdot \underline{a}) v^2 + \gamma^3 \underline{v} \cdot \underline{a}$$

$$= -\gamma^5 (\underline{v} \cdot \underline{a}) + \gamma^5 \underline{v} \cdot \underline{a} \left(\frac{v^2}{c^2} + \frac{1}{\gamma^2} \right)$$

$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow$

$$= -\gamma^5 (\underline{v} \cdot \underline{a}) + \gamma^5 \underline{v} \cdot \underline{a} \left(\underbrace{\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}}_{1} \right)$$

$$= -\gamma^5 (\underline{v} \cdot \underline{a}) + \gamma^5 (\underline{v} \cdot \underline{a})$$

$$= \underline{\underline{0}} \qquad \therefore \underline{U \cdot A = 0}$$

~~$D = B =$~~

interval $\Delta S^2 = (D-B)\cdot(D-B)$

$= -c^2(t_d - t_b)^2 + |\underline{x}_d - \underline{x}_b|^2$

If time-like $\Delta S^2 < 0$

$\therefore \quad c^2(t_d - t_b)^2 > |\underline{x}_d - \underline{x}_b|^2$

Consider $D = \begin{pmatrix} ct_d \\ x_d \\ 0 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} ct_b \\ x_b \\ 0 \\ 0 \end{pmatrix}$ in S

If in S' they are simultaneous, then

$t_d' = t_b'$ $\qquad t_d' = \gamma\left(t_d - \frac{v x_d}{c^2}\right)$

$\qquad\qquad t_b' = \gamma\left(t_b - \frac{v x_b}{c^2}\right)$

$\therefore \quad t_d - \frac{v x_d}{c^2} = t_b - \frac{v x_b}{c^2}$

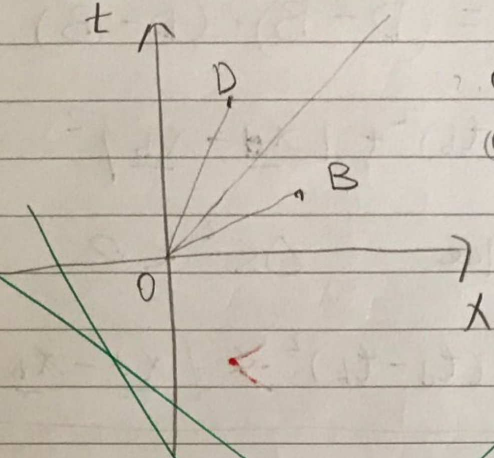
$\therefore c^2(t_d - t_b) = v\,(x_d - x_b)$

$\therefore |v| = \frac{|t_d - t_b|}{|x_d - x_b|} c^2$

$\because$ time like $\therefore \quad c^2|t_d - t_b|^2 > |x_d - x_b|^2$

$\therefore \frac{|t_d - t_b|}{|x_d - x_b|} > \frac{1}{c}$ $\qquad \therefore |v| > \frac{1}{c}\cdot c^2 = c$

∴ This is impossible



$c^2 t_d t_b = x_d x_b$

$(ct_d)(ct_b) = x_d x_b$

$D \cdot B = 0 \rightarrow (ct_d)(ct_b) = |x_d||x_b|\cos\alpha$

$\cos\alpha = \dfrac{(ct_d)(ct_b)}{|x_d|\,|x_b|} \leq 1$

$\therefore \dfrac{|x_d|}{ct_d} \geq \dfrac{ct_b}{|x_b|}$

- OD and OB are symmetric about the 45° photon worldline

So OB is the line of simultaneity for an observer whose worldline is OD, and vice versa

~~* If B is space time like, D is space like (vice versa)~~

- proper acceleration is the acceleration of an object measured by an observer instantaneously at rest with respect to the object.

- pure force is the force that does not change the rest mass of an object.

If $D \cdot B = 0$

$$D_0 B_0 = |\underline{D}| |\underline{B}| \cos\theta$$

$$\cos\theta = \frac{D_0 B_0}{|\underline{D}| |\underline{B}|} < 1$$

$\therefore$ D, B cannot be both time-like

- World line is $x^2 - t^2 = L^2$ ~~$\to t = \sqrt{x^2-L^2}$~~

$V = \frac{dx}{dt}$ $\therefore$ $2x\frac{dx}{dt} - 2t = 0$.

~~$\therefore \frac{dx}{dt} = \frac{t}{x} = \frac{t}{\sqrt{x^2-L^2}}$~~

$x = \sqrt{L^2 - t^2}$ $\therefore V = \frac{dx}{dt} = \frac{t}{x} = \boxed{\frac{t}{\sqrt{L^2-t^2}}}$

~~$\gamma(v) = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\frac{t^2}{L^2-t^2}}} = \frac{1}{\sqrt{L^2-t^2}}$~~

$t = \sqrt{x^2-L^2}$ $\therefore$ $\boxed{V = \frac{\sqrt{x^2-L^2}}{x}}$

$\gamma(v) = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\frac{x^2-L^2}{x^2}}} = \frac{1}{\sqrt{\frac{\cancel{x^2}-\cancel{x^2}+L^2}{x^2}}} = \frac{x}{L}$

$\because x\frac{dx}{dt} - t = 0$ $\therefore \frac{dx}{dt}\cdot\frac{dx}{dt} + x\frac{d^2x}{dt^2} - 1 = 0$

$\therefore v^2 + xa = 1$ $a = \frac{1-v^2}{x}$

~~$\because v^2 =$~~ $\because v^2 = \frac{x^2-L^2}{x^2}$ $\therefore 1 - v^2 = \frac{L^2}{x^2}$

$\therefore \boxed{a = \frac{L^2}{x^3}}$ $\gamma^3 a = \left(\frac{x}{L}\right)^3\left(\frac{L^2}{x^3}\right) = \frac{1}{L}$

$= \text{const}$

$\underline{f} = \frac{d\underline{p}}{dt} = \frac{d}{dt}(\gamma m \underline{v})$

∴ $\gamma m \underline{v} = \underline{f}t + \underline{c}$

At $t=0$, $\underline{v}=0$ ∴ $\underline{c}=\underline{0}$

∴ $\gamma m\underline{v} = \underline{f}t$ $\underline{f}$ and $\underline{v}$ are parallel

∴ ~~their~~ their magnitudes $\gamma m v = ft$

∵ $\beta = \frac{v}{c}$ ∴ $\cancel{\gamma m}$ $\gamma\beta mc = ft$

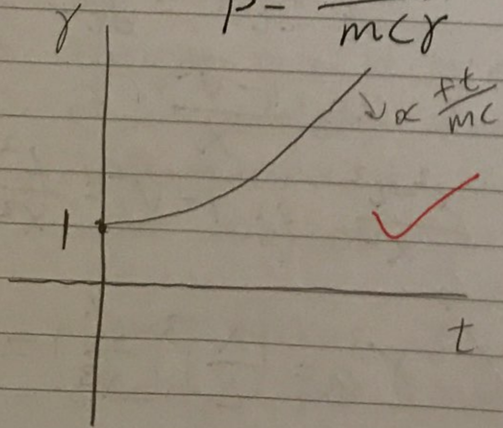
$$\therefore \begin{cases} \gamma\beta = \frac{ft}{mc} \\ \gamma^2(1-\beta^2) = 1 \end{cases} \qquad \because \cancel{ft}\ \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

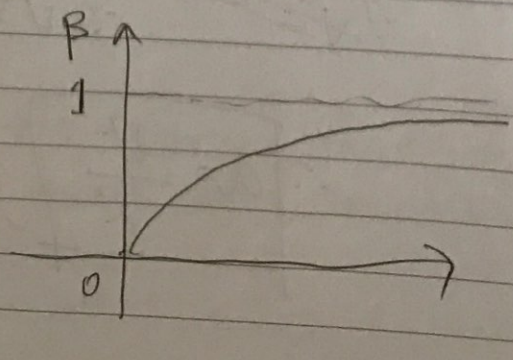
∴ $\gamma^2 - (\gamma\beta)^2 = 1$ ∵ $\gamma\beta = \frac{ft}{mc}$

∴ $\gamma^2 = 1 + \frac{f^2t^2}{m^2c^2}$

$\beta = \frac{ft}{mc\gamma}$

$$\therefore \boxed{\begin{aligned} \gamma &= \sqrt{1+\left(\frac{ft}{mc}\right)^2} \\ \beta &= \frac{ft/mc}{\sqrt{1+(ft/mc)^2}} \end{aligned}}$$





- electric force is pure force

$$\therefore \frac{dE}{dt} = \cancel{f \cdot u} \; \underline{f} \cdot \underline{u} \qquad \because E = \gamma m c^2$$

→ force

velocity

rate of change of energy

$$\therefore \frac{d\gamma}{dt} = \frac{\underline{f} \cdot \underline{u}}{mc^2}$$

in this case $\underline{f}$ and $\underline{u}$ are parallel

$$\because \underline{f} = e\underline{E} \qquad \therefore \frac{d\gamma}{dt} = \frac{eEu}{mc^2}$$

$$d\gamma = \frac{eE}{mc^2} \underbrace{u\,dt}_{dx} \qquad \therefore \frac{d\gamma}{dx} = \frac{eE}{mc^2}$$

$$\therefore \cancel{\gamma_f - \gamma_i} \; \gamma(x) - \gamma(x=0) = \frac{eEx}{mc^2}$$

- electron ~~se~~ accelerates from rest.

$$\therefore \gamma(0) = \cancel{0}\, 1$$

$$\therefore \gamma(L) = 1 + \frac{eEL}{mc^2} = 1 + \frac{(e)(5MV/m)(10m)}{0.511 MeV}$$

$$= 1 + \frac{50}{0.511} = \underline{98.85}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \underline{0.99995}$$

If now $E = (1 - 20\%)(5MeV/m) = 4MV/m$

$$\gamma'(L) = 1 + \frac{40}{0.511} = 79.28$$

V6B2 Q4

$\beta' = \sqrt{1-\frac{1}{\gamma^2}} = 0.99992$

β changes by $\frac{|\beta'-\beta|}{\beta} \times 100\% =$
(reduces)

$= \underline{\underline{0.003\%}}$

time takes $ft = \gamma m V = \gamma m \beta c$

$f = eE$

$\therefore \quad t = \frac{\gamma\beta m c}{eE} \simeq \frac{\gamma m c}{eE}$ ($\beta \sim 1$)

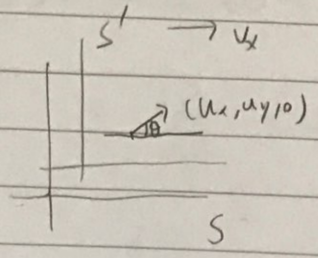
$= \frac{(98.85)\, 0.511\, \cancel{MeV}/c^{\cancel{2}} \cdot \cancel{c}}{e(5\cancel{MV}/m)}$

$= \frac{98.85 \times 0.511}{5} \times \frac{1}{c} \cdot \frac{s}{\cancel{m}} \cdot \cancel{m}$

$= \frac{98.85 \times 0.511}{5} \times \frac{1}{3\times10^8}\ s$

$= \underline{\underline{3.37 \times 10^{-8}\ s}}$

16B2 Q2



Lorentz transformation

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

$$u_x' = \frac{dx'}{dt'} = \frac{dx'/dt}{dt'/dt}$$

$$= \frac{\gamma\left(\frac{dx}{dt} - v\right)}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy'/dt}{dt'/dt} = \frac{dy/dt}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

$\underline{u} = \underline{u_{\parallel}} + \underline{u_{\perp}}$

$\Rightarrow u_y = |u|\sin\theta \qquad u_x = |u|\cos\theta$

$$\tan\theta' = \frac{u_y'}{u_x'} = \frac{\frac{u_y}{\gamma\left(\cancel{1 - \frac{vu_x}{c^2}}\right)}}{\frac{u_x - v}{\cancel{1 - \frac{u_x v}{c^2}}}} = \frac{u_y}{\gamma(u_x - v)}$$

$$\cancel{= \frac{|u|\sin\theta}{\gamma_v(|u|}} = \frac{u\sin\theta}{\gamma_v(u\cos\theta - v)}$$

$$\gamma_v = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$|u| = u$

— electron in EM field, $\underline{f} = -e(\underline{E} + \underline{v} \times \underline{B})$

$\because$ pure force $\therefore \frac{dE}{dt} = \underline{f} \cdot \underline{v} \qquad \because E = \gamma m_0 c^2$

$$\therefore \frac{d\gamma}{dt} = \frac{\underline{f} \cdot \underline{v}}{m_0 c^2}$$

$\underline{f} \cdot \underline{v} = -e(\underline{E} + \underline{v} \times \underline{B}) \cdot \underline{v} = -e\underline{E} \cdot \underline{v} = -e E_z V_z$

$\because \underline{E} = (0, 0, E_z)^T$

$\therefore \dfrac{d\gamma}{dt} = -\dfrac{e E_z V_z}{m_0 c^2}$

$\therefore d\gamma = -\dfrac{e E_z}{m_0 c^2} V_z dt = -\dfrac{e E_z}{m_0 c^2} dZ$

$\therefore \gamma_f - \gamma_i = -\dfrac{e E_z}{m_0 c^2}(Z_f - Z_i)$

assume at $t = t_0$, $Z = 0$. $\gamma_i = 17$, $Z_f = 1m$

$\therefore \gamma_f = \gamma_i - \dfrac{e E_z Z_f}{m_0 c^2}$

$\underline{P} = \gamma m \underline{V}$, $\underline{P_{\parallel}} = \gamma m \underline{V_{\parallel}}$, $\underline{P_{\perp}} = \gamma m \underline{V_{\perp}}$

($\parallel$ and $\perp$ here are with respect to $\hat{\underline{z}}$, not $\underline{V}$)

$\underline{V} = \underline{V_{\parallel}} + \underline{V_{\perp}}$

$\underline{E_{\perp}} = 0$, $\underline{E_{\parallel}} = \underline{E}$, $\underline{B_{\perp}} = 0$ $\underline{B_{\parallel}} = \underline{B}$

$\therefore \underline{f} = -e(\underline{E_{\parallel}} + (\underline{V_{\parallel}} + \underline{V_{\perp}}) \times \underline{B_{\parallel}})$ $\because \underline{V_{\parallel}} \times \underline{B_{\parallel}} = 0$

$\therefore \underline{f} = -e(\underline{E_{\parallel}} + \underline{V_{\perp}} \times \underline{B_{\parallel}}) = \dfrac{d\underline{P}}{dt} = \left(\dfrac{d\underline{P_{\parallel}}}{dt} + \dfrac{d\underline{P_{\perp}}}{dt}\right)$

$\frac{d\underline{P}_\perp}{dt} = \frac{d}{dt}(\underline{P} - \underline{P}_\parallel)$
$= \frac{d}{dt}(\underline{P} - (\underline{P}\cdot\hat{\underline{z}})\hat{\underline{z}})$
$= \frac{d\underline{P}}{dt} - (\frac{d\underline{P}}{dt}\cdot\hat{\underline{z}})\hat{\underline{z}} = \frac{d\underline{P}}{dt} - (\frac{d\underline{P}}{dt})_\parallel$
$= (\frac{d\underline{P}}{dt})_\perp$

$\underline{P}_\parallel = \gamma m \underline{V}_\parallel = \gamma m V_z \hat{\underline{z}}$

$\therefore \frac{d\underline{P}_\parallel}{dt} = \frac{d}{dt}(\gamma m V_z)\hat{\underline{z}} \quad (\frac{d\hat{\underline{z}}}{dt} = 0)$

$\therefore \frac{d\underline{P}_\parallel}{dt}$ always along $\hat{\underline{z}}$ $(\parallel)$

$\underline{P}_\perp$ is always $\perp$ $\hat{\underline{z}}$ at all times

$\therefore \frac{d\underline{P}_\perp}{dt}$ always $\perp \hat{\underline{z}}$ $(\perp)$

$\because$ $\underline{E}_\parallel$ along $\hat{\underline{z}}$, $\underline{V}_\perp \times \underline{B}_\parallel \perp \hat{\underline{z}}$

$\therefore \frac{d\underline{P}_\parallel}{dt} = -e\underline{E}_\parallel$, $\frac{d\underline{P}_\perp}{dt} = -e\underline{V}_\perp \times \underline{B}_\parallel$

$\because \underline{P}_\perp = \gamma m \underline{V}_\perp$ $\therefore$ ~~$\frac{d}{dt}(P_\perp^2)$~~

$\therefore \frac{d}{dt}(P_\perp^2) = 2\underline{P}_\perp \cdot \frac{d\underline{P}_\perp}{dt} = -2\gamma m e \underbrace{\underline{V}_\perp\cdot(\underline{V}_\perp \times \underline{B}_\parallel)}_{0}$

$= 0$

$\therefore |P_\perp|$ remains constant

At $t = t_0$, $\frac{|P_\parallel|}{|P_\perp|} = 1$ $\therefore \frac{\gamma m V_z}{\gamma m V_\perp} = 1 = \frac{V_z}{V_\perp}$

$\therefore V_z = V_\perp$ at $t=0$

At $t=0$ $\gamma = 17 = \frac{1}{\sqrt{1-\beta^2}}$ $(\beta = \frac{v}{c})$

$\therefore 1-\beta^2 = \frac{1}{17^2}$ $\therefore$ ~~$V_0 = c$~~ $\beta = \sqrt{1-\frac{1}{17^2}} = 0.998$

$\therefore \cancel{v_z(0) = v_\perp(0) = 0}$

$\cancel{v(t_0)}$ $\quad v_{t_0} = \cancel{0.998c} = \frac{12\sqrt{2}}{17}c$

$$v_{t_0}^2 = v_z^2(t_0) + v_\perp^2(t_0) = 2v_\perp^2(t_0) = (0.998)^2c^2$$

$$\therefore \quad v_\perp(t_0) = \frac{0.998c}{\sqrt{2}} = \cancel{0.706c}\ \frac{12\sqrt{2}}{\sqrt{2}\,17}c = \frac{12}{17}c$$

$$|p_\perp| = \gamma m_0 v_\perp = \gamma(t_0)\, m_0\, v_\perp(t_0)$$

$$= \cancel{(17 \times 0.706)m_0c = 12m_0c}$$

$$= 17 \times m_0 \times \frac{12}{17}c = 12m_0c$$

At $\quad z = 1m. \quad \gamma = \gamma_f \quad v = v_\perp = v_f \quad v_z = 0$

$$\gamma_f v_f m_0 = \frac{v_f}{\sqrt{1 - v_f^2/c^2}} m_0 = 12m_0c = |p_\perp| \quad (p_{\parallel} = 0 \text{ at } z = 1m)$$

$$\therefore \quad v_f^2 = 12^2\left(1 - \frac{v_f^2}{c^2}\right)c^2$$

$$\therefore \quad v_f^2 = 144c^2 - 144v_f^2$$

$$\therefore \quad 145v_f^2 = 144c^2 \quad \therefore \quad v_f = \frac{12}{\sqrt{145}}c$$

$$\gamma_f = \frac{1}{\sqrt{1 - v_f^2/c^2}} = \underline{\underline{12.04}}$$

$\therefore\ 12.04 - 17 = -\dfrac{e E_z (1m)}{0.511\,MeV}$

$\therefore\ e E_z (1m) = (0.511 \times 10^6)(17 - 12.04)$

$= 2.535 \times 10^6\,eV$

$$\boxed{E_z = 2.535 \times 10^6\ V/m}$$

$$\underline{f} = -e(\underline{E_{//}} + \underline{V_\perp} \times \underline{B_{//}}) = \gamma m_0 \underline{a} + \frac{\underline{f}\cdot\underline{V}}{c^2}\underline{V}$$

$$\left(\underline{f} = \frac{d}{dt}(\gamma m_0 \underline{V}) = m_0\left(\gamma \frac{d\underline{V}}{dt} + \underline{V}\frac{d\gamma}{dt}\right) = \gamma m_0 \underline{a} + m_0 \underline{V}\frac{\underline{f}\cdot\underline{V}}{m_0 c^2} \right.$$

$$\left. = \gamma m_0 \underline{a} + \frac{\underline{f}\cdot\underline{V}}{c^2}\underline{V} \right).$$

$\therefore\ -e(\underline{E_{//}} + \underline{V_\perp} \times \underline{B_{//}}) = \gamma m_0 (\underline{a_{//}} + \underline{a_\perp}) + \dfrac{e E_z V_z}{c^2}(\underline{V_{//}} + \underline{V_\perp})$

$\because\ \underline{E_{//}}, \underline{a_{//}}, \underline{V_{//}}$ along $\hat{\underline{z}}$ always

$\underline{V_\perp} \times \underline{B_{//}}, \underline{a_\perp}, \underline{V_\perp} \perp \hat{\underline{z}}$ always

$\therefore\ -e\underline{E_{//}} = \gamma m_0 \underline{a_{//}} - \dfrac{e E_z V_z}{c^2}\underline{V_{//}}$

$-e\underline{V_\perp} \times \underline{B_{//}} = \gamma m \underline{a_\perp} - \dfrac{e E_z V_z}{c^2}\underline{V_\perp}$

$\because\ \underline{V_\perp} \perp \underline{V_\perp} \times \underline{B_{//}}$ let $\underline{V_\perp} \times \underline{B_{//}}$ along direction $\hat{\underline{r}}$

$\hat{\underline{r}} = \dfrac{\underline{V_\perp} \times \hat{\underline{z}}}{|\underline{V_\perp} \times \hat{\underline{z}}|}$

and $\underline{V_\perp}$ along $\hat{\underline{v}} = \dfrac{\underline{V_\perp}}{|\underline{V_\perp}|}$

then $\underline{V_\perp} \times \underline{B} = V_\perp B \, \underline{\hat{r}}$ $\quad \underline{V_\perp} = V_\perp \underline{\hat{v}}$

$\underline{\hat{r}} \perp \underline{\hat{v}}$, let $\underline{a_\perp} = a_r \underline{\hat{r}} + a_v \underline{\hat{v}}$

∴ $-e V_\perp B \underline{\hat{r}} = \gamma m (a_r \underline{\hat{r}} + a_v \underline{\hat{v}}) - \frac{e E_z V_z}{c^2} V_\perp \underline{\hat{v}}$

∴ $-e V_\perp B \underline{\hat{r}} = \gamma m a_r \underline{\hat{r}}$

$\gamma m a_v \underline{\hat{v}} = \frac{e E_z V_z V_\perp}{c^2} \underline{\hat{v}}$

∴ $e V_\perp B = \gamma m |a_r|$

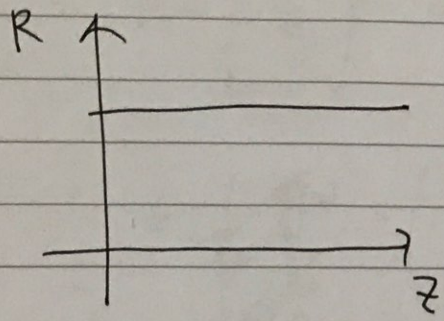
$|a_r|$ is the magnitude of acceleration perpendicular to both velocity and $\underline{B}$

∴ $|a_r| = \frac{V_\perp^2}{R}$ (R = ~~Larm~~ Larmor radius)

$e \cancel{V_\perp} B = \frac{\gamma V_\perp^2}{R} m_0$ ∴ $R = \frac{\gamma m_0 V_\perp}{eB}$

$R = \frac{|P_\perp|}{eB}$ ∵ $|P_\perp|$ is constant

$\therefore$ R remains constant with z



— two photons $\qquad \hbar = c = 1$

$$P_1 = \begin{pmatrix} E_1 \\ E_1 \\ 0 \\ 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} E_2 \\ E_2\cos\theta \\ E_2\sin\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \omega \\ \omega \\ 0 \\ 0 \end{pmatrix} \qquad = \begin{pmatrix} \omega \\ \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

$$P_{tot} = P_1 + P_2$$

$$E_{cm}^2 = -P_{tot} \cdot P_{tot}$$

$$= -P_1^2 - P_2^2 - 2P_1 \cdot P_2$$

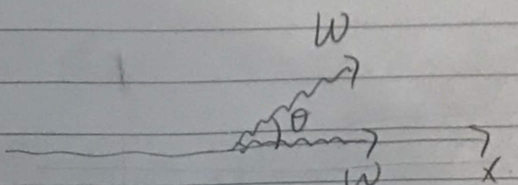
$$= -0 - 0 - 2P_1 \cdot P_2 \qquad \left(\begin{array}{l} P_1^2 = P_2^2 = 0 \\ \text{for photons} \end{array}\right)$$

$$= -2\left[-\omega^2 + \omega^2\cos\theta\right]$$

$$= 2\omega^2(1-\cos\theta)$$

$\therefore \cancel{E_{cm} = \sqrt{2}\,\omega\sqrt{}}$

$$\underline{\underline{E_{cm} = \hbar\omega\sqrt{2(1-\cos\theta)}}} \qquad \theta \in \cancel{[0, 2\pi]}$$



velocity of cm frame is

$$\underline{V}_{cm} = \frac{\underline{P}_{tot}}{E_{tot}} \qquad P_1 = \begin{pmatrix} \omega \\ \omega \\ 0 \\ 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} \omega \\ \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

$$\therefore P_{tot} = \begin{pmatrix} E_{tot} \\ \underline{P}_{tot} \end{pmatrix} \qquad \therefore \underline{P}_{tot} = \begin{pmatrix} \omega(1+\cos\theta) \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

$$E_{tot} = 2\omega$$

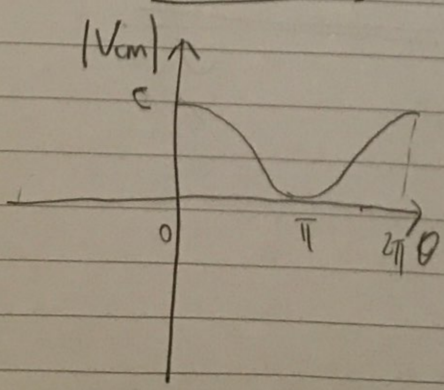
~~V_{cm} =~~

$$\therefore \underline{V}_{cm} = \frac{1}{2\cancel{\omega}} \begin{pmatrix} \cancel{\omega}(1+\cos\theta) \\ \cancel{\omega}\sin\theta \\ 0 \end{pmatrix} c$$

$$\underline{V}_{cm} = \frac{c}{2} \begin{pmatrix} 1+\cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

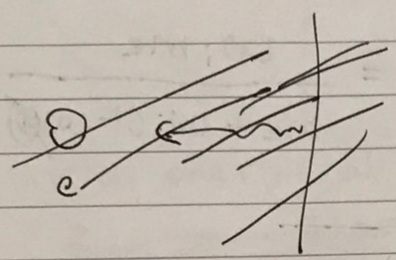
$$|V_{cm}| = \frac{c}{2}\sqrt{(1+\cos\theta)^2 + \sin^2\theta} = \frac{1}{2}c\sqrt{2+2\cos\theta}$$

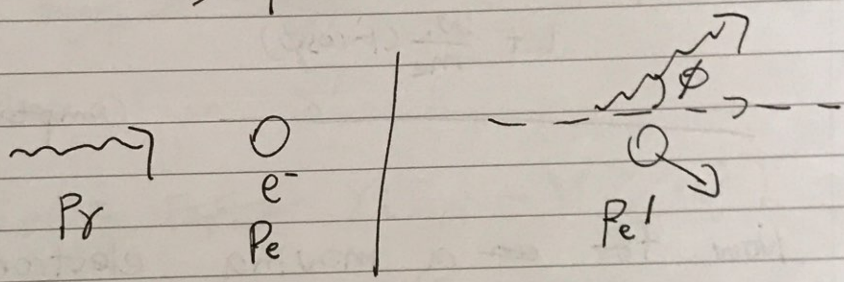
$$= \frac{1}{2}c\sqrt{2(1+\cos\theta)}$$



- inverse compton scattering

We start by deriving the Compton effect formula in electron rest frame





4 - vector $\quad P_\gamma + P_e = P_\gamma' + P_e'$

$$P_\gamma = \begin{pmatrix} P_\gamma \\ \underline{P_\gamma} \end{pmatrix} \quad P_e = \begin{pmatrix} m_e \\ 0 \end{pmatrix} \quad P_\gamma' = \begin{pmatrix} P_\gamma' \\ \underline{P_\gamma'} \end{pmatrix} \quad P_e' = \begin{pmatrix} E_e' \\ \underline{P_e'} \end{pmatrix}$$

$$P_e' \cdot P_e' = (P_\gamma + P_e - P_\gamma')^2$$

$$\cancel{-m_e^2} = \cancel{-m_e^2} + 2P_\gamma \cdot P_e - 2P_\gamma \cdot P_\gamma' - 2P_e \cdot P_\gamma'$$

$$\therefore \quad P_\gamma \cdot P_e = P_\gamma \cdot P_\gamma' + P_e \cdot P_\gamma'$$

$$\therefore -P_\gamma m_e = -P_\gamma P_\gamma' + \underline{P_\gamma} \cdot \underline{P_\gamma'} - m_e P_\gamma'$$

$$\underline{P_\gamma} \cdot \underline{P_\gamma'} = P_\gamma P_\gamma' \cos\phi$$

$$\therefore \quad P_\gamma m_e = P_\gamma P_\gamma'(1 - \cos\phi) + m_e P_\gamma'$$

$\therefore$

$$P_\gamma' = \frac{P_\gamma m_e}{P_\gamma(1-\cos\theta) + m_e} \qquad c=\hbar=1$$

$\because P_\gamma' = \omega_f$, $P_\gamma = \omega_i$

$\therefore$ ~~ω_f~~

$$\omega_f = \frac{\omega_i m_e}{m_e + \omega_i(1-\cos\cancel{\theta})}$$

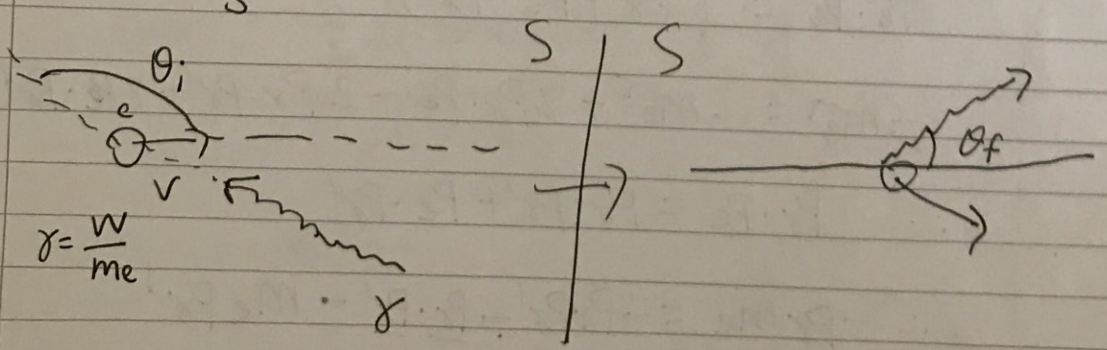
$$\omega_f = \frac{\omega_i}{1 + \frac{\omega_i}{m_e}(1-\cos\phi)}$$

Compton Scattering

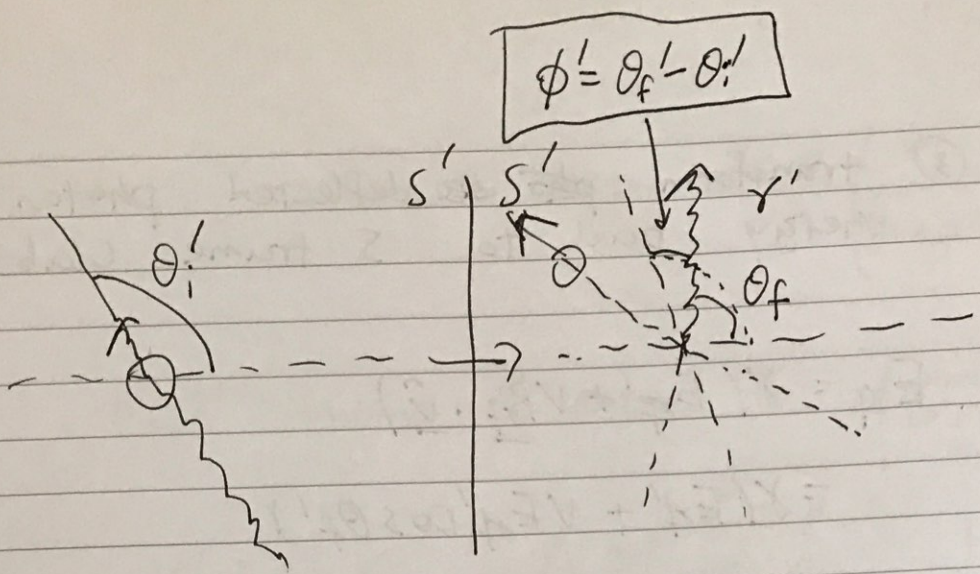
Now for ~~an~~ a moving electron, the photon can gain energy.

We do calculation by

① transform photon energy from lab frame S to electron rest frame S'



θ_i, θ_f are the incident and scattered angle, with respect to x-direction of the ~~p~~ photon in S.



① initial energy of photon in S is $E_{\gamma i}$

② ~~$E_{\gamma f} = E_{\gamma i}$~~

then in S' is

$$E_{\gamma i}' = \cancel{E_{\gamma i}} \, \gamma(E_{\gamma i} - V \underline{P_{\gamma i}} \cdot \underline{\hat{v}})$$

$$= \gamma(E_{\gamma i} - V E_{\gamma i} \cos\theta_i)$$

$$= E_{\gamma i} \gamma (1 - V \cos\theta_i)$$

② in S', use Compton effect

$$E_{\gamma f}' = \frac{E_{\gamma i}'}{1 + \frac{E_{\gamma i}'}{m_e}(1 - \cos(\phi'))}$$

$$= \frac{E_{\gamma i}'}{1 + \frac{E_{\gamma i}'}{m_e}(1 - \cos(\theta_f' - \theta_i'))}$$

③ transform ~~pho~~ deflected photon energy back to S frame (lab)

$$E_{\gamma f} = \gamma\left(E_{\gamma f}' + v\underline{p_{\gamma f}'} \cdot \underline{\hat{v}} \right)$$

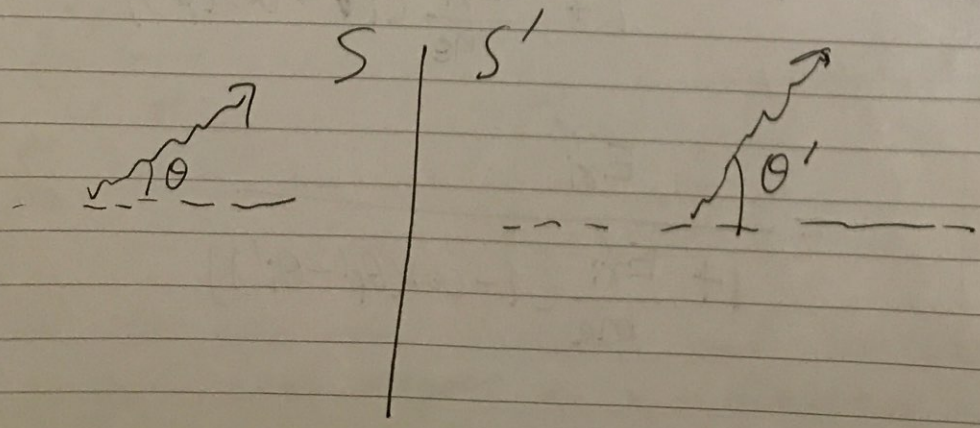
$$= \gamma\left(E_{\gamma f}' + vE_{\gamma f}'\cos\theta_f' \right)$$

$$= \gamma E_{\gamma f}'\left(1 + v\cos\theta_f' \right)$$

$$= \frac{\gamma E_{\gamma i}'\left(1 + v\cos\theta_f' \right)}{1 + \frac{E_{\gamma i}'}{m_e}\left(1 - \cos(\theta_f' - \theta_i') \right)}$$

$$= \boxed{\frac{E_{\gamma i}\,\gamma^2\left(1 + v\cos\theta_f' \right)\left(1 - v\cos\theta_i \right)}{1 + \frac{E_{\gamma i}\,\gamma}{\gamma m_e}\left(1 - v\cos\theta_i \right)\left(1 - \cos(\theta_f' - \theta_i') \right)}}$$

Now, consider the relation between $\theta_{i,f}'$ and $\theta_{i,f}$



in S, wave 4 vector $K = \begin{pmatrix} \omega \\ \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$

in S', wave 4 vector $K' = \begin{pmatrix} \omega' \\ \omega'\cos\theta' \\ \omega'\sin\theta' \\ 0 \end{pmatrix}$

$$K' = \Delta K \qquad \Delta \doteq \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\therefore \quad \omega' = \gamma\omega - \gamma\beta\omega\cos\theta$$
$$\omega'\cos\theta' = -\gamma\beta\omega + \gamma\omega\cos\theta$$

$$\therefore \quad (\cancel{\gamma\omega} - \cancel{\gamma}\beta\cancel{\omega}\cos\theta)\cos\theta' = -\cancel{\gamma}\beta\cancel{\omega} + \cancel{\gamma\omega}\cos\theta$$

$$\therefore \quad (1-\beta\cos\theta)\cos\theta' = \cos\theta' - \beta$$

$c=1 \rightarrow \beta = V$

$$\therefore \quad \boxed{\cos\theta'_{i,f} = \frac{\cos\theta_{i,f} - V}{1 - V\cos\theta_{i,f}}}$$

- $E_{\gamma i}$ = time component of 4-vector K

$= K^0$

- electron energy $= W$ $\quad \therefore \gamma = \dfrac{W}{m_e}$

→ maximum energy of photon after scattering is the maximum value of

$E_{\gamma f}$ for all possible θ_i, θ_f

This is complicated for general θ W and V and K^0

→ I wrote a matlab code for this, result is interesting.

→ but if we use the numbers in the problem

$$W = 2\,GeV, \quad m_e = 0.511\,MeV$$

$$\therefore \gamma = \frac{2\times10^9}{0.511\times10^6} = 3914 \gg 1$$

$$\therefore V \sim C \sim 1$$

$$K^0 = E_{\gamma i} = E_{photon} = \frac{hc}{\lambda} = \frac{6.63\times10^{-34}\times3\times10^8}{(1\times10^{-2}m)}$$

$$= 1.99\times10^{-23}J = 1.243\times10^{-4}eV$$

$$\therefore K^0\gamma \approx 0.5\,eV \ll 0.511\,MeV = m_e$$

$$\therefore K^0\gamma \ll m_e \quad \therefore \frac{K_0\gamma}{m_e} \ll 1$$

→ the denominator of $E_{\gamma f}$ can be treated as 1

and $V \sim 1$

$\rightarrow \quad E_{\gamma f} \simeq K^{\circ}\gamma^{2}(1+V\cos\theta_f')(1-V\cos\theta_i)$

this clearly has maximum when

$\theta_f' = 0^{\circ}$ and $\theta_i = 180^{\circ}$

$\Rightarrow$ corresponds to $(\theta_i, \theta_f) = (180^{\circ}, 0^{\circ})$

$(\theta_i', \theta_f') = (180^{\circ}, 0^{\circ})$

$\therefore \phi' = \phi = -180^{\circ}$

$\rightarrow$ This means the maximum scattered photon energy is obtained when the photon is bounced straight back along the original path.

In this case

$E_{\gamma f, max} \simeq K^{\circ}(\gamma^{2})(1+1)(1+1)$

$= \underline{\underline{4\gamma^{2}K^{\circ}}} = \underline{\underline{4\left(\frac{W}{m_e}\right)^{2}K^{\circ}}}$

- with numbers

$E_{\gamma f, max} = 4(3914)^{2}(1.243\times10^{-4}\,eV)$

$= 7616\,eV$

$= \boxed{7.62\times10^{3}\,eV} = 1.63\times10^{-10}\,m$

$= 0.16\,nm$

- For fixed $\theta_i = 180°$ (head-on collision)

$$E_{\gamma f} = \frac{\gamma^2 E_{\gamma i}(1+v\cos\theta_f')(1+v)}{1+\frac{\gamma E_{\gamma i}}{m_e}(1+v)(1+\cos\theta_f')} \qquad \cos\theta_f' = \frac{\cos\theta_f - v}{1-v\cos\theta_f}$$

$$\therefore\ 1+v\cos\theta_f' = \frac{1-\cancel{v\cos\theta_f}+\cancel{v\cos\theta_f}-v^2}{1-v\cos\theta_f} = \frac{1-v^2}{1-v\cos\theta_f}$$

$$1+\cos\theta_f' = \frac{1-v\cos\theta_f+\cos\theta_f-v}{1-v\cos\theta_f} = \frac{(1-v)(1+\cos\theta_f)}{1-v\cos\theta_f}$$

$$\therefore E_{\gamma f} = \frac{\gamma^2 E_{\gamma i}(1-v^2)(1+v)}{(1-v\cos\theta_f)\left[1+\frac{\gamma E_{\gamma i}}{m_e}(1+v)\frac{(1-v)(1+\cos\theta_f)}{1-v\cos\theta_f}\right]}$$

$$= \frac{\gamma^2 E_{\gamma i}(1-v^2)(1+v)}{1-v\cos\theta_f+\frac{\gamma E_{\gamma i}}{m_e}(1+v)(1-v)(1+\cos\theta_f)} \qquad \because \gamma^2(1-v^2)=1$$

$$= \frac{E_{\gamma i}(1+v)}{1-v\cos\theta_f+\frac{\gamma E_{\gamma i}}{m_e}(1+v)(1-v)(1+\cos\theta_f)} \qquad \times\ \gamma(1+v)m_e$$

$$= \frac{E_{\gamma i}\,\gamma m_e(1+v)^{\cancel{2}}}{(1-v\cos\theta_f)\gamma m_e\cancel{(1+v)}+E_{\gamma i}\underbrace{\cancel{\gamma^2(1-v^2)}}_{1}\cancel{(1+v)}(1+\cos\theta_f)}$$

$$= \frac{\gamma m_e E_{\gamma i}(1+v)}{\gamma m_e(1-v\cos\theta_f)+E_{\gamma i}(1+\cos\theta_f)}$$

$$= \frac{\gamma m_e E_{\gamma i}(1+v)}{(E_{\gamma i}+\gamma m_e)+(E_{\gamma i}-v\gamma m_e)\cos\theta_f}$$

If $E_{\gamma i} < v\gamma m_e$, then $\cos\theta_f = 1$ $(\theta_f = 0)$

so denominator is smallest.

$\therefore$ bounce straight back

Appendix

"Brute-force" method of Q2 part ii (2016 B2):

- $\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2} \underline{v} \cdot \underline{\dot{v}}$ $\left(\underline{\dot{v}} = \frac{dv}{dt}\right)$

$$\frac{d\underline{p}}{dt} = -e(\underline{E} + \underline{v} \times \underline{B})$$

$v = (v_x, v_y, v_z)^T$ $\underline{B} = (0, 0, B)^T$, $E = (0, 0, E_z)^T$

$$\underline{v} \times \underline{B} = \begin{vmatrix} i & j & k \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = (Bv_y, -Bv_x, 0)^T$$

$$\therefore \frac{d\underline{p}}{dt} = -e(Bv_y, -Bv_x, E_z)^T$$

$$\therefore \underline{p} = \gamma m_0 \underline{v} = \gamma m_0 (v_x, v_y, v_z)^T$$

where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \frac{v_x^2 + v_y^2 + v_z^2}{c^2}\right)^{-\frac{1}{2}}$

$$\therefore \cancel{\gamma \frac{dv_x}{dt} = -\omega v_y}$$

$$\cancel{\gamma \frac{dv_y}{dt} = \omega v_x}$$

$$\cancel{\gamma \frac{dv_z}{dt} = -A}$$

constants

$\omega \equiv \frac{eB}{m_0}$ $A \equiv \frac{eE_z}{m_0}$

$\omega, A > 0$

$$\frac{d}{dt}(\gamma v_x) = -\omega v_y \quad (1)$$

$$\frac{d}{dt}(\gamma v_y) = \omega v_x \quad (2)$$

$$\frac{d}{dt}(\gamma v_z) = -A \quad (3)$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_x^2 + v_y^2 + v_z^2}{c^2}}} \quad (*)$$

$\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2} \underline{v} \cdot \underline{\dot{v}}$

③ $\gamma \frac{dV_z}{dt} + V_z \frac{d\gamma}{dt} = -A$

$$\frac{d\gamma}{dt} = \frac{\gamma^2}{c^2}\left[\gamma V_x \frac{dV_x}{dt} + \gamma V_y \frac{dV_y}{dt} + \gamma V_z \frac{dV_z}{dt}\right]$$

$= \cancel{\frac{\gamma^2}{c^2}[\omega}$ ① $\gamma \frac{dV_x}{dt} + V_x \frac{d\gamma}{dt} = -\omega V_y$

② $\gamma \frac{dV_y}{dt} + V_y \frac{d\gamma}{dt} = \omega V_x$

$$\therefore \frac{d\gamma}{dt} = \frac{\gamma^2}{c^2}\left[\cancel{-\omega V_y V_x} - V_x^2 \frac{d\gamma}{dt} + \cancel{\omega V_x V_y} - V_y^2 \frac{d\gamma}{dt} - AV_z - V_z^2 \frac{d\gamma}{dt}\right]$$

$$= \frac{\gamma^2}{c^2}\left[-(V^2)\frac{d\gamma}{dt} - AV_z\right]$$

$$\therefore \left(1 + \frac{\gamma^2 V^2}{c^2}\right)\frac{d\gamma}{dt} = -AV_z \frac{\gamma^2}{c^2}$$

$$\left(1 - \frac{V^2}{c^2}\right) = \frac{1}{\gamma^2} \quad \therefore \quad \frac{V^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

$$\therefore \left(1 + \left(1 - \frac{1}{\gamma^2}\right)\gamma^2\right)\frac{d\gamma}{dt} = -AV_z \frac{\gamma^2}{c^2}$$

$$\therefore \cancel{\gamma^2}\frac{d\gamma}{dt} = -\frac{A}{c^2}\cancel{\gamma^2} V_z$$

$$\therefore \boxed{\frac{d\gamma}{dt} = -\frac{A}{c^2}V_z \equiv -KV_z} \quad \left(K \equiv \frac{A}{c^2} = \frac{eE_z}{m_0 c^2}\right)$$

$\therefore \; d\gamma = -K V_z \, dt = -K \, dz$

$\therefore \; \Delta\gamma = -K \Delta z$

$\therefore \; \gamma(z) - \underbrace{\gamma(z=0)}_{17} = -K(z-0)$

$\therefore \gamma(z) = 17 - \dfrac{eE_z z}{m_0 c^2}$

$$\boxed{\gamma(1m) = 17 - \frac{eE_z(1m)}{m_0 c^2}}$$

focus on ①, ②

$V_x ① + V_y ② \Rightarrow \gamma\left(V_x \dfrac{dV_x}{dt} + V_y \dfrac{dV_y}{dt}\right)$

$+ (V_x^2 + V_y^2)\dfrac{d\gamma}{dt} = 0$

Define $V_\perp = \sqrt{V_x^2 + V_y^2}$ $\quad \therefore \; \dfrac{d}{dt}(V_\perp^2) = 2V_x \dfrac{dV_x}{dt} + 2V_y \dfrac{dV_y}{dt}$

$\therefore \; \frac{1}{2}\gamma \dfrac{d}{dt}(V_\perp^2) + V_\perp^2 \dfrac{d\gamma}{dt} = 0$

$\therefore \; \gamma V_\perp \dfrac{dV_\perp}{dt} + V_\perp^2 \dfrac{d\gamma}{dt} = 0$

$\therefore \; \gamma \dfrac{dV_\perp}{dt} + V_\perp \dfrac{d\gamma}{dt} = 0$

$\therefore \; \boxed{\dfrac{d}{dt}(\gamma V_\perp) = 0} \Rightarrow \boxed{V_\perp \gamma \equiv b}$

$b = \text{const}$

$\therefore \dfrac{1}{\sqrt{1-\frac{V_\perp^2+V_z^2}{c^2}}} V_\perp = b$

$\therefore V_\perp^2 = b^2\left(1-\dfrac{(V_\perp^2+V_z^2)}{c^2}\right)$

$\therefore V_\perp^2 = b^2 - \dfrac{b^2}{c^2}V_\perp^2 - \dfrac{b^2}{c^2}V_z^2$

$\therefore \left(1+\dfrac{b^2}{c^2}\right)V_\perp^2 + \left(\dfrac{b^2}{c^2}\right)V_z^2 = b^2$

$\times\left(\dfrac{c}{b}\right)^2 \Rightarrow \boxed{\left(1+\dfrac{c^2}{b^2}\right)V_\perp^2 + V_z^2 = c^2}$

define ~~$m^2 = 1+$~~ $\boxed{S^2 = 1+\dfrac{c^2}{b^2}}$. then

$\boxed{S^2 V_\perp^2 + V_z^2 = c^2}$

$\because \dfrac{d}{dt}(\gamma V_z) = -A$ ③

$\therefore \gamma V_z = -At + B$

If at $t=t_0$, $V_z = 0$, then $B = At_0$

$\therefore \boxed{\gamma V_z = A(t_0-t)}$

$\because \gamma V_z = A(t_0-t)$
$\gamma V_\perp = b$

$\boxed{\alpha \equiv \dfrac{A}{b}}$

$\therefore \dfrac{V_\perp}{V_z} = \dfrac{b}{A(t_0-t)} = \dfrac{1}{\alpha(t_0-t)}$

$$\therefore \quad V_z = \alpha(t_0 - t)\, V_\perp \quad , \quad V_\perp = \frac{V_z}{\alpha(t_0 - t)}$$

$$\cancel{\therefore \quad \left[s^2 \alpha^2 (t_0 - t)^2 + 1 \right] V_\perp^2 = c^2}$$

$$\cancel{\therefore \quad V_\perp = \frac{c}{\sqrt{1 + \alpha^2 s^2 (t_0 - t)^2}}}$$

$$\therefore \quad s^2 V_\perp^2 + \alpha^2 (t_0 - t)^2 V_\perp^2 = c^2$$

$$\therefore \quad V_\perp = \frac{c}{\sqrt{s^2 + \alpha^2 (t_0 - t)^2}} = \frac{c/s}{\sqrt{1 + (\frac{\alpha}{s})^2 (t_0 - t)^2}}$$

define $\delta \equiv \frac{\alpha}{s}$, $g \equiv \frac{c}{s}$

$$\boxed{V_\perp = \frac{g}{\sqrt{1 + \delta^2 (t_0 - t)^2}}}$$

$(s = \frac{\alpha}{\delta} > 1$ $\therefore \alpha > \delta)$

$(c = gs = \frac{g\alpha}{\delta})$

$$s^2 \frac{V_z^2}{(\alpha(t_0 - t))^2} + V_z^2 = c^2$$

$$\therefore \quad \left(1 + \frac{1}{\delta^2 (t_0 - t)^2}\right) V_z^2 = c^2$$

$$\therefore \quad V_z = \frac{c}{\sqrt{1 + \frac{1}{\delta^2 (t_0 - t)^2}}} = \frac{c \cdot \delta (t_0 - t)}{\sqrt{1 + \delta^2 (t_0 - t)^2}}$$

$c\delta = \frac{c\alpha}{s} = \alpha g$

$$= \boxed{\frac{\alpha (t_0 - t)\, g}{\sqrt{1 + \delta^2 (t_0 - t)^2}}}$$

Now for $V_x(t)$ & $V_y(t)$

$\because V_x^2(t) + V_y^2(t) = V_\perp^2(t)$

$\therefore$ let $\cancel{V_x(t) = V_\perp(t)\sin(f(t))}$

$\cancel{V_y(t) = V_\perp(t)\cos(f(t))}$

$$V_x(t) = V_\perp(t)\cos(f(t)) \qquad (\gamma V_\perp = b)$$
$$V_y(t) = V_\perp(t)\sin(f(t))$$

$$\frac{d}{dt}(\gamma V_x) = -\omega V_y$$

$$\therefore -\gamma \cancel{V_\perp} \cancel{\sin(f(t))} \frac{df}{dt} = -\omega \cancel{V_\perp} \cancel{\sin(f(t))}$$

$$\therefore \underline{\frac{df}{dt} = \frac{\omega}{\gamma}} = \omega\sqrt{1-\frac{v^2}{c^2}} = \omega\sqrt{1-\frac{V_\perp^2+V_z^2}{c^2}}$$

similarly $\frac{d}{dt}(\gamma V_y) = \omega V_x$

$$\gamma \cancel{V_\perp} \cancel{\cos(f(t))} \frac{df}{dt} = \omega \cancel{V_\perp} \cancel{\cos(f(t))}$$

$$\frac{df}{dt} = \frac{\omega}{\gamma}$$

$$V_\perp^2 + V_z^2 = \underbrace{S^2 V_\perp^2 + V_z^2}_{c^2} - (S^2-1)V_\perp^2$$

$$= c^2 - (S^2-1)V_\perp^2 \qquad \cancel{\boxed{k^2 = S^2-1}} \quad \boxed{c^2k^2 = S^2-1}$$

$$= c^2 - c^2k^2 V_\perp^2 = c^2(1-k^2V_\perp^2)$$

$$\therefore \quad 1 - \frac{V_\perp^2 + V_z^2}{c^2} = 1 - (1 - k^2 V_\perp^2) = (k V_\perp)^2$$

$$\therefore \quad \frac{df}{dt} = (\omega k) V_\perp$$

$$\therefore \quad \cancel{f(t) = C + \int}$$

$$f(t) = C + \omega k \int V_\perp(t)\, dt$$

$$\int V_\perp(t)\, dt = \int \frac{q}{\sqrt{1+\delta^2 (t_0 - t)^2}}\, dt$$

let $\delta(t_0 - t) = \tan\theta$

$-\delta dt = \sec^2\theta\, d\theta$

$dt = -\frac{1}{\delta} \sec^2\theta\, d\theta$

$\cancel{1+\tan}$

$$= -\frac{q}{\delta} \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}}$$

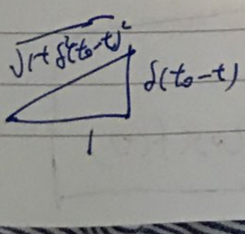
$$= -\frac{q}{\delta} \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = -\frac{q}{\delta} \int \sec\theta\, d\theta$$

$$= -\frac{q}{\delta} \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\, d\theta$$

$$= -\frac{q}{\delta} \int \frac{d(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} = -\frac{q}{\delta} \ln \left| \sec\theta + \tan\theta \right|$$

$$= -\frac{q}{\delta} \ln \left(\sqrt{1+\delta^2 (t_0 - t)^2} + \delta(t_0 - t) \right)$$

$\cancel{f(t) =}$



$f(0) = 0 \quad \rightarrow$

$$\boxed{f(t) = \omega k\left[\frac{q}{\delta}\right]\left[\ln \frac{\sqrt{1+\delta^2 t_0^2} + \delta t_0}{\sqrt{1+\delta^2(t_0-t)^2} + \delta(t_0-t)}\right]}$$

$f(0)=0 \quad \because \qquad V_x(0) = V_\perp(0) \qquad V_y(0) = 0 \quad (\text{WLG})$

$$V_\perp = \frac{1}{\omega k}\frac{df}{dt}$$

$\therefore \; V_x(t) = V_\perp(t)\cos(f(t)) \qquad V_y = V_\perp(t)\sin(f(t))$

$$= \frac{1}{\omega k}\frac{df}{dt}\cos(f) \qquad = \frac{1}{\omega k}\frac{df}{dt}\sin(f)$$

$$x(t) = \int V_x(t)\,dt = \frac{1}{\omega k}\int \frac{df}{dt}\cos(f)\,dt$$

$$= \frac{1}{\omega k}\int \cos(f)\,df = \cancel{\frac{1}{\omega k}\cos(} \; \frac{1}{\omega k}\sin(f) + C'$$

$x(0) = x_0 \quad \therefore \quad C' = x_0$

$$\therefore \; \underline{x = \frac{1}{\omega k}\sin(f(t)) + x_0}$$

$$y(t) = \int V_y(t)\,dt = \frac{1}{\omega k}\int \sin(f)\,df = -\frac{1}{\omega k}\cos(f) + C''$$

$y(0) = y_0 \quad \therefore \quad -\frac{1}{\omega k} + C'' = y_0 \qquad C'' = y_0 + \frac{1}{\omega k}$

$$\therefore \; y = \left(y_0 + \frac{1}{\omega k}\right) - \frac{1}{\omega k}\cos(f)$$

$$\therefore \; \cancel{(x-x_0)} = \boxed{\begin{aligned} &(x-x_0) = \frac{1}{\omega k}\sin(f(t)) \\ &\cancel{(y-y_0)}\left(y - \left(y_0 + \frac{1}{\omega k}\right)\right) = \frac{1}{\omega k}\cos(f(t)) \end{aligned}}$$

$$\therefore \quad (x-x_0)^2 + \left(\cancel{y} - \left(y_0 + \frac{1}{\omega k}\right)\right)^2 = \frac{1}{\omega^2 k^2} = R^2$$

$$R = \frac{1}{\omega k} \qquad \frac{1}{\gamma} = k V_\perp \quad \therefore \frac{1}{k} = \gamma V_\perp = \text{const}$$

$$R = \frac{\gamma V_\perp}{\omega} = \frac{\gamma V_\perp}{\frac{eB}{m}} = \boxed{\frac{\gamma m V_\perp}{eB}}$$

circular motion $\Rightarrow$ <u>unchanged Larmor Radius</u>

$$\cancel{\gamma = \frac{1}{kV_\perp} \qquad \frac{d\gamma}{dt} = -\frac{1}{kV_\perp^2}\frac{dV_\perp}{dt} = -KV_z}$$

$$\cancel{\frac{d(\gamma V_\perp)}{dt} = \frac{d\gamma}{dt}V_\perp + \gamma\frac{dV_\perp}{dt}}$$

$$\cancel{= \gamma\frac{dV_\perp}{dt} - KV_\perp V_z}$$

$$z(t) = \int V_z\,dt = \alpha\xi \int \frac{(t_0-t)}{\sqrt{1+\delta^2(t_0-t)^2}}\,dt = c\int \frac{\delta(t_0-t)}{\sqrt{1+\delta^2(t_0-t)^2}}\,dt$$

let $u = \delta(t_0-t)$

$du = -\delta\,dt$

$\therefore dt = -\frac{1}{\delta}du$

$$= -\frac{c}{\delta}\int \frac{u\,du}{\sqrt{1+u^2}} = -\frac{c}{2\delta}\int \frac{dJ}{\sqrt{J}}$$

let $J = 1+u^2$

$dJ = 2u\,du$

$u\,du = \frac{1}{2}dJ$

$$= -\frac{c}{\delta}\sqrt{J} = -\frac{c}{\delta}\sqrt{1+\delta^2(t_0-t)^2} + C^{(2)}$$

$$C^{(2)} = \frac{c}{\delta}\sqrt{1+\delta^2 t_0^2} \quad \text{for } z(0)=0$$

$$\therefore \boxed{z(t) = \frac{c}{\delta}\left(\sqrt{1+\delta^2 t_0^2} - \sqrt{1+\delta^2(t_0-t)^2}\right).}$$

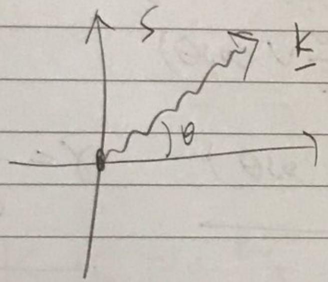
16B2 Q3

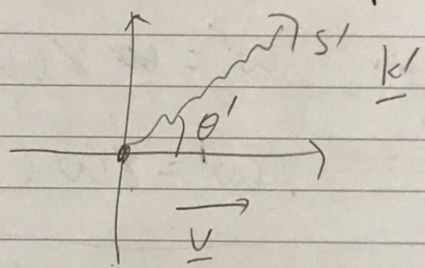
- phenomena

headlight effect

- ~~headlight effect~~
- doppler redshift transverse doppler
- stellar aberration (headlight effect)
- Appearance of superluminal motion

- relativistic Doppler shift is the difference in frequency between the rest frame of the source and the rest frame of the observer.





S' moving v with respect to S

4-vectors

K : 4-wave vector of light ~~in S~~

U : 4-velocity of the source

$K \cdot U = K' \cdot U'$ (invariant)

$K' = K$ in $S' = \begin{pmatrix} \omega'/c \\ \underline{k}' \end{pmatrix}$ $\therefore K' \cdot U'$

$U' = U$ in $S' = \begin{pmatrix} c \\ 0 \end{pmatrix}$ $= -\omega'$

$$K \cdot U = \begin{pmatrix} \omega/c \\ \underline{k} \end{pmatrix} \cdot \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$$

$$= -\gamma\omega + \gamma \underline{k} \cdot \underline{v} = \cancel{} - \gamma(\omega - \underline{k} \cdot \underline{v})$$

$$= -\omega'$$

$$\therefore \quad \omega' = \cancel{} \gamma(\omega - \underline{k} \cdot \underline{v})$$

photon $\underline{k} = \frac{\omega}{c} \hat{\underline{k}}$ $\qquad \hat{\underline{k}} \cdot \underline{v} = v\cos\theta$

$$\therefore \quad \omega' = \cancel{} \gamma(\omega - \frac{\omega}{c} v\cos\theta)$$

$$\underline{\omega' = \gamma\omega(1 - v/c \cos\theta)} \qquad \underline{\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

— ~~gr~~ phase / group velocity

~~v~~ phase velocity $= \underline{v_{ph}} = \frac{\omega}{|\underline{k}|} \hat{\underline{k}} = \boxed{\frac{\omega}{|\underline{k}|^2} \underline{k}}$

group velocity $= \underline{v_{gr}} = \underline{\nabla_k} \omega = \boxed{\frac{\partial \omega}{\partial \underline{k}}}$,

If $\underline{k}$ defined to be in x-direction

$$\underline{v_{ph} = \frac{\omega}{k}, \quad v_{gr} = \frac{d\omega}{dk}}$$

2016

- refractive index n

$$\omega = \frac{c}{n}|\underline{k}| \quad (n \text{ may be a function of } (\omega), (k))$$
$$= \frac{c}{n}k$$

$\therefore$ 4 wave vector $K = \begin{pmatrix} \omega/c \\ \underline{k} \end{pmatrix}$

$$V_{ph} = \frac{\omega}{|\underline{k}|} = \frac{c}{n}$$

$$V_{gr} = \frac{d\omega}{dk} = c\frac{d}{dk}\left(\frac{k}{n}\right).$$

$$= \frac{c}{n} + ck\left(\frac{d}{dk}\frac{1}{n}\right) = \frac{c}{n} - \frac{ck}{n^2}\frac{dn}{dk}$$

$$K \cdot K = -\frac{\omega^2}{c^2} + k^2 = -\left(\frac{c^2}{n^2}\right)k^2 \cdot \frac{1}{c^2} + k^2$$

$$= k^2\left(1 - \frac{1}{n^2}\right)$$

If $n = 1$, then $K \cdot K = 0$

$$\therefore -\frac{\omega^2}{c^2} + k^2 = 0$$

differentiate this ~~wrt~~ with k

$$-\frac{1}{c^2} \cdot 2\omega\frac{d\omega}{dk} + 2k = 0.$$

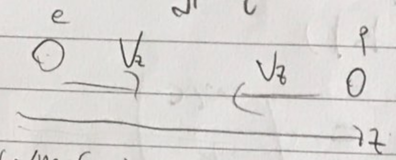
$$\therefore \omega\frac{d\omega}{dk} = kc^2$$

$$\therefore \frac{\omega}{k} \cdot \frac{d\omega}{dk} = c^2 \Rightarrow V_{gr}V_{ph} = c^2$$

— e^-, p collision $P = \begin{pmatrix} E/c \\ P_z \\ P_y \\ P_x \end{pmatrix}$

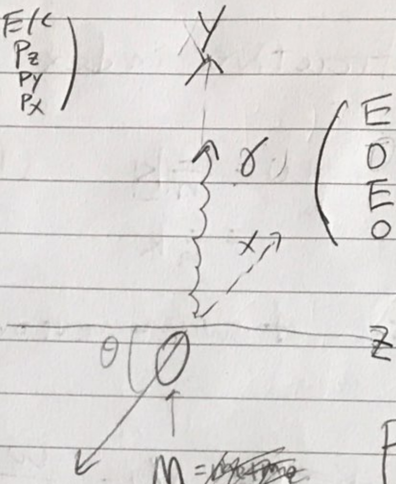
$P_e = \begin{pmatrix} E_e \\ P_e \\ 0 \end{pmatrix}$ $\quad \gamma_z = \frac{1}{\sqrt{1-\frac{V_z^2}{c^2}}}$ $\quad$ S | S

$P_p = \begin{pmatrix} E_p \\ -P_p \\ 0 \\ 0 \end{pmatrix}$



$P_e = \begin{pmatrix} \gamma_z m_e c \\ \gamma_z m_e V_z \\ 0 \\ 0 \end{pmatrix}$ $\quad P_p = \begin{pmatrix} \gamma_z M_p c \\ -\gamma_z M_p V_z \\ 0 \\ 0 \end{pmatrix}$

$P_{tot} = P_e + P_p = \begin{pmatrix} E_{tot} \\ P_{tot} \\ 0 \end{pmatrix}$



$\begin{pmatrix} E \\ 0 \\ E \\ 0 \end{pmatrix} = P_\gamma$

$E = \frac{hc}{\lambda} = \frac{2\pi}{\lambda}$

$(\hbar = c = 1)$

$M = \cancel{m_e + m_p}$

$= m_e + m_p$

$P_M = \begin{pmatrix} E_M \\ P_M \end{pmatrix} = \begin{pmatrix} E_M \\ -P_M\cos\theta \\ -P_M\sin\theta \\ 0 \end{pmatrix}$

4-momentum conservation:

$$P_e + P_p = P_\gamma + P_M$$

$$P_M^2 = (P_e + P_p - P_\gamma)^2$$

$$\therefore -M^2 \cancel{(m_e + m_p)^2} = (P_e + P_p)^2 - 2(P_e + P_p)\cdot P_\gamma + \underbrace{P_\gamma^2}_{=0}$$

$$(P_e + P_p)^2 = -E_{tot}^2 + P_{tot}^2 = \cancel{(\gamma_z m_e \; E_{tot})^2}$$

$$-E_{tot}^2 + (\gamma_z V_z)^2 (m_p - m_e)^2$$

$$\therefore -M^2 = -E_{tot}^2 + \gamma_z^2 V_z^2 (m_p - m_e)^2 - 2(P_e + P_p)\cdot P_\gamma$$

$P_e + P_p = \begin{pmatrix} E_{tot} \\ P_{tot} \\ 0 \\ 0 \end{pmatrix}$, $\quad P_\gamma = \begin{pmatrix} E \\ 0 \\ E \\ 0 \end{pmatrix}$

$$\therefore 2(P_e + P_p)\cdot P_\gamma = -2E_{tot}E + 0$$

$$= -2E_{tot}E$$

$\therefore\ -M^2 = -E_{tot}^2 + \gamma_z^2 V_z^2 (M_p - m_e)^2 + 2E_{tot}E$

$\therefore\ E_{tot}^2 - 2E_{tot}E = M^2 + \gamma_z^2 V_z^2 (M_p - m_e)^2$

$\therefore\ E_{tot}^2 - 2E_{tot}E + E^2 = M^2 + \gamma_z^2 V_z^2 (m_p - m_e)^2 + E^2$

$\therefore\ (E_{tot} - E)^2 = M^2 + \gamma_z^2 V_z^2 (m_p - m_e)^2 + E^2$

$\cancel{\therefore\ -M^2 = E_e}$

$(P_e + P_\gamma)^2 = P_{tot} \cdot P_{tot} = -(m_e + m_p)^2$

$(P_e \cdot P_p) \cdot P_\gamma = \begin{pmatrix} E_{tot} \\ P_{tot} \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} E \\ 0 \\ E \\ 0 \end{pmatrix}$

$= -E_{tot}E \quad , \quad P_\gamma \cdot P_\gamma = 0$

$\therefore\ -M^2 = -(m_e + m_p)^2 + 2E_{tot}E$

$P_\gamma \cdot P_\gamma = 0 \qquad 2(P_e + P_p) \cdot P_\gamma = 2\begin{pmatrix} E_{tot} \\ P_{tot} \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} E \\ 0 \\ E \\ 0 \end{pmatrix}$

$= -2E_{tot}E = -2(E_e + E_p)E$

$(P_e + P_p)^2 = P_e^2 + P_p^2 + 2P_e \cdot P_p = -m_e^2 - m_p^2 + 2\begin{pmatrix} E_e \\ P_e \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} E_p \\ -P_p \\ 0 \\ 0 \end{pmatrix}$

$= -m_e^2 - m_p^2 + 2(-E_eE_p + (P_e)(-P_p))$

$= -(m_e^2 + m_p^2 + 2(E_eE_p + P_eP_p))$

$\therefore -M^2 = -(m_e^2 + m_p^2 + 2E_eE_p + 2p_ep_p) + 2(E_e + E_p)E$

$\because \quad E_e = \gamma_z m_e c^2 \quad , \quad E_p = \gamma_z m_p c^2 \qquad \left[\gamma_z = \left(1 - \frac{v_z^2}{c^2}\right)^{-\frac{1}{2}}\right.$

$\therefore \quad \frac{E_p}{E_e} = \frac{m_p}{m_e} \qquad \because \quad p_p = \gamma_z m_p v_z$

$p_e = \gamma_z m_e v_z$

$\therefore \quad \frac{p_p}{p_e} = \frac{m_p}{m_e} \quad \Rightarrow \quad E_p = \frac{m_p}{m_e}E_e, \quad p_p = \frac{m_p}{m_e}p_e$

$E_e^2 - p_e^2 = m_e^2 \qquad \therefore \cancel{E_e^2 + p_e^2 =}$

$\therefore \quad p_e^2 = E_e^2 - m_e^2 \qquad \therefore E_e^2 + p_e^2 = 2E_e^2 - m_e^2$

~~Similarly~~ $\cancel{E_p^2 + p_p^2}$

$\therefore \quad M^2 = m_e^2 + m_p^2 + 2\frac{m_p}{m_e}(E_e^2 + p_e^2) - \cancel{2E\,2\frac{m_p}{}}$

$- 2\left(1 + \frac{m_p}{m_e}\right)E_e E$

$\therefore \quad M^2 = m_e^2 + m_p^2 + 2\frac{m_p}{m_e}(2E_e^2 - m_e^2) - 2\left(1 + \frac{m_p}{m_e}\right)E_eE$

$M^2 = \underbrace{(m_e^2 + m_p^2 - 2m_pm_e)}_{(m_p - m_e)^2} + 4\frac{m_p}{m_e}E_e^2 - 2E\left(1 + \frac{m_p}{m_e}\right)E_e$

$\therefore$

$\cancel{M^2 = (m_p - m_e)^2}$

$0 = \frac{4m_p}{m_e}E_e^2 - 2E\left(1 + \frac{m_p}{m_e}\right)E_e + \left[(m_p - m_e)^2 - M^2\right].$

$\therefore \cancel{E_e =}$

$\therefore 4m_p E_e^2 - 2E(m_p+m_e)E_e + [(m_p-m_e)^2 - M^2]m_e = 0$

$$\therefore E_e = \frac{1}{8m_p}\left[2E(m_p+m_e) \pm \left(4E^2(m_p+m_e)^2 - 16m_p m_e\left((m_p-m_e)^2-M^2\right)\right)^{\frac{1}{2}}\right]$$

~~take the~~ $M = m_p + m_e$

~~inside square root~~ $\therefore (m_p-m_e)^2 - (m_p+m_e)^2 = 4m_p m_e$.

$E_{tot} = E_e + E_p = E_e\left(1+\frac{m_p}{m_e}\right)$ (ignore negative root)

$$\therefore E_{tot} = \frac{1}{8}\left(\frac{1}{m_e}+\frac{1}{m_p}\right)\left[2E(m_p+m_e) + \sqrt{4E^2(m_p+m_e)^2 + 64m_p^2 m_e^2}\right]$$

where $E = \frac{hc}{\lambda}$

(or use $M = m_p + m_e - 13.6\,eV$ as the minimum of M

$\rightarrow$ ~~hean~~ Hence the min of E_{tot})

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- EM wave

$\underline{E} = (0, E_y, 0)$ $\quad \underline{B} = (B_x, 0, 0)$, $\underline{k} = (0, 0, k)$

$E_y = E_0 \cos(\omega t - kz)$ $\quad \cancel{E_y = f}$

$B_x = B_0 \cos(\omega t - kz)$

Maxwell's equations with ~~idex of~~ index of refraction n. $\left(n \equiv \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}} \right.$, $\therefore \left. \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} \right)$

$\mu\epsilon = \frac{n^2}{c^2}$

$\underline{\nabla} \cdot \underline{E} = 0$

$\underline{\nabla} \cdot \underline{B} = 0$

$\underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t}$

$\underline{\nabla} \times \underline{B} = \mu\epsilon \frac{\partial \underline{E}}{\partial t} = \frac{n^2}{c^2} \frac{\partial \underline{E}}{\partial t}$

$\underline{\nabla} \cdot \underline{E} = \frac{\partial E_y}{\partial y} = 0$ ✓ $\quad \underline{\nabla} \cdot \underline{B} = \frac{\partial B_x}{\partial x} = 0$ ✓

$$\underline{\nabla} \times \underline{E} = \begin{vmatrix} i & j & k \\ \partial_x & 0 & \partial_z \\ 0 & E_y & 0 \end{vmatrix} = \hat{\underline{i}} \, (-\partial_z E_y)$$

$$= -\hat{\underline{x}} \, (\partial_z E_y) = -kE_0 \sin(\omega t - kz) \, \hat{\underline{x}}$$

$$\frac{\partial \underline{B}}{\partial t} = \cancel{-\omega B_0 \cos(\omega t - kz)} \; -\omega B_0 \sin(\omega t - kz) \hat{\underline{x}}$$

$\therefore \; -kE_0 = \omega B_0 \qquad \therefore \frac{\omega}{k} = -\frac{E_0}{B_0}$

$$\underline{\nabla} \times \underline{B} = \begin{vmatrix} i & j & k \\ 0 & 0 & \partial_z \\ B_x & 0 & 0 \end{vmatrix} = -j(0 - \partial_z B_x) = \partial_z B_x \, \underline{\hat{y}}$$

$= +B_0 k \sin(\omega t - kz) \, \underline{\hat{y}}$

$$\frac{n^2}{c^2} \frac{\partial \underline{E}}{\partial t} = \frac{-n^2}{c^2} \omega E_0 \sin(\omega t - kz) \, \underline{\hat{y}}$$

$\therefore \quad B_0 k = -\frac{n^2}{c^2} \omega E_0$

$\therefore \quad \frac{n^2}{c^2} \frac{\omega}{k} = \frac{k}{\omega} \qquad \therefore \left(\frac{\omega}{k}\right)^2 = \left(\frac{n}{c}\right)^2$

$\rightarrow \quad \underline{\frac{\omega}{k} = \frac{c}{n}}$

$\because \; \underline{B} = \underline{\nabla} \times \underline{A} = (B_x, 0, 0)$

$= (B_x(z), 0, 0)$

$$\therefore \; = \begin{vmatrix} i & j & k \\ 0 & 0 & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

$\therefore \; A_x = 0 \quad \because \; B_y = 0$ we only need

$\underline{A} = (0, A_y, 0) \qquad \underline{\nabla} \times \underline{A} = \underline{\hat{x}}(-\partial_z A_y)$

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$\therefore \quad -\partial_z A_y = B_x = B_0 \cos(\omega t - kz)$

$$\therefore \quad A_y = -\int B_0 \cos(\omega t - kz)\, dz$$

$$= \frac{B_0}{k} \sin(\omega t - kz)$$

$$\therefore \underline{A} = \begin{pmatrix} 0 \\ \frac{B_0}{k}\sin(\omega t - kz) \\ 0 \end{pmatrix}$$

$$\frac{\partial \underline{A}}{\partial t} = \frac{\omega}{k} B_0 \cos(\omega t - kz)$$

$$= -\frac{E_0}{B_0} B_0 \cos(\omega t - kz)$$

$$= -E_0 \cos(\omega t - kz)$$

$$\therefore \underline{E} = -\frac{\partial \underline{A}}{\partial t} = E_0 \cos(\omega t - kz)$$

$$\because \underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \quad \therefore \quad \underline{\nabla}\phi = 0$$

$$\therefore \phi = \text{const} \underset{\text{set to}}{=} 0$$

check Lorenz gauge $\partial^\nu A_\nu = 0$

$$\partial^\nu = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$A_\nu = (-\phi/c, A_x, A_y, A_z)$$

$\therefore \partial^{\mu} A_{\nu} = 0 \rightarrow \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \underline{\nabla} \cdot \underline{A} = 0$

if $\phi = 0$, $\underline{A} = \frac{B_0 \hat{y}}{k} \sin(\omega t - kz)$ ✓

$\because \underline{E_0} \times \underline{B_0} = \begin{vmatrix} i & j & k \\ 0 & E_0 & 0 \\ B_0 & 0 & 0 \end{vmatrix} = \cancel{-\hat{z} B_x E_y} - \hat{z} E_0 B_0$

$\therefore |\underline{E_0} \times \underline{B_0}| = |E_0 B_0| = 1$

$\because \left|\frac{E_0}{B_0}\right| = \frac{\omega}{k} = \frac{c}{n}$, $|E_0||B_0| = 1$

$\therefore \cancel{\frac{n}{c}|B_0|} \ \frac{c}{n} B_0^2 = 1$

$\therefore \cancel{B_0 = \sqrt{\frac{c}{n}}, \ E_0 = -\frac{\omega}{k} B_0}$

$\cancel{= -\frac{n}{c}}$

$B_0 = \sqrt{\frac{n}{c}}$, $E_0 = -\frac{\omega}{k} B_0 = -\frac{c}{n}\sqrt{\frac{n}{c}}$

$= -\sqrt{\frac{c}{n}}$

$\therefore \underline{\phi = 0}$ $\quad \underline{A} = \frac{\hat{y}}{k}\sqrt{\frac{c}{n}} \sin(\omega t - kz)$

4-wave vector $K = \begin{pmatrix} \omega/c \\ \underline{k} \end{pmatrix}$

$\because \underline{k} = (0, 0, k)$ $\therefore$ $K = \begin{pmatrix} \omega/c \\ 0 \\ 0 \\ k \end{pmatrix}$

EM field invariant quantities

$$D = B^2 - \frac{E^2}{c^2} \quad , \quad \alpha = \frac{\underline{B} \cdot \underline{E}}{c}$$

$\because \alpha = 0$, $\therefore$ it is possible for either $\underline{E}$ or $\underline{B}$ to vanish

$$D = B^2 - \frac{E^2}{c^2} = \cos^2(\omega t - kz)\left(B_0^2 - \frac{E_0^2}{c^2}\right)$$

$$= \cos^2(\omega t - kz)\left(\frac{n}{c} - \frac{1}{c^2} \cdot \frac{c}{n}\right)$$

$$= \frac{1}{c}\cos^2(\omega t - kz)\left(n - \frac{1}{n}\right)$$

$\because n > 1 \quad \therefore D > 0$

$\therefore$ only Electric field is possible to vanish.

Transformation of EM field

$$\underline{E}'_{\parallel} = \underline{E}_{\parallel} \qquad \underline{E}'_{\perp} = \gamma(\underline{E}_{\perp} + \underline{v} \times \underline{B})$$

$$\underline{B}'_{\parallel} = \underline{B}_{\parallel} \qquad \underline{B}'_{\perp} = \gamma\left(\underline{B}_{\perp} - \frac{\underline{v} \times \underline{E}}{c^2}\right)$$

$\because \underline{E}' = 0 \quad \therefore \underline{E}'_{\parallel} = 0 \quad , \quad \underline{E}'_{\perp} = 0$

$\underline{E}'_{\parallel} = 0$, need to choose a $\underline{v}$ such that

$\underline{E}_{\parallel} = (\underline{E} \cdot \hat{\underline{v}})\hat{\underline{v}} = 0 \quad \therefore \quad \underline{E} \perp \underline{v}$

$$\underline{E}'_\perp = \gamma(\underline{E}_\perp + \underline{v} \times \underline{B}) = 0$$

$$\therefore \underline{E}_\perp = -\underline{v} \times \underline{B}$$

$$\because \underline{E}_\parallel = \underline{E}'_\parallel = 0 \qquad \therefore \underline{E}_\perp = \underline{E}$$

$$\therefore \underline{E} = -\underline{v} \times \underline{B}$$

$$\because \underline{E} = E_y \hat{\underline{y}} \qquad \underline{B} = B_x \hat{\underline{x}}$$

$$\therefore \underline{v} = -v\hat{\underline{z}}$$

So that $\cancel{v\hat{z}}\ (-v)\hat{\underline{z}} \times B_x \hat{\underline{x}} = \cancel{= v\hat{z}(\hat{z}\times\hat{x})}$

$$= \cancel{v\hat{z}}\ vB_x(\hat{\underline{z}} \times \hat{\underline{x}}) = vB_x \hat{\underline{y}} = E_y \hat{\underline{y}}$$

$$\cancel{v = \frac{|E|}{|B|}\hat{y}}$$

$$\therefore \cancel{\underline{v} = -\frac{|E|}{|B|}\hat{z}} \qquad \underline{v} = -\frac{E_y}{B_x}\hat{\underline{z}}$$

$$\underline{v} = -\frac{E_0 \cos(\omega t - kz)}{B_0 \cos(\omega t - kz)}\hat{\underline{z}}$$

$$= -\frac{E_0}{B_0}\hat{\underline{z}}$$

check.

In lab frame.

$$\because \boxed{\underline{V} = -\frac{E_0}{B_0}\hat{\underline{z}}}.$$

$$\therefore \quad \underline{E}_{\parallel} = 0 \qquad \underline{E}_{\perp} = E_y\hat{\underline{y}}$$

$$\underline{B}_{\parallel} = 0 \qquad \underline{B}_{\perp} = B_x\hat{\underline{x}}$$

$$\underline{E}_{\parallel}' = \underline{E}_{\parallel} = 0$$

$$\underline{E}_{\perp}' = \gamma(\underline{E}_{\perp} + \underline{V}\times\underline{B})$$

$$= \gamma\left(E_y\hat{\underline{y}} - \frac{E_0}{B_0}\hat{\underline{z}}\times B_x\hat{\underline{x}}\right)$$

$$= \gamma\left(E_y\hat{\underline{y}} - \frac{E_y}{B_x}B_x(\underbrace{\hat{\underline{z}}\times\hat{\underline{x}}}_{\hat{\underline{y}}})\right)$$

$$= \hat{\underline{y}}\,\gamma(E_y - E_y) = 0$$

16B2 Q4

$$A^\mu = \begin{pmatrix} \phi/c \\ \underline{A} \end{pmatrix} \qquad \underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \qquad \underline{B} = \underline{\nabla} \times \underline{A}$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$\partial^c F^{ab} + \partial^a F^{bc} + \partial^b F^{ca}$$

$$= \partial^c \partial^a A^b - \partial^c \partial^b A^a + \partial^a \partial^b A^c - \partial^a \partial^c A^b + \partial^b \partial^c A^a - \partial^b \partial^a A^c$$

$$= 0$$

$$\cancel{J_\nu = \mu \begin{pmatrix} \rho c \\ \underline{j} \end{pmatrix}}$$

$$J_\nu = -\mu_0 \begin{pmatrix} -\rho c \\ \underline{j} \end{pmatrix} = -\mu_0 \rho_0 U_\nu$$

4-velocity
(~~contra~~ covariant)

$$F_{\mu\nu} = \begin{pmatrix} 0 & & -\underline{E}/c & \\ & 0 & B_z & -B_y \\ \underline{E}/c & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{pmatrix} \qquad \partial^\mu = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right)$$

covariant

$$\partial^\mu F_{\mu\nu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right) \begin{pmatrix} 0 & & -\underline{E}/c & \\ & 0 & B_z & -B_y \\ \underline{E}/c & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{pmatrix}$$

$$= \left(\frac{\underline{\nabla}\cdot\underline{E}}{c}, +\frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} - \underline{\nabla} \times \underline{B} \right)$$

$\because$ $\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$ (M1)

$\because c^2 = \frac{1}{\mu_0 \varepsilon_0}$

$\varepsilon_0 c^2 = \frac{1}{\mu_0}$

$$\therefore \frac{\nabla \cdot \underline{E}}{c} = \frac{\rho}{\varepsilon_0 c} = \frac{\rho c}{\varepsilon_0 c^2}$$

$$\therefore \frac{\nabla \cdot \underline{E}}{c} = \mu_0 \rho c$$

$$\because \nabla \times \underline{B} = \mu_0 \underline{j} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t}$$

$$\therefore \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} - \nabla \times \underline{B} = -\mu_0 \underline{j}$$

$$\therefore \partial^\nu F_{\mu\nu} = (\mu_0 \rho c, -\mu_0 \underline{j})$$

$$= -\mu_0(-\rho c, \underline{j}) = J_\nu$$

$$\therefore \partial^\nu F_{\mu\nu} = J_\nu$$

Lorentz force per unit volume is

$$\cancel{f_\nu = \rho F^{\mu\nu} U_\nu}$$

$$\cancel{f^\mu = \rho(\underline{E} + \underline{V} \times \underline{B})}$$

$$\cancel{= \rho F^{\mu\nu} U_\nu}$$

~~contravariant~~

(the contravariant version)

rest charge density

$$f^{\mu} = \cancel{\quad} \rho_0 F^{\mu\nu} U_{\nu} \Rightarrow f^{\nu} = \rho_0 F^{\nu\lambda} U_{\lambda}$$

turn to covariant

$$f_{\mu} = g_{\mu\nu} \rho_0 F^{\nu\lambda} U_{\lambda}$$

$$= \cancel{g_{\mu\nu}} \rho_0 F_{\mu}{}^{\lambda} U_{\lambda}$$

$$= \cancel{\rho_0 g\, \rho_0 g^{\alpha\lambda}}$$

$$= \rho_0 g^{\lambda\alpha} F_{\mu\alpha} g_{\lambda\beta} U^{\beta} = g^{\lambda\alpha} g_{\lambda\beta} F_{\mu\alpha} U^{\beta}$$

$$= \rho_0 \delta^{\alpha}{}_{\beta} F_{\mu\alpha} U^{\beta} = \cancel{\rho_0 F} \rho_0 \delta^{\alpha}{}_{\lambda} \delta^{\lambda}{}_{\beta} F_{\mu\alpha} U^{\beta}$$

$$= \rho_0 F_{\mu\lambda} U^{\lambda}$$

$$= \rho_0 U^{\lambda} F_{\mu\lambda} = \rho_0 U^{\nu} F_{\mu\nu}$$

$$= \cancel{J_{\nu} F} - J^{\nu} F_{\mu\nu} = J^{\nu}(-F_{\mu\nu})$$

$$= J^{\nu} F_{\nu\mu}$$

Here

$J_{\nu} = -\rho_0 U_{\nu}$

$J^{\nu} = -\rho_0 U^{\nu}$

$$f_0 = J^{\nu} F_{\nu 0} = J^0 \underbrace{F_{00}}_{0} + \overbrace{J^1}^{-j_x} \underbrace{F_{10}}_{-E_x/c} + \overbrace{J^2}^{-j_y} \underbrace{F_{20}}_{-E_y/c} + \overbrace{J^3}^{-j_z} \underbrace{F_{30}}_{-\frac{E_z}{c}}$$

$$= \frac{\rho_0}{c}(\underline{j} \cdot \underline{E})$$

is the rate of work done on charge per unit volume.

— $W^\mu = \begin{pmatrix} \underline{S}\cdot\underline{P} \\ \frac{E\underline{S}}{c} \end{pmatrix}$ $\quad E = \gamma m_0 c^2 \quad \frac{E}{m_0 c} = \gamma c$

$$U_W^\mu = \frac{W^\mu}{m_0 c|\underline{S}|} = \begin{pmatrix} \frac{\underline{S}\cdot\underline{P}}{m_0 c|\underline{S}|} \\ \frac{\gamma \underline{S}}{|\underline{S}|} \end{pmatrix}$$

4 momentum $P = \begin{pmatrix} E/c \\ \underline{P} \end{pmatrix} = \begin{pmatrix} \gamma m c \\ \gamma m \underline{v} \end{pmatrix}$

in rest frame (S'), $\underline{P}' = 0$, $\underline{v}' = 0$, $\underline{S}' = \underline{S_0}$, $\gamma = 1$

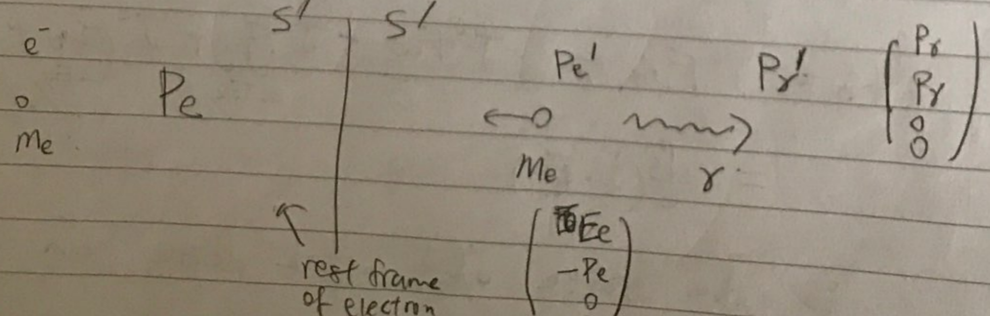
∴ and, the values of $U_W^\mu P_\mu$ is invariant

$$\therefore \quad U_W^\mu P_\mu = U_W'^\mu P_\mu' = \begin{pmatrix} 0 \\ \frac{\underline{S_0}}{|\underline{S_0}|} \end{pmatrix} \cdot \begin{pmatrix} mc \\ 0 \end{pmatrix}$$

$$= \underline{\underline{0}}$$

— electron is an elementary particle (invariant rest mass)

ie (a) $n = 1$ ∴ photon speed $= c$



let $c = \hbar = 1$

$P_e = P_e' + P_\gamma'$ $\qquad P_e = \begin{pmatrix} m_e \\ \underline{0} \end{pmatrix}$

$\therefore P_e' = P_\gamma' - P_e$ $\qquad P_\gamma = \begin{pmatrix} P_\gamma \\ P_\gamma \\ 0 \\ 0 \end{pmatrix}$ photon $E = P$. if velocity $= c$

$\therefore P_e' \cdot P_e' = (P_\gamma' - P_e)^2$ $\qquad P_e' = \begin{pmatrix} E_e' \\ -P_e' \\ 0 \\ 0 \end{pmatrix}$.

$$\therefore -\cancel{m_e^2} = \underbrace{P_\gamma' \cdot P_\gamma'}_{0} + \underbrace{P_e \cdot P_e}_{-\cancel{m_e^2}} - 2P_\gamma' \cdot P_e.$$

$$-\cancel{m_e^2} - 2P_\gamma' \cdot P_e$$

$\therefore P_\gamma' \cdot P_e = 0$

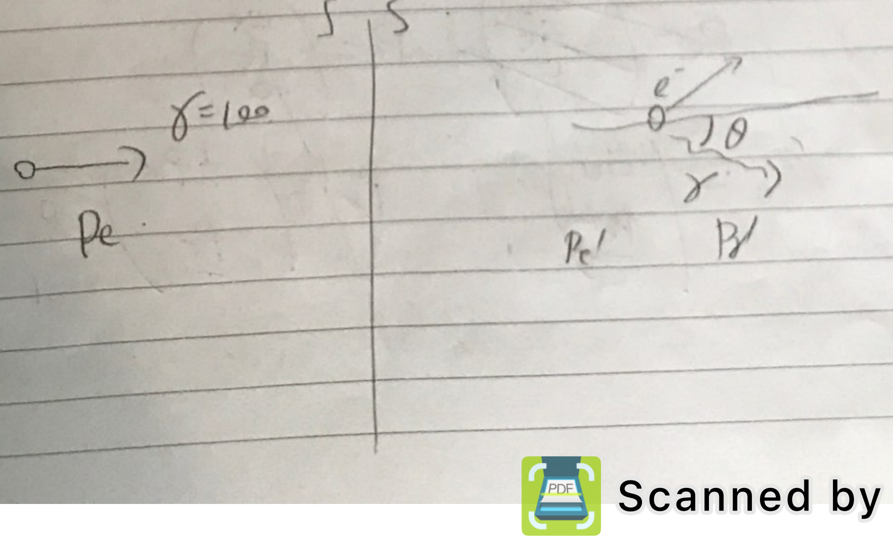
$$\therefore \begin{pmatrix} P_\gamma \\ P_\gamma \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} m_e \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$\therefore P_\gamma m_e = 0 \rightarrow \underline{\underline{P_\gamma = 0}}$

photon has zero energy

cannot ~~emitt~~ emit a photon in any frame since the P_γ vector is null.

(b)



In this case the difference is that

~~$P_e = P_e' + P_\gamma'$~~ velocity of photon is

$$V_p = \frac{c}{n} = \frac{c}{2} , \quad \frac{\omega}{k} = \frac{c}{n} = \frac{c}{2}$$

$E_\gamma \neq \hbar\omega$. $\underline{P_\gamma} = \hbar\underline{k}$

~~$E_\gamma = \hbar\omega$~~

$$E_\gamma = \frac{\omega}{k} P_\gamma = \frac{c}{2} P_\gamma = \frac{1}{2} P_\gamma$$

$$\cancel{P_\gamma' \cdot P_\gamma' = \begin{pmatrix} P_\gamma \\ \frac{1}{2}P_\gamma \end{pmatrix} \cdot \begin{pmatrix} P_\gamma \\ \frac{1}{2}P_\gamma \end{pmatrix}}$$

$$\cancel{= -P_\gamma^2 + \frac{1}{4}P_\gamma^2 = -\frac{3}{4}P_\gamma}$$

$$P_\gamma' \cdot P_\gamma' = \begin{pmatrix} \frac{1}{2}P_\gamma \\ \underline{P_\gamma} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}P_\gamma \\ \underline{P_\gamma} \end{pmatrix}$$

$$= -\frac{1}{4}P_\gamma^2 + P_\gamma^2 = \frac{3}{4}P_\gamma^2$$

For electron

$\gamma_e \simeq 100$ $\therefore$ $V_e \simeq c$ $\quad |\underline{P_e}| \simeq E_e$

$\therefore$ $\cancel{P_e = \begin{pmatrix} E_e \\ E_e \\ 0 \\ 0 \end{pmatrix}}$ $\quad P_e = \begin{pmatrix} E_e \\ P_e \\ 0 \\ 0 \end{pmatrix}$

$P_e' = \begin{pmatrix} E_e' \\ \underline{P_e'} \end{pmatrix}$ $\quad P_\gamma' = \begin{pmatrix} \frac{1}{2}P_\gamma \\ \underline{P_\gamma} \end{pmatrix}$

∴ $P_\gamma' \cdot P_e = \cancel{\frac{3}{8} P_\gamma^2} \quad \frac{3}{8}\, \underline{P_\gamma} \cdot \underline{P_\gamma}$

∴ $\begin{pmatrix} E_e \\ \underline{P_e} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} P_\gamma \\ \underline{P_\gamma} \end{pmatrix} = \frac{3}{8}\, \underline{P_\gamma} \cdot \underline{P_\gamma}$

∴ $-\frac{1}{2} E_e P_\gamma + \underline{P_e} \cdot \underline{P_\gamma} = \frac{3}{8} P_\gamma^2$

$\underline{P_e} \cdot \underline{P_\gamma} = P_e P_\gamma \cos\theta \simeq E_e P_\gamma \cos\theta$

$\gamma = 100$

∴ $E_e \cancel{P_\gamma} (\cos\theta - \frac{1}{2}) = \frac{3}{8} P_\gamma^{\cancel{2}}$

∴ $\cancel{E_e (\cos\theta - \frac{1}{2}) = \frac{3}{8}} P_\gamma$

photon energy and momentum

$\cancel{E_\gamma = hc}$ $\quad E_\gamma = \hbar\omega = h\nu$, $P_\gamma = \frac{h}{\lambda}$

$V_p = \lambda\nu$ = phase velocity of light.

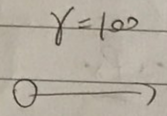
in medium $n=2$, $V_p = \frac{c}{n} = \frac{c}{2} = \frac{1}{2}$

$\therefore \quad \lambda\nu = \frac{1}{2} \qquad \therefore \quad \lambda = \frac{1}{2\nu}$

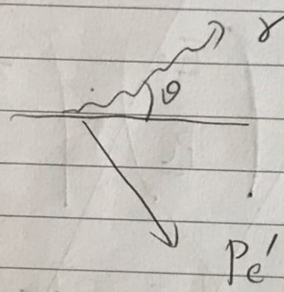
$\therefore \quad P_\gamma = \frac{h}{\lambda} = 2h\nu \qquad \text{let} \quad c = h = 1$

$\therefore \quad P_\gamma = 2\nu \qquad E_\gamma = h\nu = \nu$

$\therefore \quad P_\gamma$ 4vector is $P_\gamma = \begin{pmatrix} \nu \\ 2\underline{\nu} \end{pmatrix}$



$P_e = \begin{pmatrix} E_e \\ \underline{p_e} \end{pmatrix}$



$P_\gamma = \begin{pmatrix} \nu \\ 2\underline{\nu} \end{pmatrix}$

~~4=4~~ 4 momentum conservation

$P_e = P_\gamma' + P_e' \qquad \therefore P_e' = P_e - P_\gamma'$

$\therefore \quad \cancel{\underbrace{P_e' \cdot P_e'}_{-m_e^2}} = \cancel{\underbrace{P_e \cdot P_e}_{-m_e^2}} + P_\gamma' \cdot P_\gamma' - 2P_e \cdot P_\gamma'$

$\therefore \quad 2P_e \cdot P_\gamma' = P_\gamma' \cdot P_\gamma'$

$P_e \cdot P_\gamma' = \begin{pmatrix} E_e \\ \underline{P_e} \end{pmatrix} \cdot \begin{pmatrix} \nu \\ 2\underline{\nu} \end{pmatrix}$

$= -E_e\nu + 2\underline{P_e} \cdot \underline{\nu} = -E_e\nu + 2P_e\nu\cos\theta$

$$P_\gamma' \cdot P_\gamma' = -\nu^2 + 4\nu^2 = 3\nu^2$$

$$\therefore 2(2P_e\cancel{\nu}\cos\theta - E_e\cancel{\nu}) = 3\nu^{\cancel{2}}$$

$$\therefore 2P_e\cos\theta - E_e = \frac{3\nu}{4}$$

$$\because P_e = \gamma m_e v \qquad E_e = \gamma m_e c^2$$
$$= \gamma m_e c\beta \qquad = \gamma m_e.$$
$$= \gamma m_e \beta$$

$$\therefore P_e = \beta E_e \qquad \left(\beta = \frac{v_e}{c}\right)$$ velocity of electron

$$\therefore E_e(2\beta\cos\theta - 1) = \frac{3\nu}{4}$$

$$\therefore 2\beta\cos\theta - 1 = \frac{3\nu}{4E_e}$$

$$\therefore \cos\theta = \frac{1}{2\beta}\left(1 + \frac{3\nu}{4E_e}\right)$$

$\nu \sim$ photon energy

— most energetic photons, gamma rays, are typically $\sim$ 1 MeV

but $E_e \sim \gamma m_e \sim 100 \times 0.5$ MeV ~ 50 MeV

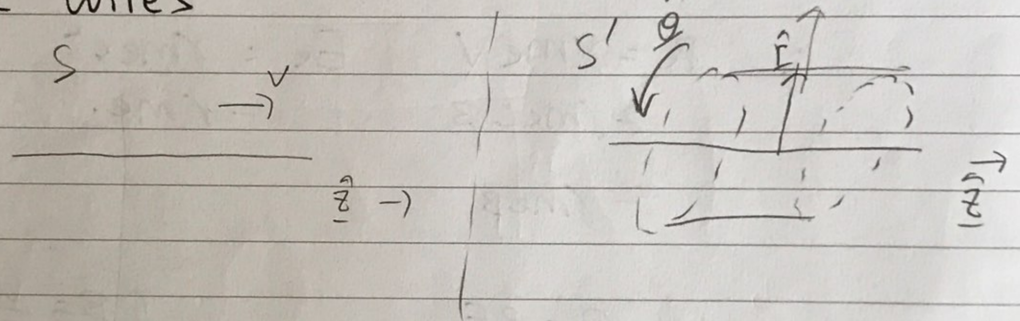
$$\therefore \frac{\nu}{E_e} \ll 1$$

$\therefore$ we have $\cos\theta \simeq \frac{1}{2\beta}$

$\because \gamma = 100 \quad \therefore \beta \simeq 1$

$\therefore \cos\theta = \cancel{1}\,\frac{1}{2} \rightarrow \underline{\underline{\theta = 60^\circ}}$

— the wires



(1) in S', wire is stationary, rest charge density (ρ_0)

use Gauss's law $\oiint \underline{E}' \cdot \vec{n}\, ds = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho_0\, dl$

$\therefore$ By symmetry $\underline{E}' = E'\hat{\underline{r}}$

$(2\pi r l) E' = \frac{1}{\varepsilon_0}\rho l$

$$\therefore \boxed{\underline{E}' = \frac{\rho_0 \hat{\underline{r}}}{2\pi\varepsilon_0 r}}$$

$\because \rho = \gamma\rho_0$

$$\therefore \boxed{\underline{E}' = \frac{\rho\hat{\underline{r}}}{2\pi\varepsilon_0\gamma r}}$$

Stationary charges $\underline{\underline{\underline{B}' = 0}}$

(ii) Lorentz transformation of ~~EM~~ EM fields

$$\underline{E}'_{\parallel} = \underline{E}_{\parallel} \qquad \underline{B}'_{\parallel} = \underline{B}_{\parallel}$$

$$\underline{E}'_{\perp} = \gamma(\underline{E}_{\perp} + \underline{V} \times \underline{B}) \qquad \underline{B}'_{\perp} = \gamma(\underline{B}_{\perp} - \frac{\underline{V} \times \underline{E}}{c^2})$$

$$\underline{E}_{\perp} = \gamma(\underline{E}'_{\perp} - \underline{V} \times \underline{B}')$$

$$\underline{B}_{\perp} = \gamma(\underline{B}'_{\perp} + \frac{\underline{V} \times \underline{E}'}{c^2}).$$

transform from S' back to S

$$\because \underline{V} = V\hat{\underline{z}} \qquad \underline{E}' = \underline{E}'_{\perp} = \frac{\rho}{2\pi\varepsilon_0 r}\hat{\underline{r}}$$

$$\underline{E}'_{\parallel} = 0$$

$$\underline{B}'_{\perp} = \underline{B}'_{\parallel} = 0 \qquad \therefore \cancel{\underline{V} \times \underline{E}' = 0}$$

$$\underline{V} \times \underline{B}' = 0$$

$$\therefore \underline{V} \times \underline{E}' = V\hat{\underline{z}} \times \frac{\rho_0}{2\pi\varepsilon_0 r}\hat{\underline{r}}$$

$$= \frac{\rho_0 V \hat{\underline{\theta}}}{2\pi\varepsilon_0 r}$$

$$\therefore \underline{E}_{\parallel} = \underline{E}'_{\parallel} = 0$$

$$\underline{E}_{\perp} = \gamma(\underline{E}'_{\perp} - \underline{V} \times \underline{B}') = \gamma \underline{E}'_{\perp}$$

$$\therefore \underline{E} = \frac{\gamma\rho_0}{2\pi\varepsilon_0 r}\hat{\underline{r}} \qquad \therefore \boxed{\underline{E} = \frac{\rho\hat{\underline{r}}}{2\pi\varepsilon_0 r}}$$

$$\underline{B}_{\parallel} = \underline{B}'_{\parallel} = 0$$

$$\underline{B}_{\perp} = \cancel{B} \gamma\left(\underline{B}'_{\perp} + \frac{1}{c^2}\underline{v}\times\underline{E}'\right)$$

$$= \frac{\gamma}{c^2}\frac{\rho_0 v \hat{\underline{\theta}}}{2\pi\varepsilon_0 r}$$

$$\therefore \boxed{\frac{\rho v \hat{\theta}}{2\pi c^2 \varepsilon_0 r} = \underline{B}}$$

~~Ex~~

→ Invariants of EM field

$$D = B^2 - \frac{E^2}{c^2}$$

$$\alpha = \frac{E \cdot B}{c}$$

in S' $\cancel{E}$ $\underline{B}' = 0$ $\therefore$ $\alpha = 0$

$$D = \cancel{B}\, 0^2 - \frac{E'^2}{c^2} < 0$$

If we can find a ~~fre~~ frame in which $\underline{E}$ $\cancel{B}$ is 0, then

$$D = B^2 - \frac{0^2}{c^2} = B^2 > 0$$

contradiction

$\therefore$ No frame has $\underline{E} = 0$

$\rightarrow$ <u>cannot</u> be purely magnetic.