A13484W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B5: GENERAL RELATIVITY AND COSMOLOGY

TRINITY TERM 2015

Saturday, 20 June, 9.30 am – 11.30 am

10 minutes reading time

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

1. Show that if you extremize the action for a test particle $S = -\int \mathrm{d}\lambda\, g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$ (where λ is the affine parameter and $g_{\alpha\beta}$ the metric), you will obtain the correct expressions for the connection coefficients for a general metric. [4]

Consider the "global rain" metric,

$$\mathrm{d}s^2 = -c^2\left(1-\frac{r_\mathrm{s}}{r}\right)\mathrm{d}\bar{t}^2 + 2\sqrt{\frac{r_\mathrm{s}}{r}}\,c\,\mathrm{d}\bar{t}\,\mathrm{d}r + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\ ,$$

where $(\bar{t}, r, \theta, \phi)$ are space-time coordinates and r_s is a constant. Show that the non-zero components of the inverse metric are

$$g^{\bar{t}\bar{t}} = -\frac{1}{c^2}\ ,\ g^{\bar{t}r} = \frac{1}{c}\sqrt{\frac{r_\mathrm{s}}{r}}\ ,\ g^{rr} = \left(1-\frac{r_\mathrm{s}}{r}\right)\ ,\ g^{\theta\theta} = \frac{1}{r^2}\ ,\ g^{\phi\phi} = \frac{1}{r^2\sin^2\theta}\ .$$

Using the definition of the connection coefficients (or otherwise), show that the radial geodesic equation is

$\Gamma^r_{\bar{t}\bar{t}}$ $\Gamma^r_{r\bar{t}}$

$$\ddot{r} - \frac{1}{2r_\mathrm{s}}\left(\frac{r_\mathrm{s}}{r}\right)^2\dot{r}^2 + \frac{c^2}{2r_\mathrm{s}}\left(1-\frac{r_\mathrm{s}}{r}\right)\left(\frac{r_\mathrm{s}}{r}\right)^2\dot{\bar{t}}^2 - \frac{c}{r_\mathrm{s}}\left(\frac{r_\mathrm{s}}{r}\right)^{\frac{5}{2}}\dot{r}\dot{\bar{t}}$$
$$- \left(1-\frac{r_\mathrm{s}}{r}\right)r\dot{\theta}^2 - \left(1-\frac{r_\mathrm{s}}{r}\right)r\sin^2\theta\,\dot{\phi}^2 = 0\ .$$

Γ^r_{rr} $\Gamma^r_{\theta\theta}$ $\Gamma^r_{\phi\phi}$

[10]

Compare what happens to this metric at $r = r_\mathrm{s}$ with what happens to the Schwarzschild metric in the usual coordinates. By looking only at light-like radial geodesics, explain why, if $r < r_\mathrm{s}$, photons always fall inwards [hint: show that $\mathrm{d}r/\mathrm{d}\bar{t} < 0$]. [6]

Consider a change of coordinates such that $\bar{t} = \bar{t}(r,t)$ where

$$\frac{\partial\bar{t}}{\partial t} = 1\ ,$$
$$\frac{\partial\bar{t}}{\partial r} = \sqrt{\frac{r_\mathrm{s}}{r}}\left(1-\frac{r_\mathrm{s}}{r}\right)^{-1}\frac{1}{c}\ .$$

Rewrite the global rain metric in terms of t and r, and show that it is equivalent to the Schwarzschild metric. [5]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\frac{r_s}{r}\left(1-\frac{r_s}{r}\right)^{-1}+1$

$\frac{r_s}{r}+1$ $\quad \frac{1+\frac{r_s}{r}}{1-\frac{r_s}{r}}dr^2$

$\frac{r_s}{r}+\left(1-\frac{r_s}{r}\right)$

2. Consider a conformally flat space-time with metric given in Cartesian coordinates by

$$ds^2 = e^{\frac{2\varphi}{c^2}}\,\eta_{\alpha\beta}\,dx^\alpha dx^\beta \ ,$$

where φ is a scalar function of space-time coordinates and $\eta_{\alpha\beta}$ is the Minkowski metric. Show that the connection coefficients take the form

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{c^2}\left[\partial_\beta\varphi\,\delta^\mu{}_\alpha + \partial_\alpha\varphi\,\delta^\mu{}_\beta - \partial^\mu\varphi\,\eta_{\alpha\beta}\right] ,$$

where $\partial^\mu = \eta^{\mu\nu}\,\partial_\nu$. [7]

The Ricci tensor is given by

$$R_{\nu\beta} \equiv \partial_\mu\Gamma^\mu{}_{\beta\nu} - \partial_\beta\Gamma^\mu{}_{\mu\nu} + \Gamma^\mu{}_{\mu\epsilon}\Gamma^\epsilon{}_{\nu\beta} - \Gamma^\mu{}_{\epsilon\beta}\Gamma^\epsilon{}_{\nu\mu} \ .$$

Assume that $\varphi/c^2 \ll 1$, and show that the Einstein tensor takes the form

$$G_{\alpha\beta} = \frac{2}{c^2}\left(\partial_\mu\partial^\mu\varphi\,\eta_{\alpha\beta} - \partial_\alpha\partial_\beta\varphi\right) \ .$$

[6]

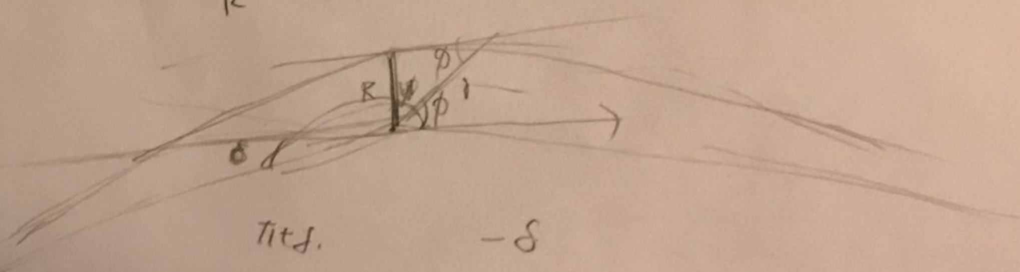
Rewrite the metric into spherical coordinates, (t, r, θ, ϕ), and assuming that φ is a function of r only, show that for an equatorial orbit the geodesic equations for t and ϕ take the form

$$e^{\frac{2\varphi}{c^2}}\,\dot{t} = d \quad \text{and} \quad e^{\frac{2\varphi}{c^2}}\,r^2\dot{\phi} = \ell \ ,$$

where $\dot{f} \equiv df/d\lambda$ for any function $f(\lambda)$ (λ is the affine parameter), and d and ℓ are integration constants. Write down an expression for the null condition for photons in this metric. [5]

Assume that $\varphi = -GM/r$, where M is a constant and r is the distance from the origin. Show that there is no light deflection around $r = 0$. [7]

$u_0 = \frac{\sin\phi}{R}$ $\qquad r = \frac{R}{\sin\phi}$



Tit f. $\qquad -\delta$

3. The Riemann tensor is defined to be

$$R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^{\mu}{}_{\beta\nu} - \partial_\beta \Gamma^{\mu}{}_{\alpha\nu} + \Gamma^{\mu}{}_{\alpha\epsilon}\Gamma^{\epsilon}{}_{\nu\beta} - \Gamma^{\mu}{}_{\epsilon\beta}\Gamma^{\epsilon}{}_{\nu\alpha} \ ,$$

where $\Gamma^{\mu}{}_{\beta\nu}$ is the connection coefficient tensor. Show that

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu = R^{\mu}{}_{\nu\alpha\beta} V^\nu \tag{1}$$

for any contravariant vector V^μ. $\nabla_\mu U^\nu + \nabla^\nu U_\mu = 0$ [7]

A "Killing" vector, U^μ, satisfies the condition $\nabla_\mu U_\nu + \nabla_\nu U_\mu = 0$. We define the commutator between two vectors to be

$$W^\mu \equiv [U, V]^\mu = U^\nu \nabla_\nu V^\mu - V^\nu \nabla_\nu U^\mu \ .$$

Using equation 1 (and the symmetry of the Riemann tensor, $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$), show that the commutator of two Killing vectors is also a Killing vector. [8]

Consider a tensor $T_{\mu\nu} = \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^\sigma \nabla_\sigma \phi$, where ϕ is a scalar function of the space-time coordinates. Show that

$$\nabla^\mu T_{\mu\nu} = R_\nu{}^\sigma \, \nabla_\sigma \phi \ .$$

[6]

Assume now that $\nabla^\mu T_{\mu\nu} = 0$. If k^ν is a Killing vector, show that

$$\nabla^\mu (T_{\mu\nu} k^\nu) = 0 \ .$$

[4]

4. Consider an inflationary universe that undergoes three phases of expansion: an initial inflationary phase in which the pressure, P, and the density, ρ, satisfy $P = -\rho c^2$ up until the scale factor $a = a_1$, followed by a radiation phase in which $P = \frac{1}{3}\rho c^2$ up until the scale factor $a = a_2$, followed by a matter phase in which $P = 0$ up until the scale factor $a = 1$. Find an expression for the Hubble rate and deceleration rate, as a function of the scale factor a, in each of these regimes (neglecting all other non-dominant components of the energy density). Solve the Friedman-Robertson-Walker (FRW) equation in each one of the three phases. [7]

Explain why the expansion rate in such a universe is slower when $a < a_1$ than in a universe where there is *no* initial inflationary phase (i.e. a universe where there is no period of inflation for $a < a_1$ and that has exactly the same expansion rate as a function of the scale factor for $a > a_1$). [Hint: assume continuity in the Hubble rate at $a = a_1$ for the inflationary universe]. [5]

Assume that the initial scale factor of the universe is $a_{\rm in}$. Find an expression for the age of the universe in terms of an integral over the Hubble rate. Taking the limit $a_{\rm in} \to 0$, show that the inflationary universe must be older than the non-inflationary universe. [6]

Find an expression for the particle horizon in each one of the phases of the inflationary universe. Comparing the physical size of a length scale of fixed comoving size with the particle horizon, explain the qualitative difference between what happens for $a < a_1$ and $a > a_1$. [7]

$$\begin{cases} 3\left(\frac{\dot a}{a}\right)^2 = 8\pi G\rho \\ 3\frac{\ddot a}{a} = -4\pi G\left(\rho + \frac{3P}{c^2}\right) \\ \dot\rho + 3\frac{\dot a}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \end{cases}$$

15B5Q1

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad \text{(geodesic)}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)$$

(affine connection)

extremise action $S = -\int d\lambda\, g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \equiv -\int d\lambda\, L$

gives $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu}$

$$\frac{\partial L}{\partial x^\mu} = \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\dot{x}^\alpha \dot{x}^\beta$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = g_{\alpha\beta}\frac{\partial}{\partial \dot{x}^\mu}(\dot{x}^\alpha \dot{x}^\beta) = g_{\alpha\beta}\left(\dot{x}^\alpha \underbrace{\frac{\partial \dot{x}^\beta}{\partial \dot{x}^\mu}}_{\delta^\beta_\mu} + \dot{x}^\beta \underbrace{\frac{\partial \dot{x}^\alpha}{\partial \dot{x}^\mu}}_{\delta^\alpha_\mu}\right)$$

$$= g_{\alpha\beta}(\dot{x}^\alpha \delta^\beta_\mu + \dot{x}^\beta \delta^\alpha_\mu)$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{d}{d\tau}(g_{\alpha\beta}\dot{x}^\alpha \delta^\beta_\mu) + \frac{d}{d\tau}(g_{\alpha\beta}\dot{x}^\beta \delta^\alpha_\mu)$$

$$\cancel{= \frac{d}{d\tau}(g_{\beta\mu}\dot{x}^\alpha) + \frac{d}{d\tau}(g_{\alpha\mu}\dot{x}^\beta)}$$

$$\cancel{= \dot{x}^\alpha \frac{dg_{\alpha\mu}}{d\tau} + g_{\alpha\mu}\frac{d\dot{x}^\alpha}{d\tau} + \frac{dg_{\alpha\mu}}{d\tau}\dot{x}^\beta + g_{\alpha\mu}\frac{d\dot{x}^\beta}{d\tau}}$$

$$\cancel{= \dot{x}^\alpha \frac{\partial g_{\beta\mu}}{\partial x^\lambda}\frac{dx^\lambda}{d\tau} + g_{\beta\mu}\frac{d\dot{x}^\alpha}{d\tau} + \dot{x}^\beta \frac{\partial g_{\alpha\mu}}{\partial x^\lambda}\frac{dx^\lambda}{d\tau} + g_{\alpha\mu}\frac{d\dot{x}^\beta}{d\tau}}$$

$$\cancel{= \frac{\partial L}{\partial x^\mu} = \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\dot{x}^\alpha \dot{x}^\beta}$$

$$\cancel{\therefore\ \dot{x}^\alpha \dot{x}^\lambda \frac{\partial g_{\beta\mu}}{\partial x^\lambda} + \frac{d\dot{x}^\alpha}{d\tau}g_{\beta\mu} + \dot{x}^\beta \dot{x}^\lambda \frac{\partial g_{\alpha\mu}}{\partial x^\lambda} + g_{\alpha\mu}\frac{d\dot{x}^\beta}{d\tau}}$$

$$\cancel{= \dot{x}^\alpha \dot{x}^\beta \frac{\partial g_{\alpha\beta}}{\partial x^\mu}}$$

$$\cancel{\frac{d\dot{x}^\alpha}{d\tau} g_{\beta\nu} + \frac{d\dot{x}^\beta}{d\tau} g_{\alpha\nu} = \frac{d\dot{x}^\alpha}{d\tau} g_{\beta\nu} + dx}$$

$$= \frac{d}{d\tau}\left(g_{\alpha\nu}\dot{x}^\alpha + g_{\beta\nu}\dot{x}^\beta \right) \qquad (g_{\mu\nu} = g_{\nu\mu})$$

$$= 2\frac{d}{d\tau}\left(g_{\alpha\nu}\dot{x}^\alpha \right)$$

$$= 2\frac{dg_{\alpha\nu}}{d\tau}\dot{x}^\alpha + \cancel{2\dot{x}^\alpha}\, 2g_{\alpha\nu}\frac{d\dot{x}^\alpha}{d\tau}$$

$$= 2\frac{\partial g_{\alpha\nu}}{\partial x^\lambda}\cancel{d\dot{x}}\frac{dx^\lambda}{d\tau}\dot{x}^\alpha + 2g_{\alpha\nu}\frac{d\dot{x}^\alpha}{d\tau}$$

$$= \cancel{2\partial g}\; 2\frac{\partial g_{\alpha\nu}}{\partial x^\lambda}\dot{x}^\lambda\dot{x}^\alpha + 2g_{\alpha\nu}\frac{d\dot{x}^\alpha}{d\tau}$$

$$= 2\frac{\partial g_{\alpha\nu}}{\partial x^\lambda}\dot{x}^\lambda\dot{x}^\alpha + 2g_{\alpha\nu}\ddot{x}^\alpha$$

$$\therefore \cancel{2g}\; 2\frac{\partial g_{\alpha\nu}}{\partial x^\lambda}\dot{x}^\lambda\dot{x}^\alpha + 2g_{\alpha\nu}\ddot{x}^\alpha = \frac{\partial g_{\alpha\lambda}}{\partial x^\nu}\dot{x}^\alpha\dot{x}^\lambda$$

$$= \frac{\partial g_{\alpha\nu}}{\partial x^\lambda}\dot{x}^\lambda\dot{x}^\alpha + \frac{\partial g_{\lambda\nu}}{\partial x^\alpha}\dot{x}^\lambda\dot{x}^\alpha + 2g_{\alpha\nu}\ddot{x}^\alpha$$

$$\therefore g_{\alpha\nu}\ddot{x}^\alpha + \frac{1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\lambda}}{\partial x^\nu} \right)\dot{x}^\alpha\dot{x}^\lambda = 0$$

$$\Downarrow$$

multiply by $g^{\sigma\nu}$

$$\therefore g^{\sigma\nu}g_{\alpha\nu}\ddot{x}^\alpha + \frac{g^{\sigma\nu}}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\lambda}}{\partial x^\nu} \right)\dot{x}^\alpha\dot{x}^\lambda = 0$$

$$\ddot{x}^\alpha g^{\sigma\nu}g_{\alpha\nu} = \delta^\sigma_\alpha\ddot{x}^\alpha = \ddot{x}^\sigma$$

$$\therefore \quad \ddot{x}^\sigma + \frac{g^{\sigma\nu}}{2} \underbrace{\left(\frac{\partial g_{\alpha\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\lambda}}{\partial x^\nu} \right)}_{\Gamma^\sigma_{\lambda\alpha} \text{ = connection coefficient}} \dot{x}^\lambda \dot{x}^\alpha = 0$$

$$\therefore \quad \underline{\ddot{x}^\sigma + \Gamma^\sigma_{\lambda\alpha} \dot{x}^\lambda \dot{x}^\alpha = 0}$$

Global Rain metric :

$$ds^2 = -c^2\left(1 - \frac{r_s}{r}\right) d\bar{t}^2 + 2\sqrt{\frac{r_s}{r}}\, c\, d\bar{t}\, dr$$

$$+ dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\because \quad ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$\therefore \quad g_{\bar{t}\bar{t}} = -\left(1 - \frac{r_s}{r}\right)c^2 \qquad g_{\bar{t}r} = \sqrt{\frac{r_s}{r}}\, c = g_{r\bar{t}}$$

$$g_{rr} = 1 \qquad g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2 \sin^2\theta$$

$$\therefore \text{ matrix } \quad g_{\mu\nu} = \begin{pmatrix} g_{\bar{t}\bar{t}} & g_{\bar{t}r} & g_{\bar{t}\theta} & g_{\bar{t}\phi} \\ g_{r\bar{t}} & g_{rr} & g_{r\theta} & g_{r\phi} \\ g_{\theta\bar{t}} & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\bar{t}} & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} \quad \text{(symmetric)}$$

$$= \begin{pmatrix} -\left(1 - \frac{r_s}{r}\right)c^2 & \sqrt{\frac{r_s}{r}}\, c & 0 & 0 \\ \sqrt{\frac{r_s}{r}}\, c & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

$g^{\mu\nu}$ is the inverse

inverse of $\begin{pmatrix} -(1-\frac{r_s}{r})c^2 & \sqrt{\frac{r_s}{r}}c \\ \sqrt{\frac{r_s}{r}}c & 1 \end{pmatrix}$ is:

~~det~~ $\cancel{-(1-\frac{r_s}{r}) - \frac{r_s}{r} = -1}$

$\therefore \quad \det = -(1-\frac{r_s}{r})c^2 - \frac{r_s}{r}c^2$

$= -c^2$

$\therefore \quad g^{\mu\nu}_{2\times2} = \frac{-1}{c^2}\begin{pmatrix} 1 & -\sqrt{\frac{r_s}{r}}c \\ -\sqrt{\frac{r_s}{r}}c & -(1-\frac{r_s}{r})c^2 \end{pmatrix}$

$\Rightarrow \quad g^{\bar t\bar t} = -\frac{1}{c^2}$ $\quad g^{\bar r\bar r} = (1-\frac{r_s}{r})$

$g^{\bar t r} = g^{r\bar t} = \frac{1}{c}\sqrt{\frac{r_s}{r}}$

Inverse of $\begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$

$\therefore \quad g^{\theta\theta} = \frac{1}{r^2}, \quad g^{\phi\phi} = \frac{1}{r^2\sin^2\theta}$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{g^{\lambda\sigma}}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right)$$

$$\ddot{x}^{\lambda} + \Gamma^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$

in the case of r coordinate :

$(\cancel{\Leftarrow} \; x^{\lambda} = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi))$

$$\ddot{r} + \cancel{\;} \Gamma^{r}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$

non zero $\Gamma^{r}_{\mu\nu}$ are

$$\Gamma^{r}_{rr}, \quad \Gamma^{r}_{tt}, \quad \Gamma^{r}_{rt} = \Gamma^{r}_{tr}$$

$$\Gamma^{r}_{\theta\theta} \quad \Gamma^{r}_{\phi\phi}$$

$$\cancel{\Gamma^{r}_{rr} = \frac{g^{rr}}{2}\left(\partial\right.}$$

$$\Gamma^{r}_{rr} = \frac{g^{r\sigma}}{2}\left(\frac{\partial g_{r\sigma}}{\partial r} + \frac{\partial g_{r\sigma}}{\partial r} - \frac{\partial g_{rr}}{\partial x^{\sigma}}\right)$$

$$= \frac{g^{rr}}{2}\frac{\partial g_{rr}}{\partial r} + \cancel{\frac{g^{rr}}{2}\frac{\partial g_{rr}}{\partial r}} - \cancel{\frac{g^{rr}}{2}\frac{\partial g_{rr}}{\partial r}}$$

$$+ \frac{g^{rt}}{2}\frac{\partial g_{rt}}{\partial r} + \frac{g^{rt}}{2}\frac{\partial g_{rt}}{\partial r} - \cancel{\frac{g^{rt}}{2}\times\frac{\partial g_{rr}}{\partial t}}$$

$$= \cancel{\frac{g^{rr}}{2}}\frac{\partial g_{rr}}{\partial r} + g^{rt}\frac{\partial g_{rt}}{\partial r}$$

$$\cancel{= \frac{1}{2}\left(-\frac{1}{c^2}\right)\left(-\frac{r_s}{r^2}\right) + \sqrt{\frac{r_s}{r}}\,c\left(\right.}$$

$$
\begin{aligned}
&\cancel{= \frac{1}{2}\left(1-\frac{r_s}{r}\right)\left(-\frac{r_s}{r^2}\right) + \frac{1}{c}\sqrt{\frac{r_s}{r}}\, c \left(+\frac{1}{2}\right)\left(\frac{r_s}{r}\right)^{-\frac{1}{2}}\left(-\frac{r_s}{r^2}\right).} \\
&\cancel{= -\frac{r_s}{2r^2} + \frac{1}{2}\frac{r_s^2}{r^3} - \frac{1}{2}\frac{r_s}{r}} \\
&\cancel{= \frac{1}{2}\left(1-\frac{r_s}{r}\right)\left(-\frac{r_s}{r^2}\right) + \frac{1}{c}\sqrt{\frac{r_s}{r}}\, c} \\
&= g_{r\bar{t}}\frac{\partial g_{r\bar{t}}}{\partial r} = \frac{1}{c}\sqrt{\frac{r_s}{r}}\, c \left(+\frac{1}{2}\right)\left(\frac{r_s}{r}\right)^{-\frac{1}{2}}\left(-\frac{r_s}{r^2}\right) \\
&= -\frac{r_s}{2r^2} = -\frac{1}{2r_s}\left(\frac{r_s}{r}\right)^2 \\
&\cancel{\Gamma^r_{\bar{t}\bar{t}} = \frac{g^{rr}}{2}\left(\frac{\partial g_{\bar{t}r}}{\partial \bar{t}} + \frac{\partial g_{\bar{t}r}}{\partial r} - \frac{\partial g_{\bar{t}r}}{\partial r}\right)} \\
&\qquad \cancel{+ \frac{g^{r\bar{t}}}{2}\left(\frac{\partial g_{\bar{t}\bar{t}}}{\partial \bar{t}} + \frac{\partial g_{\bar{t}\bar{t}}}{\partial r} - \frac{\partial g_{\bar{t}r}}{\partial \bar{t}}\right)} \\
&\cancel{= \frac{g^{r\bar{t}}}{2}\left(\frac{\partial g_{\bar{t}\bar{t}}}{\partial r}\right)} \\
&\cancel{= \frac{1}{2}\left(\frac{1}{c}\sqrt{\frac{r_s}{r}}\right)\left(-\right.} \\
&\Gamma^r_{\bar{t}\bar{t}} = \frac{g^{rr}}{2}\left(\cancel{\frac{\partial g_{\bar{t}r}}{\partial \bar{t}}} + \cancel{\frac{\partial g_{\bar{t}r}}{\partial \bar{t}}} - \frac{\partial g_{\bar{t}\bar{t}}}{\partial r}\right) \\
&\qquad + \frac{g^{r\bar{t}}}{2}\left(\cancel{\frac{\partial g_{\bar{t}\bar{t}}}{\partial \bar{t}}} + \cancel{\frac{\partial g_{\bar{t}\bar{t}}}{\partial \bar{t}}} - \cancel{\frac{\partial g_{\bar{t}\bar{t}}}{\partial \bar{t}}}\right) \\
&= -\frac{1}{2} g^{rr}\frac{\partial g_{\bar{t}\bar{t}}}{\partial r} = \cancel{-}\frac{1}{2}\left(1-\frac{r_s}{r}\right)\left(\cancel{-}c^2\right)\left(\frac{r_s}{r^2}\right).
\end{aligned}
$$

$$= \frac{c^2}{2r_s}\left(1-\frac{r_s}{r}\right)\left(\frac{r_s}{r}\right)^2$$

$$\Gamma^r_{r\bar{t}} = \frac{g^{rr}}{2}\left(\cancel{\frac{\partial g_{rr}}{\partial \bar{t}}} + \cancel{\frac{\partial g_{\bar{t}r}}{\partial r}} - \cancel{\frac{\partial g_{r\bar{t}}}{\partial r}}\right)$$

$$+ \frac{g^{r\bar{t}}}{2}\left(\cancel{\frac{\partial g_{r\bar{t}}}{\partial \bar{t}}} + \frac{\partial g_{\bar{t}\bar{t}}}{\partial r} - \cancel{\frac{\partial g_{r\bar{t}}}{\partial \bar{t}}}\right)$$

$$= \frac{g^{r\bar{t}}}{2}\frac{\partial g_{\bar{t}\bar{t}}}{\partial r}$$

$$= \frac{1}{c}\left(\frac{r_s}{r}\right)^{\frac{1}{2}}\left(-c^2\right)\left(\frac{r_s^2}{r^2}\right)\frac{1}{r_s}$$

$$= -\frac{c}{r_s}\left(\frac{r_s}{r}\right)^{5/2}$$

$$\cancel{\Gamma^r_{r\theta} = \frac{g^{rr}}{2}\left(\frac{\partial g_{rr}}{\partial \theta} + \frac{\partial g_{\theta r}}{\partial r} - \frac{\partial g_{r\theta}}{\partial r}\right)}$$

$$\cancel{+ g^{r\bar{t}}}$$

$$\Gamma^r_{\theta\theta} = \frac{g^{rr}}{2}\left(\cancel{\frac{\partial g_{\theta r}}{\partial \theta}} + \cancel{\frac{\partial g_{\theta r}}{\partial \theta}} - \frac{\partial g_{\theta\theta}}{\partial r}\right)$$

$$+ \frac{g^{r\bar{t}}}{2}\left(\cancel{\frac{\partial g_{\theta\bar{t}}}{\partial \theta}} + \cancel{\frac{\partial g_{\theta\bar{t}}}{\partial \theta}} - \cancel{\frac{\partial g_{\theta\theta}}{\partial \bar{t}}}\right)$$

$$= -\frac{g^{rr}}{2}\frac{\partial g_{\theta\theta}}{\partial r} = -\left(1-\frac{r_s}{r}\right)\frac{1}{2}(2r)$$

$$= -\left(1-\frac{r_s}{r}\right)r$$

$$\Gamma^r_{\phi\phi} = \frac{g^{rr}}{2}\left(\cancel{\frac{\partial g_{\phi r}}{\partial \phi}} + \cancel{\frac{\partial g_{\phi r}}{\partial \phi}} - \frac{\partial g_{\phi\phi}}{\partial r}\right)$$

$$+ \frac{g^{r\bar t}}{2}\left(\cancel{\frac{\partial g_{\phi \bar t}}{\partial \phi}} + \cancel{\frac{\partial g_{\phi \bar t}}{\partial \phi}} - \cancel{\frac{\partial g_{\phi\phi}}{\partial \bar t}}\right)$$

$$= \frac{1}{\cancel{2}}\left(1-\frac{r_s}{r}\right)(-)\left(\cancel{2} r \sin^2\theta\right)$$

$$= \underline{\underline{-\left(1-\frac{r_s}{r}\right) r \sin^2\theta}}.$$

$\therefore$ $0 = \ddot r + \Gamma^{\lambda}_{\mu\nu}\dot x^\mu \dot x^\nu$ becomes

$$\ddot r - \frac{1}{2r_s}\left(\frac{r_s}{r}\right)^2 \dot r^2 + \frac{c}{2r_s}\left(1-\frac{r_s}{r}\right)\left(\frac{r_s}{r}\right)^2 \dot{\bar t}^2 - \frac{c}{r_s}\left(\frac{r_s}{r}\right)^{3/2}\dot r \dot{\bar t}$$
$$- \left(1-\frac{r_s}{r}\right) r\dot\theta^2 - \left(1-\frac{r_s}{r}\right) r \sin^2\theta\, \dot\phi^2 = 0$$

– At $r=r_s$

Schwarzschild metric

$$\cancel{ds^2 = -c^2\left(1-\frac{r_s}{r}\right)d\bar t^2 + 2\sqrt{\frac{r_s}{r}}\, c\, d\bar t\, dr}$$

$$ds^2 = -c^2\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

has coordinate singularity at $r=r_s$.
g_{rr} term $\to \infty$

whereas in Global rain metric there is no coordinate singularity at $r=r_s$.

- light geodesic $ds^2 = 0$

for radial photon $d\theta, d\phi = 0$

$\therefore \quad 0 = -c^{\cancel{2}}(1-\frac{r_s}{r})\, d\bar{t}^{\cancel{2}} + 2\sqrt{\frac{r_s}{r}}\, c\, \cancel{d\bar{t}}\, dr$

$\Rightarrow$

$$c(1-\frac{r_s}{r})\, d\bar{t} = 2\sqrt{\frac{r_s}{r}}\, dr$$

$$\therefore \quad \frac{dr}{d\bar{t}} = \frac{c}{2}\sqrt{\frac{r}{r_s}}\left(1-\frac{r_s}{r}\right)$$

If $r<r_s$, $\frac{dr}{d\bar{t}}$ always <0

$\therefore$ photons fall inwards.

- change of coordinates

$$\bar{t} = \bar{t}(r,t)$$

$$\therefore \quad d\bar{t} = \frac{\partial \bar{t}}{\partial t}dt + \frac{\partial \bar{t}}{\partial r}dr$$

$$= dt + \sqrt{\frac{r_s}{r}}\left(1-\frac{r_s}{r}\right)^{-1}\frac{1}{c}\, dr$$

$$d\bar{t}^2 = dt^2 + 2\sqrt{\frac{r_s}{r}}\left(1-\frac{r_s}{r}\right)^{-1}\frac{1}{c}\, dt\, dr + \frac{r_s}{r}\left(1-\frac{r_s}{r}\right)^{-2}\frac{1}{c^2}\, dr^2$$

$$\therefore \quad -c^2(1-\frac{r_s}{r})\, d\bar{t}^2 + 2\sqrt{\frac{r_s}{r}}\, c\, d\bar{t}\, dr + dr^2$$

$$= -c^2(1-\frac{r_s}{r})\, dt^2 \cancel{- 2c\sqrt{\frac{r_s}{r}}\, dt\, dr} - \frac{r_s}{r}(1-\frac{r_s}{r})^{-1}dr^2$$

$$\cancel{+ 2\sqrt{\frac{r_s}{r}}\, c\, dt\, dr} + 2\frac{r_s}{r}(1-\frac{r_s}{r})^{-1}dr^2 + dr^2$$

$$= -c^2\left(1-\frac{r_s}{r}\right)dt^2 + \frac{r_s}{r}\left(1-\frac{r_s}{r}\right)^{-1}dr^2 + dr^2$$

$$= -c^2\left(1-\frac{r_s}{r}\right)dt^2 + \left[\cancel{\frac{r_s}{r}} + \left(1-\cancel{\frac{r_s}{r}}\right)\right]\left(1-\frac{r_s}{r}\right)^{-1}dr^2$$

$$= -c^2\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2$$

($r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ is unchanged)

$\Rightarrow$ recovers Schwarzschild metric.

15B5Q2

$$ds^2 = e^{2\varphi/c^2}\eta_{\alpha\beta}\,dx^\alpha d\alpha^\beta = g_{\alpha\beta}\,dx^\alpha dx^\beta$$

$$\therefore\ g_{\alpha\beta} = e^{\frac{2\varphi}{c^2}}\eta_{\alpha\beta}$$

$$\Gamma^\nu_{\alpha\beta} = \frac{g^{\nu\sigma}}{2}\left(\frac{\partial g_{\alpha\sigma}}{\partial x^\beta} + \frac{\partial g_{\beta\sigma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right)$$

$\because\ g_{\alpha\beta}$ is diagonal (as $\eta_{\alpha\beta}$)

$$\therefore\ g^{\alpha\beta} = e^{-\frac{2\varphi}{c^2}}\eta^{\alpha\beta} \qquad (\partial_\lambda \eta_{\alpha\beta} = 0)$$

$$\therefore\ \Gamma^\nu_{\alpha\beta} = \frac{e^{-\frac{2\varphi}{c^2}}\eta^{\nu\sigma}}{2}\Big[\partial_\beta\big(e^{\frac{2\varphi}{c^2}}\eta_{\alpha\sigma}\big) + \partial_\alpha\big(e^{\frac{2\varphi}{c^2}}\eta_{\beta\sigma}\big) - \partial_\sigma\big(e^{\frac{2\varphi}{c^2}}\eta_{\alpha\beta}\big)\Big].$$

$$= \frac{\cancel{e^{-\frac{2\varphi}{c^2}}}\eta^{\nu\sigma}}{\cancel{2}}\Big[\eta_{\alpha\sigma}\frac{\cancel{2}}{c^2}\cancel{e^{\frac{2\varphi}{c^2}}}\partial_\beta\varphi + \eta_{\beta\sigma}\frac{\cancel{2}}{c^2}\cancel{e^{\frac{2\varphi}{c^2}}}\partial_\alpha\varphi - \eta_{\alpha\beta}\frac{\cancel{2}}{c^2}\cancel{e^{\frac{2\varphi}{c^2}}}\partial_\sigma\varphi\Big].$$

$$= \frac{1}{c^2}\Big[\partial_\beta\varphi\,\underbrace{\eta^{\nu\sigma}\eta_{\alpha\sigma}}_{\delta^\nu_\alpha} + \partial_\alpha\varphi\,\underbrace{\eta^{\nu\sigma}\eta_{\beta\sigma}}_{\delta^\nu_\beta} - \underbrace{\eta^{\nu\sigma}\partial_\sigma}_{\partial^\nu}\varphi\,\eta_{\alpha\beta}\Big]$$

$$= \frac{1}{c^2}\Big[\partial_\beta\varphi\,\delta^\nu_\alpha + \partial_\alpha\varphi\,\delta^\nu_\beta - \partial^\nu\varphi\,\eta_{\alpha\beta}\Big].$$

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g R \qquad (R = R^{\alpha}{}_{\alpha}).$$

$$\cancel{R} \quad R_{\nu\beta} = \partial_\mu \Gamma^{\mu}_{\beta\nu} - \partial_\beta \Gamma^{\mu}_{\mu\nu} + \Gamma^{\mu}_{\mu\epsilon}\Gamma^{\epsilon}_{\nu\beta} - \Gamma^{\mu}_{\epsilon\beta}\Gamma^{\epsilon}_{\nu\mu}$$

$$\cancel{= \partial_\mu \frac{1}{c^2}\left[\partial_\nu \varphi \delta^{\mu}_{\beta} + \partial_\mu \varphi \delta^{\mu}_{\nu} - \right]}$$

$$+ \cancel{\frac{1}{c^4}\left[\partial_\epsilon \varphi \delta^{\mu}_{\nu} + \partial_\nu \varphi \delta^{\mu}_{\epsilon} - \partial\right.}$$

$$= \frac{1}{c^2}\Big(\partial_\mu \partial_\nu \varphi \delta^{\mu}_{\beta} + \cancel{\partial_\mu \partial_\beta \varphi \delta^{\mu}_{\nu}} - \partial_\mu \partial^\mu \varphi \eta_{\beta\nu}$$

$$- \cancel{\partial_\beta \partial_\mu \varphi \delta^{\mu}_{\nu}} - \partial_\beta \partial_\nu \varphi \delta^{\mu}_{\mu} + \partial_\beta \partial^\mu \varphi \eta_{\mu\nu}\Big)$$

neglect

$$+ \frac{1}{c^4}\Big[\left(\partial_\epsilon \varphi \delta^{\mu}_{\mu} + \partial_\mu \varphi \delta^{\mu}_{\epsilon} - \partial^\mu \varphi \eta_{\mu\epsilon}\right)$$

$$\times \left(\partial_\beta \varphi \delta^{\epsilon}_{\nu} + \partial_\nu \varphi \delta^{\epsilon}_{\beta} - \partial^\epsilon \varphi \eta_{\nu\beta}\right)$$

$$- \left(\partial_\beta \varphi \delta^{\mu}_{\epsilon} + \partial_\epsilon \varphi \delta^{\mu}_{\beta} - \partial^\mu \varphi \eta_{\epsilon\beta}\right)$$

$$\times \left(\partial_\mu \varphi \delta^{\epsilon}_{\nu} + \partial_\nu \varphi \delta^{\epsilon}_{\mu} - \partial^\epsilon \varphi \eta_{\mu\nu}\right).\Big]$$

$$\underset{\frac{\varphi}{c^2} \ll 1}{\approx} \frac{1}{c^2}\Big(\partial_\mu \partial_\nu \varphi \delta^{\mu}_{\beta} + \cancel{\partial_\mu \partial_\beta \varphi \delta^{\mu}_{\nu}} - \partial_\mu \partial^\mu \varphi \eta_{\beta\nu}$$

$$- \partial_\beta \partial_\nu \varphi \delta^{\mu}_{\mu} + \partial_\beta \partial^\mu \varphi \eta_{\mu\nu}\Big).$$

$$R = g^{\nu\beta} R_{\nu\beta} = e^{-\frac{2\varphi}{c^2}} \eta^{\nu\beta} R_{\nu\beta}$$

$$= \frac{1}{c^2} e^{-\frac{2\varphi}{c^2}} \eta^{\nu\beta} \left(\partial_\mu \partial_\nu \varphi \, \delta^\mu_\beta - \partial_\mu \partial^\mu \varphi \, \eta_{\beta\nu} - \partial_\beta \partial_\nu \varphi \, \delta^\mu_\mu + \partial_\beta \partial^\mu \varphi \, \eta_{\mu\nu} \right).$$

$$\cancel{= \frac{1}{c^2} e^{-\frac{2\varphi}{c^2}} \left(\partial_\mu \partial^\beta \varphi \, \delta^\mu_\beta - \partial_\mu \partial^\mu \varphi \, \delta^\nu_\nu - \partial_\beta \partial^\beta \varphi \, \delta^\mu_\mu + \partial_\beta \partial^\mu \varphi \, \delta^\beta_\nu \right)}$$

$$\cancel{= \frac{2}{c^2} e^{-\frac{2\varphi}{c^2}} \left(\partial_\mu \partial^\beta \varphi \, \delta^\mu_\beta - \partial_\mu \partial^\mu \varphi \, \delta^\nu_\nu \right)}$$

$$\cancel{\tfrac{1}{2} g}$$

$$= \frac{1}{c^2} e^{-\frac{2\varphi}{c^2}} \Big(\overbrace{\eta^{\nu\mu} \partial_\mu \partial_\nu \varphi}^{1} - \partial_\mu \partial^\mu \varphi \, \overbrace{\delta^\nu_\nu}^{4} - \eta^{\nu\beta} \partial_\beta \underbrace{\partial_\nu \varphi \, \delta^\mu_\mu}_{4} + \eta^{\mu\beta} \underbrace{\partial_\beta \partial_\nu \varphi}_{1} \Big)$$

$$= -\frac{6}{c^2} e^{-\frac{2\varphi}{c^2}} \left(\partial_\mu \partial^\mu \varphi \right)$$

$$\therefore -\frac{1}{2} g_{\alpha\beta} R = \frac{3}{c^2} e^{-\frac{2\varphi}{c^2}} \left(\partial_\mu \partial^\mu \varphi \right) \eta_{\alpha\beta}$$

$$\simeq \frac{3}{c^2} \partial_\mu \partial^\mu \varphi \, \eta_{\alpha\beta}$$

$$R_{\alpha\beta} = \frac{1}{c^2} \left(\partial_\mu \partial_\alpha \varphi \, \delta^\mu_\beta - \partial_\mu \partial^\mu \varphi \, \eta_{\alpha\beta} - \partial_\beta \partial_\alpha \varphi \, \delta^\mu_\mu + \partial_\beta \partial^\mu \varphi \, \eta_{\mu\alpha} \right)$$

$$= \frac{1}{c^2}\left(\partial_\alpha \partial_\beta \varphi - \partial_\mu \partial^\mu \varphi \,\eta_{\alpha\beta} - 4\partial_\alpha\partial_\beta\varphi \right.$$

$$\left. + \partial_\alpha \partial_\beta \varphi\right)$$

$$= \frac{1}{c^2}\left(-\partial_\mu \partial^\mu \varphi \,\eta_{\alpha\beta} - 2\partial_\alpha\partial_\beta\varphi\right).$$

$$\therefore G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

$$= \frac{3}{c^2}\partial_\mu \partial^\mu \varphi \,\eta_{\alpha\beta} + \frac{1}{c^2}\left(-\partial_\mu \partial^\mu \varphi \,\eta_{\alpha\beta} - 2\partial_\alpha\partial_\beta\varphi\right)$$

$$= \frac{2}{c^2}\left(\partial_\mu \partial^\mu \varphi \,\eta_{\alpha\beta} - \partial_\alpha\partial_\beta\varphi\right)$$

$$ds^2 = e^{\frac{2\varphi}{c^2}}\left(-c^2dt^2 + dx^2 + dy^2 + dz^2\right)$$

$$= e^{\frac{2\varphi}{c^2}}\left(-c^2dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \, d\phi^2\right)$$

Geodesic:

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0$$

equatorial $d\theta = 0$, $\sin\theta = 1$

$$ds^2 = e^{\frac{2\varphi}{c^2}}\left(-c^2dt^2 + dr^2 + r^2 d\phi^2\right)$$

Lagrangian $L = g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$

$$L = e^{\frac{2\varphi}{c^2}}\left(-c^2\dot{t}^2 + \dot{r}^2 + r^2\underbrace{\dot{\theta}^2}_{=0} + r^2\underbrace{\sin^2\theta}_{1}\,\dot{\phi}^2\right)$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) = \frac{\partial L}{\partial x^{\mu}}$$

$$\therefore \quad \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} \quad \Rightarrow$$

$$\frac{d}{d\lambda}\left(-2c^2 e^{\frac{2\varphi}{c^2}}\dot{t}\right) = 0$$

$$\Rightarrow \quad \underline{\underline{e^{\frac{2\varphi}{c^2}}\dot{t} = d}}$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \quad \Rightarrow$$

$$\frac{d}{d\lambda}\left(2r^2\underbrace{\sin^2\theta}_{1}\, e^{\frac{2\varphi}{c^2}}\dot{\phi}\right) = 0$$

$$\therefore \quad \underline{\underline{r^2 e^{\frac{2\varphi}{c^2}}\dot{\phi} = \ell.}}$$

null condition for photon. $ds^2 = 0$

$$\cancel{ds} \quad \therefore \quad \underline{\underline{-c^2dt^2 + dr^2 + r^2d\phi^2 = 0}} \qquad (\sin\theta = 1)$$

$$\therefore \quad -c^2\dot{t}^2 + \dot{r}^2 + r^2\dot{\phi}^2 = 0$$

- deflection ?

$$-c^2\dot{t}^2 + \dot{r}^2 + r^2\dot{\phi}^2 = 0$$

$$r^2\dot{\phi} = l e^{-\frac{2\varphi}{c}} \Rightarrow \dot{\phi}^2 = \frac{l^2 e^{-\frac{4\varphi}{c^2}}}{r^4}$$

$$\dot{t} = d e^{-\frac{2\varphi}{c}}$$

$$\therefore \quad -c^2 d^2 e^{-\frac{4\varphi}{c^2}} + \dot{r}^2 + \frac{l^2 e^{-\frac{4\varphi}{c^2}}}{r^2} = 0 \qquad ①$$

let $u = \frac{1}{r}$ $\qquad \frac{du}{d\phi} = \frac{d}{d\phi}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\phi}$

$$\therefore \left(\frac{du}{d\phi}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2$$

$$\cancel{\therefore -c^2\dot{t} + \dot{r}^2 + r^2\dot{\phi}}$$

$$① \Rightarrow \quad \dot{r}^2 = e^{-\frac{4\varphi}{c^2}}\left(c^2 d^2 - \frac{l^2}{r^2}\right) \qquad \left(\varphi = -\frac{GM}{r}\right)$$

$$\frac{①}{\dot{\phi}} \Rightarrow \left(\frac{\dot{r}}{\dot{\phi}}\right)^2 = \left(\frac{dr/d\lambda}{d\phi/d\lambda}\right)^2 = \left(\frac{dr}{d\phi}\right)^2 \qquad \dot{\phi} = \frac{l e^{-\frac{2\varphi}{c^2}}}{r^2}$$

$$\cancel{\therefore \left(\frac{dr}{d\phi}\right)^2 = \frac{r^2}{l} e^{-\frac{2\varphi}{c^2}}\left(c^2 d^2 - \frac{l^2}{r^2}\right)}$$

$$\cancel{= e^{-\frac{2\varphi}{c^2}}\left(\frac{c^2 d^2}{l^2} r^2 - 1\right)}$$

$$\cancel{= e^{-\frac{2GM}{c^2}u}\left(\frac{c^2 d^2}{l^2 u^2} - 1\right)}$$

$$\cancel{\therefore \left(\frac{du}{d\phi}\right)^2 = u^4\left(\frac{dr}{d\phi}\right)^2 = e^{-\frac{2GMu}{c^2}}\left(\frac{c^2 d^2}{l^2}u^2 - u^4\right)}$$

$$\cancel{\left(\frac{du}{d\phi}\right)^2 + u^2 = 2\frac{du}{d\phi}}$$

$$\cancel{\frac{d}{d\phi}\left(\left(\frac{du}{d\phi}\right)^2 + u^2\right) = 2\frac{du}{d\phi}\frac{d^2u}{d\phi^2}}$$

~~Differentiate this:~~

$$\cancel{2\left(\frac{du}{d\phi}\right)\frac{d^2u}{d\phi^2} = e^{-\frac{2GM}{c^2}u}\cdot\left(\frac{2c^2d^2}{l^2}u - 4u^3\right)\frac{du}{d\phi}}$$

$$\therefore \left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{l^2}\left(c^2d^2 - \frac{l^2}{r^2}\right).$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 = \frac{1}{l^2}\left(c^2d^2 - \frac{l^2}{r^2}\right)$$

$$= \frac{1}{l^2}\left(c^2d^2 - l^2u^2\right)$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2d^2}{l^2}.$$

differentiate this:

$$2\left(\frac{du}{d\phi}\right)\frac{d^2u}{d\phi^2} + 2u\frac{du}{d\phi} = 0$$

$$\therefore \left(\frac{d^2u}{d\phi^2} + u\right)\frac{du}{d\phi} = 0$$

~~At $\frac{dr}{d\phi} = 0$, $r = \frac{l}{cd}$, but at this r, $\frac{d^2r}{d\phi^2} \neq 0$~~

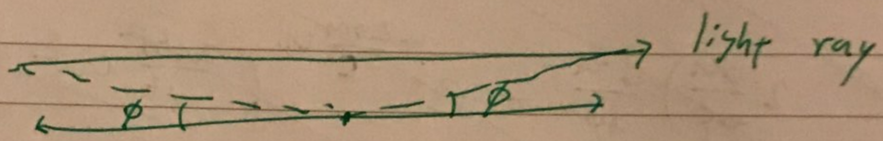
~~$\therefore$ No circular orbit $\Rightarrow \frac{du}{d\phi} \neq 0$~~

Consider unbound orbit (ignore $\frac{du}{d\phi} = 0$ case)

$$\therefore \frac{d^2u}{d\phi^2} + u^2 = 0$$

Solution $u = \frac{\sin\phi}{R}$

At $u = 0$ $(r \to \infty)$, $\phi = 0, \pi$



from π to 0, <u>no deflection</u>

~~[illegible]~~ 15BS Q3

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)V^\mu$$

$$= \nabla_\alpha\nabla_\beta V^\mu - \nabla_\beta\nabla_\alpha V^\mu$$

$$= \nabla_\alpha(\partial_\beta V^\mu + \Gamma^\mu_{\beta\nu}V^\nu) - \nabla_\beta(\partial_\alpha V^\mu + \Gamma^\mu_{\alpha\nu}V^\nu)$$

$$\cancel{= \partial_\alpha\partial_\beta V^\mu + \Gamma}$$

$$= \cancel{\partial_\alpha\partial_\beta V^\mu} + \Gamma^\mu_{\alpha\nu}\partial_\beta V^\nu - \cancel{\Gamma^\nu_{\alpha\beta}\partial_\nu V^\mu}$$

$$+ \partial_\alpha\Gamma^\mu_{\beta\nu}V^\nu + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\nu}V^\nu - \cancel{\Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\nu}V^\nu}$$

$$- \cancel{\partial_\beta\partial_\alpha V^\mu} - \Gamma^\mu_{\beta\nu}\partial_\alpha V^\nu - \cancel{\Gamma^\nu_{\beta\alpha}\partial_\nu V^\mu}$$

$$- \partial_\beta\Gamma^\mu_{\alpha\nu}V^\nu - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\nu}V^\nu - \cancel{\Gamma^\sigma_{\beta\alpha}\Gamma^\mu_{\sigma\nu}V^\nu}$$

$$= \Gamma^\mu_{\alpha\nu}\partial_\beta V^\nu + \partial_\alpha\Gamma^\mu_{\beta\nu}V^\nu + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\nu}V^\nu$$

$$- \Gamma^\mu_{\beta\nu}\partial_\alpha V^\nu - \partial_\beta\Gamma^\mu_{\alpha\nu}V^\nu - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\nu}V^\nu$$

$$= (\partial_\alpha\Gamma^\mu_{\beta\nu}V^\nu - \Gamma^\mu_{\beta\nu}\partial_\alpha V^\nu) - (\partial_\beta\Gamma^\mu_{\alpha\nu}V^\nu - \Gamma^\mu_{\alpha\nu}\partial_\beta V^\nu)$$

$$+ \Gamma^\mu_{\alpha\epsilon}\Gamma^\epsilon_{\nu\beta}V^\nu - \Gamma^\mu_{\epsilon\beta}\Gamma^\epsilon_{\nu\alpha}V^\nu$$

$$= (\partial_\alpha\Gamma^\mu_{\beta\nu})V^\nu - (\partial_\beta\Gamma^\mu_{\alpha\nu})V^\nu + (\Gamma^\mu_{\alpha\epsilon}\Gamma^\epsilon_{\nu\beta} - \Gamma^\mu_{\epsilon\beta}\Gamma^\epsilon_{\nu\alpha})V^\nu$$

$$= \underbrace{\left[\partial_\alpha\Gamma^\mu_{\beta\nu} - \partial_\beta\Gamma^\mu_{\alpha\nu} + \Gamma^\mu_{\alpha\epsilon}\Gamma^\epsilon_{\nu\beta} - \Gamma^\mu_{\epsilon\beta}\Gamma^\epsilon_{\nu\alpha}\right]}_{R^\mu{}_{\nu\alpha\beta}}V^\nu$$

$= R^{\mu}{}_{\nu\alpha\beta} V^{\nu}$

- Killing vector:

If U^{μ}, V^{μ} are Killing vectors:

$\cancel{\nabla_\sigma U \quad \nabla_\sigma U^{\mu} + \nabla}$

$$\nabla_\mu U_\nu + \nabla_\nu U_\mu = 0$$

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$$

$$W^{\mu} = [U, V]^{\mu} = U^{\nu} \nabla_\nu V^{\mu} - V^{\nu} \nabla_\nu U^{\mu}$$

~~Now~~ $\cancel{\nabla_\sigma W^{\mu} + \nabla}$

$$W_\mu = U^{\nu} \nabla_\nu V_\mu - V^{\nu} \nabla_\nu U_\mu$$

$$\nabla_\sigma W_\mu + \nabla_\mu W_\sigma$$

$$= \nabla_\sigma (U^{\nu} \nabla_\nu V_\mu) - \nabla_\sigma (V^{\nu} \nabla_\nu U_\mu) + \nabla_\mu (U^{\nu} \nabla_\nu V_\sigma)$$

$$- \nabla_\mu (V^{\nu} \nabla_\nu U_\sigma)$$

Also $\nabla_\alpha \nabla_\beta V^{\mu} - \nabla_\beta \nabla_\alpha V^{\mu} = R^{\mu}{}_{\nu\alpha\beta} V^{\nu}$

$\rightarrow \quad \nabla_\alpha \nabla_\beta V_\mu - \nabla_\beta \nabla_\alpha V_\mu = R_{\mu\nu\alpha\beta} V^{\nu}$

$\therefore \nabla_\sigma w_\mu + \nabla_\mu w_\sigma$

$= (\nabla_\sigma U^\nu)(\nabla_\nu V_\mu) + U^\nu \nabla_\sigma \nabla_\nu V_\mu$

$- (\nabla_\sigma V^\nu)(\nabla_\nu U_\mu) - V^\nu \nabla_\sigma \nabla_\nu U_\mu$

$+ (\nabla_\mu U^\nu)(\nabla_\nu V_\sigma) + U^\nu \nabla_\mu \nabla_\nu V_\sigma$

$- (\nabla_\mu V^\nu)(\nabla_\nu U_\sigma) - V^\nu \nabla_\mu \nabla_\nu U_\sigma$

Now: $(\nabla_\sigma U^\nu)(\nabla_\nu V_\mu) - (\nabla_\mu V^\nu)(\nabla_\nu U_\sigma)$

$= (\nabla_\sigma U^\nu)(\nabla_\nu V_\mu) - (\nabla_\mu V_\nu)(\nabla^\nu U_\sigma)$

$= (\nabla_\sigma U^\nu)(\nabla_\nu V_\mu) - (-\nabla_\nu V_\mu)(-\nabla_\sigma U^\nu) = \underline{\underline{0}}$

$\nabla_\nu V_\mu + \nabla_\nu V_\mu = 0$ $\qquad$ $\nabla_\sigma U_\nu + \nabla_\nu U_\sigma = 0$

$\rightarrow \nabla_\sigma U^\nu + \nabla^\nu U_\sigma = 0$

$(\nabla_\mu U^\nu)(\nabla_\nu V_\sigma) - (\nabla_\sigma V^\nu)(\nabla_\nu U_\mu)$

$= (\nabla_\mu U^\nu)(\nabla_\nu V_\sigma) - (\nabla_\sigma V_\nu)(\nabla^\nu U_\mu)$

$= (\nabla_\mu U^\nu)(\nabla_\nu V_\sigma) - (-\nabla_\nu V_\sigma)(-\nabla_\mu U^\nu) = 0$

Also

$U^\nu \nabla_\sigma \nabla_\nu V_\mu - V^\nu \nabla_\sigma \nabla_\nu U_\mu + U^\nu \nabla_\mu \nabla_\nu V_\sigma - V^\nu \nabla_\mu \nabla_\nu U_\sigma$

$$= U^\nu(\nabla_\nu\nabla_\sigma V_\mu + R_{\mu\epsilon\sigma\nu}V^\epsilon) - V^\nu(\nabla_\nu\nabla_\sigma U_\mu + R_{\mu\epsilon\sigma\nu}U^\epsilon)$$

$$+ U^\nu(\nabla_\nu\nabla_\mu V_\sigma + R_{\sigma\epsilon\mu\nu}V^\epsilon) - V^\nu(\nabla_\nu\nabla_\mu U_\sigma + R_{\sigma\epsilon\mu\nu}U^\epsilon)$$

$$= U^\nu\nabla_\nu(\underbrace{\nabla_\sigma V_\mu + \nabla_\mu V_\sigma}_{0}) - V^\nu\nabla_\nu(\underbrace{\nabla_\sigma U_\mu + \nabla_\mu U_\sigma}_{0})$$

$$+ R_{\mu\epsilon\sigma\nu}U^\nu V^\epsilon - R_{\mu\epsilon\sigma\nu}V^\nu U^\epsilon$$

$$+ R_{\sigma\epsilon\mu\nu}U^\nu V^\epsilon - R_{\sigma\epsilon\mu\nu}V^\nu U^\epsilon$$

$$= R_{\mu\epsilon\sigma\nu}U^\nu V^\epsilon - R_{\mu\nu\sigma\epsilon}U^\nu V^\epsilon$$

$$+ R_{\sigma\epsilon\mu\nu}U^\nu V^\epsilon - R_{\sigma\nu\mu\epsilon}U^\nu V^\epsilon$$

$$= U^\nu V^\epsilon(R_{\mu\epsilon\sigma\nu} - R_{\mu\nu\sigma\epsilon} + R_{\sigma\epsilon\mu\nu} - R_{\sigma\nu\mu\epsilon})$$

$$= U^\nu V^\epsilon(\cancel{R_{\mu\epsilon\sigma\nu}} - \cancel{R_{\mu\nu\sigma\epsilon}} + \cancel{R_{\mu\nu\sigma\epsilon}} - \cancel{R_{\mu\epsilon\sigma\nu}})$$

$$= \underline{\underline{0}}$$

$R_{\mu\epsilon\sigma\nu} = R_{\sigma\nu\mu\epsilon}$;

$R_{\mu\nu\sigma\epsilon} = R_{\sigma\epsilon\mu\nu}$.

$$\Rightarrow \nabla_\sigma W_\mu + \nabla_\mu W_\sigma = 0$$

$\Rightarrow$ <u>W_μ is a killing vector</u>

$$-\ T_{\mu\nu} = \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^\sigma \nabla_\sigma \phi$$

$$\cancel{\nabla^\mu T_{\mu\nu} = \nabla^\mu \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^\mu \nabla^\sigma \nabla_\sigma \phi}$$

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^\mu \nabla^\sigma \nabla_\sigma \phi$$

$$= \nabla^\mu \nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla^\sigma \nabla_\sigma \phi$$

$$= \nabla^\mu \nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla^\mu \nabla_\mu \phi$$

$$= g^{\mu\sigma} (\nabla_\sigma \nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\sigma \nabla_\mu \phi)$$

$$\cancel{= g^{\mu\sigma} (\nabla_\sigma \nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\mu \nabla_\sigma \phi)}$$

$$= g^{\mu\sigma} (\nabla_\mu \nabla_\sigma \nabla_\nu \phi - \nabla_\nu \nabla_\sigma \nabla_\mu \phi).$$

$$\cancel{= G g^{\mu\sigma}}$$

a vector

Now $\because$ $\nabla_\sigma \nabla_\nu \phi - \nabla_\nu \nabla_\sigma \phi$

$= \nabla_\sigma \partial_\nu \phi - \cancel{\nabla_\sigma \partial_\sigma \phi}\ \nabla_\nu \partial_\sigma \phi$

$= \partial_\sigma \partial_\nu \phi - \partial_\nu \partial_\sigma \phi = 0$

$\nabla_\sigma F_\nu - \nabla_\nu F_\sigma = \frac{\partial F_\nu}{\partial x^\sigma} - \frac{\partial F_\sigma}{\partial x^\nu}$

$\Rightarrow \nabla_\sigma \nabla_\nu \phi = \nabla_\nu \nabla_\sigma \phi$

$\nabla_\sigma \nabla_\mu \phi = \nabla_\mu \nabla_\sigma \phi$

$$\therefore \quad \nabla^\mu T_{\mu\nu} = g^{\mu\sigma} (\nabla_\mu \nabla_\nu \nabla_\sigma \phi - \nabla_\nu \nabla_\mu \nabla_\sigma \phi)$$

$$= g^{\mu\sigma} R_{\sigma\epsilon\mu\nu} \nabla^\epsilon \phi = R_{\epsilon\nu} \nabla^\epsilon \phi$$

$\because$ Ricci tensor $R_{\epsilon\nu}$ is symmetric $\therefore R_{\epsilon\nu} = R_{\nu\epsilon}$

$(R_{\epsilon\nu} = g^{\sigma\mu} R_{\sigma\epsilon\mu\nu} = g^{\sigma\mu} R_{\mu\nu\sigma\epsilon} = g^{\mu\sigma} R_{\mu\nu\sigma\epsilon} = R_{\nu\epsilon}).$

$$\therefore \nabla^\mu T_{\mu\nu} = R_{\sigma\nu} \nabla^\sigma \phi = R_{\sigma\nu} \nabla^\sigma \phi$$

$$= R_{\nu\sigma} \nabla^\sigma \phi = \underline{\underline{R_\nu{}^\sigma \nabla_\sigma \phi}}$$

- If $\nabla^\mu T_{\mu\nu} = 0$, ~~then~~ And $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$

$$\rightarrow \quad \nabla^\mu K^\nu + \nabla^\nu K^\mu = 0 \quad \Rightarrow \quad \nabla^\mu K^\nu = -\nabla^\nu K^\mu$$

$$\therefore \quad \nabla^\mu (T_{\mu\nu} K^\nu) = T_{\mu\nu} \nabla^\mu K^\nu = -T_{\mu\nu} \nabla^\nu K^\mu$$

$\cancel{T_{\mu\nu} = \nabla_\mu \nabla_\nu \phi}$

$$T_{\nu\mu} = \nabla_\nu \nabla_\mu \phi - g_{\mu\nu} \nabla^\sigma \nabla_\sigma \phi = \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^\sigma \nabla_\sigma \phi$$

$= T_{\mu\nu}$ proven in previous section

$$\therefore \quad \nabla^\mu (T_{\mu\nu} K^\nu) = -T_{\nu\mu} \nabla^\nu K^\mu$$

$$= -T_{\mu\nu} \nabla^\mu K^\nu = -\nabla^\mu (T_{\mu\nu} K^\nu)$$

swap μ, ν

$$\therefore \quad \underline{\underline{\nabla^\mu (T_{\mu\nu} K^\nu) = 0}}$$

15B5Q4

Conservation of stress energy tensor gives

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

$0 < a \le a_1$: $P = -\rho c^2$ $\therefore \dot{\rho} = 0$ ρ is constant.

$a_1 < a \le a_2$: $P = \frac{1}{3}\rho c^2$ $\therefore \dot{\rho} + \frac{3\dot{a}}{a}\left(\frac{4}{3}\rho\right) = 0$

$$\dot{\rho} + \frac{4\dot{a}}{a}\rho = 0 \quad \therefore \frac{1}{a^4}\frac{d}{dt}(\rho a^4) = 0$$

$$\rightarrow \quad \rho \propto \frac{1}{a^4}$$

$a_2 < a \le 1$: $P = 0$ $\therefore \dot{\rho} + \frac{3\dot{a}}{a}\rho = 0$

$$\therefore \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0 \rightarrow \rho \propto \frac{1}{a^3}$$

Hubble rate $H(a) = \frac{\dot{a}}{a}$

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho \qquad \text{(flat universe)}$$

$$\therefore H(a) = \sqrt{\frac{8\pi G\rho}{3}}$$

$0 < a \le a_1 \rightarrow$ $\rho = \text{const} = \rho_1$ (density of universe at a_1)
(so same as initial density)

$$\therefore H = \sqrt{\frac{8\pi G\rho_1}{3}}$$

$a_1 < a \le a_2 \rightarrow$ $\rho \propto \frac{1}{a^4}$ $\therefore \rho a^4 = \rho_1 a_1^4$ $\rho = \frac{\rho_1 a_1^4}{a^4}$

$$\therefore H = \sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a^4}}$$

Scanned by Photo Scanner

$a_2 < a \leq 1$: $\rho a^3 = \rho_2 a_2^3$

density at a_2

$\rho a^3 = \rho_2 a_2^3$ $\quad \rho_2 a_2^4 = \rho_1 a_1^4$ $\quad \therefore \rho_2 = \rho_1 \left(\frac{a_1}{a_2}\right)^4$

$\therefore \rho a^3 = a_2^3 \left(\frac{a_1}{a_2}\right)^4 \rho_1 = \frac{a_1^4}{a_2} \rho_1$

$\therefore \rho = \frac{\rho_1 a_1^4}{a_2 a^3}$

$\therefore H = \sqrt{\frac{8\pi G}{3} \frac{\rho_1 a_1^4}{a_2 a^3}}$

$q = -\frac{a\ddot{a}}{\dot{a}^2}$

$= \frac{\dot{a}}{a} - \frac{a}{\dot{a}}\frac{\ddot{a}}{\dot{a}}$

$= -\frac{a}{\dot{a}}\frac{a}{\dot{a}}\frac{\ddot{a}}{a} = -\left(\frac{a}{\dot{a}}\right)^2\left(\frac{\ddot{a}}{a}\right)$

- deceleration rate $= \frac{\ddot{a}}{a}$

$\therefore \left(\frac{\ddot{a}}{a}\right) = -\frac{4}{3}\pi G\left(\rho + \frac{3P}{c^2}\right)$

$0 \leq a \leq a_1$: $P = -\rho c^2$

$\left(\frac{\ddot{a}}{a}\right) = -\frac{4}{3}\pi G\left(\frac{-2\rho c^2}{c^2}\right)$

$= \frac{8}{3}\pi G \rho \cancel{c} = \frac{8}{3}\pi G \rho_1 \cancel{c}$ (accelerating)

$q = -\left(\frac{3}{8\pi G\rho_1}\right)\left(\frac{8\pi G\rho_1}{3}\right)$

$= -1$

$a_1 < a \leq a_2$ $\quad P = \frac{1}{3}\rho c^2$

$\left(\frac{\ddot{a}}{a}\right) = -\frac{4}{3}\pi G(2\rho) = -\frac{8}{3}\pi G\rho$

$= -\frac{8}{3}\pi G \frac{\rho_1 a_1^4}{a^4}$ (decelerating)

$q = -\left(\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a^4}\right)^{-1}\left(-\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a^4}\right)$

$= 1$

Scanned by Photo Scanner

$a_2 < a \leq 1 \quad P = 0$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4}{3}\pi G\rho$$

$$= -\frac{4}{3}\pi G \frac{\rho_1 a_1^4}{a_2 a^3}$$ (decelerating)

$$q = -\left(\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a_2 a^3}\right)^{-1} \times \left(-\frac{4\pi G}{3}\frac{\rho_1 a_1^4}{a_2 a^3}\right)$$

$$= \frac{1}{2}$$

- FRW equations:

$0 < a \leq a_1$:

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G\rho_1}{3}}$$

$$\therefore \int_{a_{in}}^{a} \frac{da}{a} = \int_0^t \sqrt{\frac{8\pi G\rho_1}{3}}\, dt$$

$$\therefore a = a_{in} \exp\left(+\sqrt{\frac{8\pi G\rho_1}{3}}\, t\right)$$

$a_1 < a \leq a_2$

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a^4}}$$

$$\therefore a\dot{a} = \sqrt{\frac{8\pi G}{3}\rho_1 a_1^4} = \frac{1}{2}\frac{da^2}{dt}$$

$$\therefore a^2 - a_1^2 = 2\sqrt{\frac{8\pi G}{3}\rho_1 a_1^4}\,(t - t_1)$$

↳ time at $a = a_1$

$$\therefore a = \left(a_1^2 + 2\sqrt{\frac{8\pi G}{3}\rho_1 a_1^4}\,(t - t_1)\right)^{\frac{1}{2}}$$

$a_2 < a \leq 1$

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a_2 a^3}}$$

$\dot{a}^2 a$ ~~$= \frac{8\pi G}{}$~~

$\therefore \cancel{\frac{1}{a^2}a}\ \dot{a}a^{1/2} = \sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a_2}}$

$\therefore \frac{2}{3}\frac{d}{dt}(a^{3/2}) = \sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a_2}}$

$\therefore a^{3/2} - a_2^{3/2} = \frac{3}{2}\sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a_2}}\,(t - t_2)$

↳ time at ~~t~~ $a = a_2$

$$\therefore a = \left(a_2^{3/2} + \frac{3}{2}\sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a_2}}\,(t - t_2)\right)^{\frac{2}{3}}$$

- assume the non inflationary universe has ~~t~~ exactly the same rate of expansion as a function of scale factor a ~~a~~ for $a' > a_1$

then for $a_1 < a' \le a_2$, rate of expansion ~~$\frac{\dot{a}}{a}$~~.

$\frac{\dot{a}'}{a'} = \sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a'^4}}$

Assume ~~for~~ $a_{in} \le a \le a_1$, non inflationary universe still ~~has~~ in radiation phase.

~~there~~ then, for $\frac{\dot{a}'}{a'} = \sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a'^4}}$

$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G \rho_1}{3}}$

for $a < a_1$, $\frac{\dot{a}'}{a'} > \frac{\dot{a}}{a}$

$\Rightarrow$ inflationary universe expands slower.

— age of universe $\cancel{t = \int_0^t H(t)\,dt}$

inflationary:

$\because \frac{\dot{a}}{a} = H \quad \therefore \frac{1}{a}\frac{da}{dt} = H \Rightarrow \frac{da}{a} = H\,dt$

$\therefore dt = \frac{da}{Ha}$

$\boxed{t_0 = \int_{a_{in}}^{a=1} \frac{da}{Ha}}$

$$a_1 = a_{in} \exp\left(\sqrt{\frac{8\pi G \rho_1}{3}}\, t_1\right)$$

$$\therefore \quad t_1 = \sqrt{\frac{3}{8\pi G \rho_1}} \ln\left(\frac{a_1}{a_{in}}\right)$$

$$a_2^2 - a_1^2 = 2\sqrt{\frac{8\pi G}{3}\rho_1 a_1^4}\,(t_2 - t_1)$$

$$\therefore \quad t_2 = t_1 + \frac{a_2^2 - a_1^2}{2\sqrt{\frac{8\pi G}{3}\rho_1 a_1^4}}$$

$$= \sqrt{\frac{3}{8\pi G \rho_1}} \ln\left(\frac{a_1}{a_{in}}\right) + \frac{a_2^2 - a_1^2}{2\sqrt{\frac{8\pi G}{3}\rho_1 a_1^4}}$$

$$1 - a_2^{3/2} = \frac{3}{2}\sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a_2}}\,(t_0 - t_2)$$

$$\therefore t_0 = \frac{1 - a_2^{3/2}}{\frac{3}{2}\sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a_2}}} + \frac{a_2^2 - a_1^2}{2\sqrt{\frac{8\pi G}{3}\rho_1 a_1^4}} + \sqrt{\frac{3}{8\pi G \rho_1}} \ln\left(\frac{a_1}{a_{in}}\right)$$

non inflationary

$$t_0 = \frac{1 - a_2^{3/2}}{\frac{3}{2}\sqrt{\frac{8\pi G}{3}\frac{\rho_1 a_1^4}{a_2}}} + \frac{a_2^2 - a_{in}^2}{2\sqrt{\frac{8\pi G}{3}\rho_1 a_1^4}}$$

(replaces a_1 by a_{in} and discard inflationary phase)

As $a_{in} \to 0$, $\ln\left(\frac{a_1}{a_{in}}\right) \to \infty$

Year Courses

sophy Part B

SMOLOGY

So inflationary universe is older

particle horizon $D_H = c\int \frac{dt}{a}$ $\left(H \equiv \frac{8\pi G\rho_0}{3}\right)$
=const.

— inflation : $a = a_{in} e^{Ht}$ $t_0 = \frac{1}{H}\ln\left(\frac{1}{a_{in}}\right)$

$\frac{1}{a_{in}} = e^{Ht_0}$

$$D_H = \frac{c}{a_{in}}\int_0^{t_0} e^{-Ht}\,dt$$

$$= \frac{c}{a_{in}}\frac{1}{H}\left(e^0 - e^{-Ht_0}\right)$$

$$\simeq \frac{c}{a_{in}}\frac{1}{H} = \frac{c}{a_{in}H}\cancel{e^{Ht_0}} = \frac{c}{H}e^{Ht_0}$$

~~physical horizon~~

$\therefore \cancel{D_H(t) = \frac{c}{H}e^{Ht}}$ $\cancel{D_H(t) = \frac{c}{a_{in}H}}$

physical horizon $\cancel{d_H = a(t)D_H(t) = a_{in}e^{Ht}}$ $a_{in}e^{Ht}\frac{c}{a_{in}H}$

$$d_H = \frac{c}{H}e^{Ht}$$ $= \cancel{\frac{c}{H}e^{Ht}}$

— radiation

$\cancel{D_H}$ $a = \left(\frac{t}{t_0}\right)^{\frac{1}{2}}$

$$D_H = c t_0^{1/2}\int_0^{t_0}\frac{dt}{t^{1/2}} = \cancel{\frac{1}{2}ct_0^{1/2}}\ \cancel{2ct_0}\ 2ct_0$$

$\therefore\ d_H = 2ct$ (change t_0 to t)

matter :

$$a = \left(\frac{t}{t_0}\right)^{2/3} \quad \therefore \quad D_H = c\int_0^{t_0} \frac{dt}{a}$$

$$= c\,t_0^{2/3} \int_0^{t_0} \frac{dt}{t^{2/3}} = 3ct_0$$

$$\therefore\ d_H = 3ct$$

let a comoving scale be l.

$\therefore$ physical scale $d_{phy} = a(t)\,l$.

inflation :

$$\frac{d_{phy}}{d_H} = \frac{a_{in} e^{Ht} l}{\frac{c}{H} e^{Ht}} = \frac{H}{c}(a_{in} l) = \text{const}$$

radiation : $$\frac{d_{phy}}{d_H} = \frac{(t/t_0)^{1/2} l}{2ct} = \frac{t_0^{-\frac{1}{2}} l / 2c}{\sqrt{t}} \sim \frac{1}{\sqrt{t}}$$

matter : $$\frac{d_{phy}}{d_H} = \frac{(t/t_0)^{2/3} l}{3ct} = \frac{t_0^{-2/3} l / 3c}{t^{1/3}} \sim \frac{1}{t^{1/3}}$$

for large t, $\frac{d_{phy}}{d_H} \to 0$ for radiation/matter, but not for inflation. $\therefore$ particle horizon $\gg$ physical scale for radiation/matter universe.

At very early time $\frac{d_{phy}}{d_H}$ for rad/matt is large, but we are in the inflationary phase, so "horizon problem" is solved.