α+ Great job! Keep it up!

String Theory I

Problem Set 3

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Wk 6 Thu 11-12.30.

1 $\alpha+$ $[L_m, O(\tau)] = e^{im\tau}(-i\frac{d}{d\tau} + mh) O(\tau)$

1. $\partial_\tau X^\mu(\tau) = \dot{x}(\tau)$ Boundary operator $\Rightarrow \sigma = 0$

∴ open string:

$$X^\mu(\tau) = x^\mu + l^2 p^\mu \tau + il \sum_{n\neq 0} \frac{1}{n} \alpha^\mu_n e^{-in\tau}.$$

$$\partial_\tau X^\mu(\tau) = l^2 p^\mu + il \sum_{n\neq 0} (-in) \frac{1}{n} \alpha^\mu_n e^{-in\tau}$$

$$= l\alpha^\mu_0 + l \sum_{n\neq 0} \alpha^\mu_n e^{-in\tau}.$$

$$= l \sum_{n=-\infty}^{\infty} \alpha^\mu_n e^{-in\tau}$$

$$L_m = \frac{1}{2} \sum_{-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n$$

~~$[L_m, \partial_\tau X^\mu(\tau)] = [\frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n, l\sum_{n=-\infty}^{\infty}\alpha_n]$~~

Use $[L_m, \alpha^\mu_n] = -n\alpha^\mu_{m+n}$ (From PS2).

$$\rightarrow [L_m, \partial_\tau X^\mu(\tau)] = l\,[L_m, \sum_{n=-\infty}^{\infty} \alpha^\mu_n e^{-in\tau}]$$

$$= l \sum_{n=-\infty}^{\infty} [L_m, \alpha^\mu_n] e^{-in\tau}.$$

$$= l \sum_{n=-\infty}^{\infty} (-n\alpha^\mu_{m+n}) e^{-in\tau} = -l \sum_{n=-\infty}^{\infty} n\alpha^\mu_{m+n} e^{-in\tau}.$$

$$\rightarrow e^{im\tau}(-i\frac{d}{d\tau} + mh)\partial_\tau X^\mu(\tau)$$

$$= l e^{im\tau} \sum_{n=-\infty}^{\infty} ((-i)(-in) + mh) \alpha^\mu_n e^{-in\tau}.$$

$$= -l \sum_{n=-\infty}^{\infty} (n - mh) \alpha^\mu_n e^{-i(n-m)\tau}.$$

$$= -l \sum_{n=-\infty}^{\infty} (n-m)\, \alpha_n^\mu e^{-i(n-m)\tau} - l \sum_{n=-\infty}^{\infty} m(1-h) \alpha_n^\mu e^{-i(n-m)\tau}.$$

$$= -l \sum_{(n-m)=-\infty}^{\infty} (n-m)\, \alpha_{(n-m)+m}^\mu e^{-i(n-m)\tau} - l \sum_{n=-\infty}^{\infty} m(1-h) \alpha_n^\mu e^{-i(n-m)\tau}.$$

$n-m \to n$ for first term

$$= \underbrace{-l \sum_{n=-\infty}^{\infty} n\, \alpha_{n+m}^\mu e^{-in\tau}}_{[L_m,\, \partial_\tau X^\mu(\tau)]} - l \sum_{n=-\infty}^{\infty} m(1-h) \alpha_n^\mu e^{-i(n-m)\tau}.$$

$$= [L_m, \partial_\tau X^\mu(\tau)] - l \sum_{n=-\infty}^{\infty} m(1-h)\, \alpha_n^\mu e^{-i(m-n)\tau}.$$

$$\stackrel{!}{=} [L_m, \partial_\tau X^\mu(\tau)]$$

$\Rightarrow$ second term $= 0$ $\Rightarrow$ $\boxed{h=1}$ Great!

2. Second derivative $\partial_\tau^2 X^\mu(\tau) = \partial_\tau(\partial_\tau X^\mu(\tau))$

$$= \partial_\tau \Big(l \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\tau} \Big) = l \sum_{n=-\infty}^{\infty} \alpha_n^\mu (-in) e^{-in\tau}.$$

$$= -il \sum_{n=-\infty}^{\infty} n \alpha_n^\mu e^{-in\tau}.$$

$$\rightarrow [L_m, \partial_\tau^2 X^\mu(\tau)] = -il \sum_{n=-\infty}^{\infty} [L_m, n \alpha_n^\mu e^{-in\tau}]$$

$$= -il \sum_{n=-\infty}^{\infty} n \underbrace{[L_m, \alpha_n^\mu]}_{-n\alpha_{m+n}^\mu} e^{-in\tau}.$$

$$= il \sum_{n=-\infty}^{\infty} n^2 \alpha_{m+n}^\mu e^{-in\tau}.$$

$$e^{im\tau}\left(-i\frac{d}{d\tau}+mh\right)\partial_\tau^2 X^\mu(\tau)$$

$$= e^{im\tau}\left(-i\frac{d}{d\tau}+mh\right)\left(-il\sum_{n=-\infty}^{\infty} n\alpha_n^\mu e^{-in\tau}\right)$$

$$= e^{im\tau}(-il)\sum_{n=-\infty}^{\infty}\left(n(-i)(-in)+mh\right)e^{-in\tau}\cdot\alpha_n^\mu$$

$$= e^{im\tau}(-il)\sum_{n=-\infty}^{\infty}(n^2+mh)e^{-in\tau}\cdot\alpha_n^\mu$$

$$= +il\sum_{n=-\infty}^{\infty}(n^2-mh)e^{-i(n-m)\tau}\cdot\alpha_n^\mu$$

$n \to n+m$

$$= +il\sum_{n=-\infty}^{\infty}\left((n+m)^2-mh\right)\alpha_{n+m}^\mu e^{-in\tau}$$

$$= +il\sum_{n=-\infty}^{\infty}(n^2+2mn+m^2-mh)\alpha_{n+m}^\mu e^{-in\tau}$$

$$= \left(+il\sum_{n=-\infty}^{\infty}n^2\alpha_{m+n}^\mu e^{-in\tau}\right)+il\sum_{n=-\infty}^{\infty}(m^2+2mn-mh)\alpha_{m+n}^\mu e^{-in\tau}$$

$$= [L_m, \partial_\tau^2 X^\mu(\tau)] + il\sum_{n=-\infty}^{\infty}(m^2+2mn-mh)\alpha_{n+m}^\mu e^{-in\tau}$$

$$\overset{!}{=} [L_m, \partial_\tau^2 X^\mu(\tau)]$$

$\Rightarrow\ m^2+2mn-mh=0 \ \Rightarrow\ h = m+2n \neq \text{constant}$

$\Rightarrow\ \partial_\tau^2 X^\mu(\tau)$ is not a primary field of any dimension. $\square$

Fantastic!

3: $\because [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n}$

$\therefore$ ~~$[\alpha_p^\mu, \alpha_n^\nu]$~~

$\rightarrow$ $[\alpha_p^\mu, k\cdot\alpha_n] = [\alpha_p^\mu, k_\nu \alpha_n^\nu] = k_\nu [\alpha_p^\mu, \alpha_n^\nu]$

$= p\,\delta_{p+n}\,\eta^{\mu\nu} k_\nu = p k^\mu \delta_{p+n}$.

~~$[\alpha_p^\mu, (k\cdot\alpha_n)^r]$~~

$\rightarrow$ $[\,(k\cdot\alpha_n),\ [\alpha_p^\mu, k\cdot\alpha_n]\,]$

$= [k\cdot\alpha_n,\ \underbrace{pk^\mu\delta_{p+n}}_{c\text{-number}}] = 0.$

$\rightarrow$ $[\alpha_p^\mu, (k\cdot\alpha_n)^r] =$

$(k\cdot\alpha_n)[\alpha_p^\mu, (k\cdot\alpha_n)^{r-1}] + [\alpha_p^\mu, (k\cdot\alpha_n)](k\cdot\alpha_n)^{r-1}.$

$= (k\cdot\alpha_n)^2[\alpha_p^\mu, (k\cdot\alpha_n)^{r-2}] + (k\cdot\alpha_n)[\alpha_p^\mu, (k\cdot\alpha_n)](k\cdot\alpha_n)^{r-2}$
$+ [\alpha_p^\mu, (k\cdot\alpha_n)](k\cdot\alpha_n)^{r-1}$

can be exchanged.

$= \cdots$ (use above relation)

$= r[\alpha_p^\mu, (k\cdot\alpha_n)](k\cdot\alpha_n)^{r-1}$

$= p\,\delta_{p+n} k^\mu\, r\, (k\cdot\alpha_n)^{r-1}$

$\rightarrow$ $[\alpha_p^\mu, e^{k\cdot\alpha_n}] = \sum_{r=0}^{\infty} [\alpha_p^\mu, \frac{1}{r!}(k\cdot\alpha_n)^r]$

$= \sum_{r=0}^{\infty} \frac{1}{r!}[\alpha_p^\mu, (k\cdot\alpha_n)^r]$.

$$= \sum_{r=0}^{\infty} \frac{1}{r!} p\delta_{p+n} k^\mu r (k\cdot\alpha_n)^{r-1}$$

$$= \sum_{r=1}^{\infty} \frac{1}{r!} p\delta_{p+n} k^\mu r (k\cdot\alpha_n)^{r-1} = \sum_{r=1}^{\infty} p\delta_{p+n} k^\mu \frac{1}{(r-1)!} (k\cdot\alpha_n)^{r-1}$$

$s = r-1 \Longrightarrow$ $$= p\delta_{p+n} k^\mu \underbrace{\sum_{s=0}^{\infty} \frac{1}{s!} (k\cdot\alpha_n)^s}_{e^{k\cdot\alpha_n}}$$

$$= p\delta_{p+n} k^\mu e^{k\cdot\alpha_n}$$

Similarly $[\alpha_p^\mu, e^{k\cdot\alpha_{-n}}] = p\delta_{p-n} k^\mu e^{k\cdot\alpha_{-n}}$

$\because$ $L_m = \frac{1}{2}\sum_q \alpha_{m-q}\cdot\alpha_q$

$$\therefore [L_m, e^{k\cdot\alpha_{-n}}] = \frac{1}{2}\sum_q (\alpha_{m-q}\cdot[\alpha_q, e^{k\cdot\alpha_{-n}}] + [\alpha_{m-q}, e^{k\cdot\alpha_{-n}}]\cdot\alpha_q)$$

$$= \frac{1}{2}\sum_q (\alpha_{m-q}\cdot(q\delta_{q-n} k^\mu) e^{k\cdot\alpha_{-n}} + (m-q)\delta_{m-q-n} k^\mu e^{k\cdot\alpha_{-n}}\cdot\alpha_q)$$

$$= \frac{1}{2}(n\alpha_{m-n} k^\mu e^{k\cdot\alpha_n}) + n k^\mu e^{k\cdot\alpha_{-n}}\cdot\alpha_{m-n})$$

$$= \frac{1}{2}n(\alpha_{m-n}\cdot k e^{k\cdot\alpha_n} + k e^{k\cdot\alpha_{-n}}\cdot\alpha_{m-n})$$

$$V(k,\tau) = :\exp\left(k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}\right)e^{ik\cdot(x+p\tau)}$$
$$\cdot\exp\left(-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}\right):$$

already normal ordered $\Longrightarrow$ $= \exp\left(k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}\right)e^{ik(x+p\tau)}\exp\left(-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}\right)$

$\because$ α_m and α_n do not commute only if $m=-n$

$\therefore$ all terms in $\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}$ commute with each other $\because$ all $-n$ are negative.

$$\therefore \exp\left(k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}\right) = \text{~~}\exp\left(k\cdot\frac{\alpha_{-1}}{1}e^{i\tau}\right)\text{~~}$$
$$= \prod_{n=1}^{\infty}\exp\left(k\cdot\frac{\alpha_{-n}}{n}e^{in\tau}\right) = \prod_{n=1}^{\infty}\exp\left(\frac{ke^{in\tau}}{n}\cdot\alpha_{-n}\right).$$

$$\left[L_m, \exp\left(k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}\right)\right]$$
$$= \left[L_m, \prod_{n=1}^{\infty}\exp\left(\frac{ke^{in\tau}}{n}\cdot\alpha_{-n}\right)\right]$$
$$= \left[L_m, \exp\left(ke^{i\tau}\cdot\alpha_{-1}\right)\right]\prod_{n=2}^{\infty}\exp\left(\frac{ke^{in\tau}}{n}\cdot a_{-n}\right)$$
$$+ \cdots\cdots$$

$$= \frac{1}{2}(1)\left\{\alpha_{m-1}\cdot ke^{i\tau}\exp(ke^{i\tau}\cdot\alpha_{-1}) + ke^{i\tau}\exp(ke^{i\tau}\cdot\alpha_{-1})\alpha_{m-1}\right\}$$
$$\prod_{n=2}^{\infty}\exp\left(\frac{ke^{in\tau}}{n}\cdot\alpha_{-n}\right) + \cdots$$

$$\exp\left(\frac{ke^{i\tau}}{1}\cdot\alpha_{-1}\right)\cdot\frac{1}{2}(2)\left(\alpha_{m-2}\cdot\frac{ke^{i2\tau}}{2}\exp\left(\frac{ke^{i2\tau}}{2}\cdot\alpha_{-2}\right)\right.$$
$$\left.+\frac{ke^{2i\tau}}{2}\exp\left(\frac{ke^{i2\tau}}{2}\cdot\alpha_{-2}\right)\cdot\alpha_{m-2}\right)\prod_{n\geq 3}^{\infty}\exp\left(\frac{ke^{in\tau}}{n}\cdot\alpha_{n}\right)$$
$$+\cdots$$

$$e^{im\tau}\left(-i\frac{dV}{d\tau}\right) = e^{im\tau} :-i\frac{d}{d\tau}\left(e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}} e^{ik\cdot(x+p\tau)} e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}\right):$$

$$= e^{im\tau}(-i):\left(\frac{d}{d\tau}e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}}\right)\left(e^{ik\cdot(x+p\tau)}\right)\left(e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}\right):$$

$$+e^{im\tau}(-i):\left(e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}}\right)\left(\frac{d}{d\tau}e^{ik\cdot(x+p\tau)}\right)\left(e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}\right):$$

$$+e^{im\tau}(-i):\left(e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}}\right)\left(e^{ik\cdot(x+p\tau)}\right)\left(\frac{d}{d\tau}e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}\right):$$

$$= :k\cdot\sum_{n=1}^{\infty}\alpha_{-n}e^{i(m+n)\tau}\, e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}} e^{ik\cdot(x+p\tau)} e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}:$$

$$+: e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}}\left(\frac{1}{2}(k\cdot p)e^{im\tau}e^{ik\cdot(x+p\tau)} + e^{ik\cdot(x+p\tau)}\frac{1}{2}(k\cdot p)e^{im\tau}\right)$$
$$e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}:$$

$$+e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}} e^{ik\cdot(x+p\tau)}\left(+k\cdot\sum_{n=1}^{\infty}\alpha_n e^{i(m-n)}\right)e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{-in\tau}}:$$

$$= :[L_m, V(k,\tau)]:$$

~~To find the d.~~ Note we put $\frac{d}{d\tau}e^{ik\cdot(x+p\tau)}$ in to the form above because $[x,p]\neq 0$.

To find the difference between $[L_m, V]$ and $:[L_m, V]:$, we observe the expression for $[L_m, V]$ we derived ~~earlier~~ earlier.

~~Hint~~ $[L_m, V] = [L_m, e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{in\tau}}]\, e^{ik\cdot(x+p\tau)}\, e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}$

$$+\, e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{in\tau}}\,[L_m, e^{ik\cdot(x+p\tau)}]\, e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}$$

~~$+ e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{in\tau}} e^{ik\cdot(x+p\tau)} [L_m, e^{-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}]$~~

$$+\, e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{in\tau}}\, e^{ik\cdot(x+p\tau)}\,[L_m, e^{-k\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}]$$

the ~~seco~~ second and third terms are already normal ordered

the first term has.

$$[L_m, e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{in\tau}}]$$

$$= \frac{1}{2}(\alpha_{m-1}\cdot k e^{i\tau} e^{ke^{i\tau}\cdot\alpha_{-1}} + k e^{i\tau} e^{ke^{i\tau}\cdot\alpha_{-1}}\cdot\alpha_{m-1}) \prod_{n=2}^{\infty} e^{k\frac{e^{in\tau}}{n}\cdot\alpha_n}$$

$$+\, e^{ke^{i\tau}\cdot\alpha_{-1}}\cdot\frac{1}{2}(\alpha_{m-2}\cdot ke^{2i\tau} e^{\frac{ke^{2i\tau}}{2}\cdot\alpha_{-2}}$$

$$+\, ke^{2i\tau} e^{\frac{ke^{2i\tau}}{2}\cdot\alpha_{-2}}\cdot\alpha_{m-2}) \prod_{n=3}^{\infty} e^{\frac{ke^{in\tau}}{n}\cdot\alpha_{-n}}$$

$$+ \cdots$$

$$+ \prod_{n=1}^{p-1} e^{\frac{ke^{ni\tau}}{n}\cdot\alpha_{-n}}\cdot\frac{1}{2}(\alpha_{m-p}\cdot ke^{pi\tau} e^{\frac{ke^{ip\tau}}{p}\cdot\alpha_{-p}}$$

$$+\, ke^{ip\tau} e^{\frac{ke^{ip\tau}}{p}\alpha_{-p}}\cdot\alpha_{m-p}) \prod_{n=p+1}^{\infty} e^{\frac{ke^{in\tau}}{n}\cdot\alpha_{-n}}$$

$+ \cdots\cdots$

· When $m-p \geq 0$, we have creation operators to the ~~re~~ left of annihilation operators

∴ ~~need~~ different from normal ordering.

the move annihilation operators to the right induces commutators.

~~mo~~ α_{m-p} doesn't commute with α_{p-m}

∴ If α_{p-m} is to the right of α_{m-p}, we need to induce commutators.

~~$k\cdot[\alpha_{m-p}, e^{\frac{ke^{i(m-p)}}{m-p}\cdot\alpha_{p-m}}]$~~

~~$= k(m-p)$~~

~~$[k\cdot\alpha_{m-p}\, e^{ip\tau} e^{\frac{ke^{ip\tau}}{p}\cdot\alpha_{-p}}, e^{\frac{ke^{i(m-p)}}{m-p}}]$~~

$$[ke^{ip\tau} e^{\frac{ke^{ip\tau}}{p}\cdot\alpha_{-p}}\cdot\alpha_{m-p}, e^{\frac{ke^{i(m-p)}}{m-p}\cdot\alpha_{p-m}}]$$

$$= ke^{ip\tau} e^{\frac{ke^{ip\tau}}{p}\cdot\alpha_{-p}}\cdot[\alpha_{m-p}, e^{\frac{ke^{i(m-p)\tau}}{m-p}\cdot\alpha_{p-m}}]$$

$$= k^2 e^{k\frac{ip\tau}{p}\cdot\alpha_{-p}} \underbrace{e^{ip\tau}(m-p)\frac{e^{i(m-p)\tau}}{(m-p)}} e^{ke^{\left(\frac{i(m-p)\tau}{m-p}\right)}\cdot\alpha_{pm}}$$

$$= \underbrace{e^{im\tau}}\, k^2 e^{k\frac{ip\tau}{p}\cdot\alpha_{-p}} e^{ke^{\frac{i(m-p)\tau}{m-p}}\cdot\alpha_{p-m}}$$

↓

this is the important part

each swap of α_{m-p} and α_{p-m} gives a factor of $e^{im\tau} k^2 V(k,\tau)$

→ this is also true for α_0 and $e^{ik \cdot x}$.

Now we count the number of swaps

If m is odd,

we need to swap $\alpha_{m-1}, \alpha_{m-2}, \dots \alpha_{m-(\frac{m-1}{2})}$ with their respective α_{p-m} on their right

~~∴ we have ...~~

we also need to swap ~~be~~ $\frac{1}{2}$ of α_0 term to the right of e^{ikx} this gives a total ~~contri~~ contribution of

$$\left(\frac{m-1}{2}+\frac{1}{2}\right) k^2 e^{im\tau} V = \frac{1}{2} m k^2 e^{im\tau} V.$$

If m is even, we need to swap $\alpha_{m-1}, \alpha_{m-2}, \dots, \alpha_{m-\frac{m-2}{2}}$, and $\frac{1}{2}\alpha_{\frac{m}{2}}$ term (the other $\frac{1}{2}$ is normal ordered), and $\frac{1}{2}\alpha_0$ term.

∴ this gives a total of

$$\frac{m-2}{2}+\frac{1}{2}+\frac{1}{2} = \frac{1}{2}m$$

∴ In either case, we have.

$$[L_m, V] = \underbrace{:[L_m, V]:}_{= e^{im\tau}\left(-i\frac{d}{d\tau}V\right)} + e^{im\tau}\frac{1}{2}m k\cdot k\, V$$

$$\therefore \quad [L_m, V] = e^{im\tau}\left(-i\frac{d}{d\tau} + m\left(\frac{1}{2}k\cdot k\right)\right)V.$$

$$\Rightarrow \quad h = \frac{1}{2}k\cdot k$$

Beautiful!

$\alpha +$

2 1. Consider state $\alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;p\rangle \overset{?}{=} |\phi;p\rangle$

$\because L_1 = \frac{1}{2}\sum_n \alpha_{1-n}\cdot\alpha_n$, physical condition $L_1|\phi;p\rangle \overset{?}{=} 0$.

$L_1\alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;p\rangle = \frac{1}{2}\sum_n(\alpha_{1-n}\cdot\alpha_n)\,\alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;p\rangle$

$= \frac{1}{2}(\alpha_1\cdot\alpha_0 + \alpha_0\cdot\alpha_1)\,\alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;p\rangle$

$= \frac{1}{2}(\underbrace{\alpha_1^{\mu}\alpha_{0\mu}}_{\text{commute}}\alpha_{-1}^{\nu}\tilde{\alpha}_{-1\nu} + \alpha_0^{\mu}\underbrace{\alpha_{1\mu}}_{\text{commute}}\alpha_{-1}^{\nu}\tilde{\alpha}_{-1\nu})|0;p\rangle$

$= \alpha_0^{\mu}\tilde{\alpha}_{-1}^{\nu}\alpha_{1\mu}\alpha_{-1\nu}|0;p\rangle$

$= \alpha_0^{\mu}\tilde{\alpha}_{-1}^{\nu}\underbrace{[\alpha_{1\mu},\alpha_{-1\nu}]}_{\eta^{\mu\nu}}|0;p\rangle + \alpha_0^{\mu}\tilde{\alpha}_{-1}^{\nu}\alpha_{-1\nu}\underbrace{\alpha_{1\mu}|0;p\rangle}_{=0}$

$= \alpha_0\cdot\tilde{\alpha}_{-1}|0;p\rangle = \tilde{\alpha}_{-1}\cdot\alpha_0|0;p\rangle$

$= \tilde{\alpha}_{-1}\cdot(\frac{\ell}{2}p)|0;p\rangle = \frac{\ell}{2}p\cdot\tilde{\alpha}_{-1}|0;p\rangle \neq 0$ ✓

$\therefore$ Does ~~n't~~ not satisfy.

Suppose $|\phi;p\rangle = \zeta_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}$ ~~$+\phi\,p$~~ $|0;p\rangle$.

$L_1|\phi;p\rangle = \frac{1}{2}\sum_n(\alpha_{1-n}\cdot\alpha_n)\,\zeta_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0;p\rangle$

$=$ ~~$\alpha_1\cdot\alpha_0$~~ $\alpha_1^{\rho}\alpha_{0\rho}\zeta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|0;p\rangle$

$= \zeta_{\mu\nu}\alpha_{0\rho}\alpha_1^{\rho}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|0;p\rangle$

$= \zeta_{\mu\nu}\alpha_{0\rho}\alpha_{-1}^{\mu}\underbrace{[\alpha_1^{\rho},\alpha_{-1}^{\nu}]}_{\eta^{\rho\nu}}|0;p\rangle + \zeta_{\mu\nu}\alpha_{0\rho}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}\underbrace{\alpha_1^{\rho}|0;p\rangle}_{=0}$

$$= \xi_{\mu\nu}\, \eta^{\rho\nu}\, \alpha_{0\rho}\, \alpha_{-1}^{\mu}\, |0;p\rangle$$

$$= \xi_{\mu\nu}\, \alpha_{-1}^{\mu}\, \alpha_0^{\nu}\, |0;p\rangle$$

$$= \alpha_{-1}^{\mu} \left(\xi_{\mu\nu}\, \tfrac{1}{2} \ell p^{\nu} \right) |0;p\rangle .$$

$$= \tfrac{1}{2} \ell\, \alpha_{-1}^{\mu}\, \xi_{\mu\nu} p^{\nu}\, |0;p\rangle \overset{!}{=} 0 \implies \xi_{\mu\nu} p^{\nu} = 0$$

of course $\xi_{\mu\nu} = \eta_{\mu\nu}$ doesn't satisfy this.

we choose $\xi_{\mu\nu}$ to be $\underline{\xi_{\mu\nu} = \eta_{\mu\nu} - p_{\mu}\bar{p}_{\nu} - \bar{p}_{\mu} p_{\nu}}$

such that $\bar{p}\cdot\bar{p} = 0$ and $p\cdot\bar{p} = 1$ Good!

then $\xi_{\mu\nu} p^{\nu} = \eta_{\mu\nu} p^{\nu} - p_{\mu} \underbrace{\bar{p}_{\nu} p^{\nu}}_{1} - \bar{p}_{\mu} p_{\nu} p^{\nu}$

$$= p_{\mu} - p_{\mu} - \bar{p}_{\mu} (p\cdot p)$$

$$= -\bar{p}_{\mu} (p\cdot p)$$

For a dilaton state (~~mat~~ massless)

$p\cdot p = 0 \implies \xi_{\mu\nu} p^{\nu} = 0$ satisfies the physical state condition.

state is

$$\underline{|\phi;p\rangle = (\eta_{\mu\nu} - p_{\mu}\bar{p}_{\nu} - \bar{p}_{\mu} p_{\nu})\, \alpha_{-1}^{\mu}\, \tilde{\alpha}_{-1}^{\nu}\, |0;p\rangle} .$$ Great!

$$\eta^{\mu}{}_{\mu} - p^{\mu}\bar{p}_{\mu} - \bar{p}^{\mu}p_{\mu} = D-2 = 24.$$

$\Rightarrow$ the trace part of $SO(24)$ representation

Hence dilaton is a scalar ✓

2. the dilaton vertex operator

$$V_\phi(\tau,\sigma) = \xi_{\mu\nu}\, \partial_\alpha X^\mu \partial^\alpha X^\nu :e^{ik\cdot X}:$$

where $\xi_{\mu\nu} = \eta_{\mu\nu} - p_\mu \bar{p}_\nu - \bar{p}_\mu p_\nu$, $p\cdot p = 0.$ (massless)

$$\partial_\alpha X^\mu \partial^\alpha X^\nu = \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu$$

$$= -\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu$$

define $\sigma^+ =: \tau + \sigma$, $\sigma^- =: \tau - \sigma$

$$\partial_+ = \frac{1}{2}(\partial_\tau + \partial_\sigma), \quad \partial_- = \frac{1}{2}(\partial_\tau - \partial_\sigma)$$

$$\therefore \partial_+ X^\mu \partial_- X^\nu = \frac{1}{4}(\partial_\tau + \partial_\sigma)X^\mu (\partial_\tau - \partial_\sigma)X^\nu$$

$$= \underbrace{\frac{1}{4}(\partial_\tau X^\mu \partial_\tau X^\nu - \partial_\sigma X^\mu \partial_\sigma X^\nu)}_{-\partial_\alpha X^\mu \partial^\alpha X^\nu} - \frac{1}{4}(\partial_\tau X^\mu \partial_\sigma X^\nu - \partial_\tau X^\nu \partial_\sigma X^\mu)$$

the latter part is antisymmetric w.r.t μ and ν so combine with $\xi_{\mu\nu}$, a symmetric matrix, gives zero

$$\therefore\ \partial_\alpha X^\mu \partial^\alpha X^\nu = -4\,\partial_+ X^\mu \partial_- X^\nu$$

$$\therefore\ V_\phi(\tau,\sigma) = -4\,\eta_{\mu\nu}\,\partial_+ X^\mu \partial_- X^\nu : e^{ik\cdot X}:$$

$$= -4\,\eta_{\mu\nu}\,\partial_+ X^\mu_L\, \partial_- X^\nu_R : e^{ip\cdot(X_R + X_L)}:$$

$h_{left} = 1$ $h_{Right} = 1$ $h = \frac{P\cdot P}{2} = 0$

$\Rightarrow$ conformal dimension $h = (1, 1)$ (left, right).

---

consider $:e^{iP\cdot(X_R + X_L)}:\,|0;0\rangle$.

$$= e^{\left(\frac{1}{2}P\cdot\sum_{n\geq1}\frac{\alpha_{-n}}{n}e^{2in(\tau-\sigma)}\right)}\, e^{\left(\frac{1}{2}P\cdot\sum_{n\geq1}\frac{\tilde{\alpha}_{-n}}{n}e^{2in(\tau+\sigma)}\right)}$$

$$e^{iP\cdot(X+(\alpha_0+\tilde{\alpha}_0)\tau)}$$

$$e^{\left(-\frac{1}{2}P\cdot\sum_{n\geq1}\frac{\alpha_n}{n}e^{-in\tau}\right)}\, e^{\left(-\frac{1}{2}P\cdot\sum_{n\geq1}\frac{\tilde{\alpha}_n}{n}e^{-in\tau}\right)}|0;0\rangle$$

All α_n $n\geq0$ terms leaves $|0;0\rangle$ state unchanged.

use Campbell-Baker-Hausdorff formula on $e^{iP\cdot(X+(\alpha_0+\tilde{\alpha}_0)\tau)}$ gives $e^{iPX}\, e^{(\alpha_0+\tilde{\alpha}_0)\tau}\, e^{-i\frac{P\cdot P}{2}\tau}$

But $p \cdot p = 0$, and $e^{\alpha_0} |0;0\rangle = |0;0\rangle$

$\therefore \; :e^{ip\cdot X}: |0;0\rangle$

$$= e^{\frac{1}{2} p \cdot \sum_{n\geq 1} \frac{\alpha_{-n}}{n} e^{-2in\sigma} z^{2n}} \; e^{\frac{1}{2} p \cdot \sum_{n \geq 1} \frac{\tilde{\alpha}_{-n}}{n} e^{2in\sigma} z^{2n}} .$$

$$\underbrace{e^{ip\cdot x} |0;0\rangle}_{|0;p\rangle}$$

for $z = e^{i\tau}$.

$$\lim_{\tau \to i\infty} = \lim_{z \to e^{i^2\infty} = e^{-\infty}} = \lim_{z \to 0}.$$

$$\lim_{z\to 0} :e^{ip\cdot X}: |0;0\rangle = e^{ip\cdot x} |0;0\rangle = |0;p\rangle$$

$$\therefore \lim_{\tau \to i\infty} e^{4i\tau} V_\phi(\tau, \sigma) |0;0\rangle =$$

~~$= \lim_{z\to 0} z^{-4} (-4) \partial \ldots \zeta_{\mu\nu}$~~

$$= \lim_{z\to 0} \zeta_{\mu\nu} z^{-4} \partial_+ X_L^\mu \partial_- X_R^\nu |0;p\rangle$$

(constant -4 can be absorbed into $\zeta_{\mu\nu}$)

From previous problem sets:

$$\partial_+ X_L^{\nu} = \sum_n \tilde{\alpha}_n^{\mu} e^{-2in(\tau+\sigma)} = \sum_n \tilde{\alpha}_n^{\mu} e^{-2in\sigma} z^{-2n}$$

$$\partial_- X_R^\mu = \sum_n \alpha_n^\mu e^{-2in(\tau-\sigma)} = \sum_m \alpha_m^\mu e^{2im\sigma} z^{-2m}$$

$$\therefore \lim_{\tau \to i\infty} e^{-4i\tau} V_\phi(\tau,\sigma) |0;0\rangle$$

$$= \lim_{z \to 0} \zeta_{\mu\nu} z^{-4} \sum_{m,n} e^{2i(m-n)\sigma} z^{-2(m+n)} \alpha_m^\mu \tilde{\alpha}_n^\nu |0;p\rangle$$

$$= \lim_{z \to 0} \zeta_{\mu\nu} \sum_{m,n} e^{2i(m-n)\sigma} z^{-2(m+n+2)} \alpha_m^\mu \tilde{\alpha}_n^\nu |0;p\rangle$$

observe this expression.

As $z \to 0$ if $m+n+2 < 0$, then contribution is 0 $\because z^{\text{positive}} \to 0$. $\Rightarrow$ $m+n+2 \geq 0$

If any of m or n > 0, then contribution is also 0 $\because \alpha_m |0;p\rangle = 0$ if $m > 0$, $\tilde{\alpha}_n |0;p\rangle = 0$ if $n > 0$

$\Rightarrow$ $m \leq 0,\ n \leq 0$

4 possibilities

① $m=0,\ n=0$ but $\because p \cdot p = 0$ $\therefore$ this have no contribution

②, ③ $m=0,\ n=-1$ or $m=-1,\ n=0$ but $\because \zeta_{\mu\nu} p^\nu = 0$ and $p^\mu \zeta_{\mu\nu} = 0$ $\therefore$ this have no contribution

④ only remaining contribution is

$m=-1 \quad n=-1$

and $e^{2i(m-n)\sigma}\Big|_{m=-1,\, n=-1} = 0$

$\therefore$ required state

$= \lim_{z\to 0} \zeta_{\mu\nu} z^0 \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0;p\rangle$

$= \zeta_{\mu\nu} \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0;p\rangle = |\phi;p\rangle$

Fantastic!

3

1. tachyon state — photon vertex operator — ζ_2 = polarisation vector — tachyon state

$$A_{2+1} = g_o \langle 0;k_1 | V(\zeta_2, k_2) | 0;k_3 \rangle$$

$$= g_o \langle 0;k_1 | \zeta_2 \cdot \dot{X}(0) V(k_2, 0) | 0;k_3 \rangle .$$

$= V(\zeta_2, k_2)$ at $\tau = 0$ $\quad (\dot{X} = \partial_\tau X)$

$\tau = 0$ ∵ divided by volume of gauge group

∵ $\zeta_2 \cdot k_2 = 0$
∴ $[\zeta_2 \cdot \alpha_n, e^{k_2 \cdot \alpha_{-n}}] = 0$
$\tau = 0 \Rightarrow e^{k_i p \tau} = 1$
∴ only $e^{ik \cdot x}$ term remains

$$= g_o \langle 0;k_1 | \zeta_2 \cdot \dot{X}(0) e^{ik_2 \cdot x} | 0;k_3 \rangle$$

$$= g_o \langle 0;k_1 | \zeta_2 \cdot \dot{X}(0) | 0;k_2 + k_3 \rangle$$

in $\dot{X}(0)$, all α_n ~~(n≥1)~~ $(n>0)$ terms will ~~and a ah annihil annihilate~~ annihilate $|0;k_2+k_3\rangle$ and all α_n $(n<0)$ terms will annihilate $\langle 0;k_1|$ term. only term remains is p

$$\therefore A_{2+1} = g_o \langle 0;k_1 | \zeta_2 \cdot p | 0;k_2+k_3 \rangle$$

$$= g_o (\zeta_2 \cdot (k_2 + k_3)) \langle 0;k_1 | 0;k_2+k_3 \rangle$$

$\langle 0;k_1 | 0;k_2+k_3 \rangle = \delta(k_1+k_2+k_3)$

$\zeta_2 \cdot k_2 = 0$ ∵ state 2 is a photon.

$$\Rightarrow A_{2+1} = g_o \zeta_2 \cdot k_3 \, \delta(k_1+k_2+k_3)$$

2

2.

$$A_{1+2} = g_o \langle 0; k_1 | \zeta_1 \cdot \alpha_1 \, V(k_2, 0) \, \zeta_3 \cdot \alpha_{-1} | 0; k_3 \rangle$$

$\zeta_1 \cdot \alpha_1$: photon state

$\zeta_3 \cdot \alpha_{-1}$: photon state

$V(k_2, 0)$: tachyon vertex operator at $\tau = 0$

$\tau = 0$ ∵ divided by volume of gauge group.

$$= g_o \langle 0; k_1 | \zeta_1 \cdot \alpha_1 \, e^{k_2 \cdot \alpha_{-1}} e^{ik_2 \cdot x} e^{-k_2 \cdot \alpha_1} \, \zeta_3 \cdot \alpha_{-1} | 0; k_3 \rangle$$

all other terms annihilate either $|0; k_3\rangle$ or $\langle 0; k_1|$

$\Rightarrow$ use $\zeta \cdot [\alpha_p, e^{k \cdot \alpha_{-n}}] = k \cdot \zeta \, P \delta_{p-n} \, e^{k \cdot \alpha_{-n}}$.

$$= g_o \langle 0; k_1 | e^{k \cdot \alpha_{-1}} (\zeta_1 \cdot k_2 + \zeta_1 \cdot \alpha_1) \, e^{k_2 \cdot x}$$

$$\times (-\zeta_3 \cdot k_2 + \zeta_3 \cdot \alpha_{-1}) \, e^{k \cdot \alpha_1} | 0; k_3 \rangle$$

$e^{k \cdot \alpha_{-1}}$: acting as identity.

$e^{k \cdot \alpha_1}$: acting as identity

kills $|\,\rangle$; kills $\langle\,|$

$$= g_o \langle 0; k_1 | \Big[(\zeta_1 \cdot \alpha_1)(-\zeta_3 \cdot k_2) + (\zeta_1 \cdot k_2)(\zeta_3 \cdot \alpha_{-1})$$

$$+ (\zeta_1 \cdot \alpha_1)(\zeta_3 \cdot \alpha_{-1}) - (\zeta_1 \cdot k_2)(\zeta_3 \cdot k_2) \Big] e^{ik_2 \cdot x}$$

$$| 0; k_3 \rangle$$

$$= g_0 \langle 0;k_1 | \xi_1\cdot\xi_3 \overbrace{[\alpha_1,\alpha_{-1}]}^{1} + \cancel{\xi_1\cdot\alpha_1\,\xi_3\cdot\alpha_1}$$

$$- (\xi_1\cdot k_2)(\xi_3\cdot k_2)\, | 0; k_2+k_3\rangle$$

$$= g_0\big(\xi_1\cdot\xi_3 - (\xi_1\cdot k_2)(\xi_3\cdot k_2)\big)\,\delta(k_1+k_2+k_3) \quad \square$$

✓ Clear!

3. Gauge invariance of photon.

1. replace ξ_2 by k_2.

$$A_{2+1} = g_0\, k_2\cdot k_3.$$

$$k_1+k_2+k_3 = 0$$

$$\Rightarrow k_1^2 = (-k_1)^2 = (k_2+k_3)^2$$

$$= k_2^2 + k_3^2 + 2k_2\cdot k_3$$

$k_1^2 = 2$ tachyon, $k_2^2 = 0$ photon, $k_3^2 = 2$ tachyon

$$\Rightarrow k_2\cdot k_3 = \frac{2-2}{2} = 0 \quad \Rightarrow A_{2+1} = 0$$

$\rightarrow$ gauge invariant

2. replace ξ_1 by k_1

$$A_{1+2} = g_0\big(k_1\cdot\xi_3 - k_1\cdot(-k_1-k_3)\,\xi_3\cdot(-k_1-k_3)\big)$$

$$= g_0\big(k_1\cdot\xi_3 - (\underbrace{k_1\cdot k_1}_{=0\ \text{photon}} + k_1\cdot k_3)(\xi_3\cdot k_1 + \underbrace{\xi_3\cdot k_3}_{=0\ \text{photon}})\big)$$

$$\Rightarrow \underbrace{k_2^2}_{=2\ \text{tachyon}} = (-k_2)^2 = (k_1+k_3)^2 = \underbrace{k_1^2}_{=0} + \underbrace{k_3^2}_{=0} + 2k_1\cdot k_3 \Rightarrow k_1\cdot k_3 = 1$$

✓

$\therefore A_{1+2} = g_0 (\zeta_3 \cdot k_1 - \zeta_3 \cdot k_1) = 0$

Similarly replae ζ_3 by k_3 also gives

$A_{1+2} = 0$

$\Rightarrow$ ~~Gg~~ gauge invariant.

✓ Good!

[4]

α+ 1.

$$V(k_1,\tau_1)\, V(k_2,\tau_2)$$

$$= e^{k_1\cdot \sum_{n\geq 1}^{\infty} \frac{\alpha_{-n}}{n} e^{in\tau_1}} \; e^{ik_1\cdot(x+p\tau_1)} \; e^{-k_1\cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{n} e^{-in\tau_1}}$$

① ② ③

$$e^{k_2\cdot \sum_{n\geq 1}^{\infty} \frac{\alpha_{-n}}{n} e^{in\tau_2}} \; e^{ik_2\cdot(x+p\tau_2)} \; e^{-k_2\cdot \sum_{n\geq 1}^{\infty} \frac{\alpha_n}{n} e^{-in\tau_2}}$$

④ ⑤ ⑥

= ① ② ③ ④ ⑤ ⑥

① is creation.

⑥ is annihilation

∴ ①, ⑥ correctly ordered.

need to normal order ② ⑤ and ③ ④ (they can be treated separately because they commute i.e. [②, ③] = 0 etc).

Consider :③ ④:

~~Proposal~~ Proposition: $:e^A::e^B: = :e^{A+B}: e^{\langle AB \rangle}$

where $\langle C \rangle = \langle 0 | C | 0 \rangle$ and

A, B are linear in oscillator modes.

- linearity ⟹ LHS and RHS of above ~~factor~~ factorise into separate factors for each oscillator modes ⟹ it is sufficient to prove the above formula for a single oscillator mode:

Consider $A = c_1 a^\dagger + c_2 a$
$B = c_3 a^\dagger + c_4 a$

$[a, a^\dagger] = D$ (c number)

$:e^A: = e^{c_1 a^\dagger} e^{c_2 a}$. $\quad :e^B: = e^{c_3 a^\dagger} e^{c_4 a}$.

$:e^A::e^B: = e^{c_1 a^\dagger} e^{c_2 a} e^{c_3 a^\dagger} e^{c_4 a}$,

$:e^{A+B}: = :e^{c_1 a^\dagger + c_2 a + c_3 a^\dagger + c_4 a}:$

$= e^{c_1 a^\dagger} e^{c_3 a^\dagger} e^{c_2 a} e^{c_4 a}$.

Use Campbell - Baker - Hausdorff

$e^{c_2 a} e^{c_3 a^\dagger}$

$= e^{c_2 a + c_3 a^\dagger} e^{\frac{c_2 c_3}{2}[a, a^\dagger]}$

$= e^{c_3 a^\dagger + c_2 a} e^{\frac{c_2 c_3}{2}[a, a^\dagger]}$

$= e^{c_3 a^\dagger} e^{c_2 a} e^{-\frac{c_2 c_3}{2}\underbrace{[a^\dagger, a]}_{-D}} e^{\frac{c_2 c_3}{2}\underbrace{[a, a^\dagger]}_{D}}$

$= e^{c_3 a^\dagger} e^{c_2 a} e^{c_2 c_3 D}$

$\Rightarrow :e^A::e^B: = :e^{A+B}: e^{c_2 c_3 D}$

$$\langle AB\rangle = \langle 0|(c_1 a^\dagger + c_2 a)(c_3 a^\dagger + c_4 a)|0\rangle$$

$$= c_2 c_3 \langle 0|\, a a^\dagger |0\rangle$$

$$= c_2 c_3 \langle 0|\, \underbrace{[a, a^\dagger]}_{D} + a^\dagger a |0\rangle$$

$$= c_2 c_3 D.$$

$$\Rightarrow \; :e^A: \, :e^B: = :e^{A+B}: \; e^{\langle AB\rangle}.$$

For the product ③④.

$$③④ = e^{-k_1 \cdot \sum_{n\geq 1} \frac{\alpha_n}{n} e^{-in\tau_1}} \; e^{k_2 \cdot \sum_{m\geq 1} \frac{\alpha_{-m}}{m} e^{im\tau_2}}.$$

$$\left.\begin{matrix} e^{i\tau_1} = y_1 \\ e^{i\tau_2} = y_2 \end{matrix}\right\} \Rightarrow = e^{-k_1 \cdot \sum_{n\geq 1} \frac{\alpha_n}{n} y_1^{-n}} \; e^{k_2 \cdot \sum_{m\geq 1} \frac{\alpha_{-m}}{m} y_2^{m}} = e^{\tilde{③}} e^{\tilde{④}}$$

$$\because \; [\alpha_n^\mu, \alpha_m^\nu] = n\,\delta_{m+n}\,\eta^{\mu\nu}$$

$$\therefore$$

$$\langle \hat{③}\hat{④} \rangle = \left\langle \left(-k_1 \cdot \sum_{n\geq 1} \frac{\alpha_n}{n} y_1^{-n}\right)\left(k_2 \cdot \sum_{m\geq 1} \frac{\alpha_{-m}}{m} y_2^{m}\right)\right\rangle.$$

$$= -k_1 k_2 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{y_2}{y_1}\right)^n}_{-\log(1-\frac{y_2}{y_1})} = k_1 \cdot k_2 \log\left(1 - \frac{y_2}{y_1}\right)$$

where we've made whatever assumption to make the series convergent.

→ Now consider ⓪ ② ⑤.

$$②⑤ = e^{ik_1\cdot(x+p\tau_1)}\, e^{ik_2\cdot(x+p\tau_2)}$$

$$= e^{ik_1\cdot x}\, e^{ik_1\cdot p\tau_1}\, \underbrace{e^{-\frac{1}{2}\underbrace{k_1\cdot k_1}_{2}\tau_1}}_{\frac{1}{y_1}}\, e^{ik_2\cdot x}\, e^{ik_2\cdot p\tau_2}\, \underbrace{e^{-\frac{1}{2}\underbrace{k_2\cdot k_2}_{2}\tau}}_{\frac{1}{y_2}}$$

$$= \frac{1}{y_1 y_2}\, e^{ik_1\cdot x}\, e^{ik_1\cdot p\tau_1}\, e^{ik_2\cdot x}\, e^{ik_2\cdot p\tau_2}.$$

$$\because\quad e^{ik_1\cdot p\tau_1}\, e^{ik_2\cdot x} = e^{ik_2\cdot x + ik_1\cdot p\tau_1}\, e^{\frac{1}{2}(i)^2\tau_1 \overbrace{[p,x]}^{-i} k_1\cdot k_2}$$

$$= e^{ik_2\cdot x + ik_1\cdot p\tau_1}\, e^{+\frac{1}{2}ik_1\cdot k_2\tau_1}.$$

$$= e^{ik_2\cdot x}\, e^{ik_1\cdot p\tau_1}\, e^{-\frac{1}{2}\underbrace{(i)^2}_{-1}\overbrace{[x,p]}^{i}k_1\cdot k_2}\cdot e^{\frac{i}{2}k_1\cdot k_2\tau_1}$$

$$= e^{ik_2\cdot x}\, e^{ik_1\cdot p\tau_1}\, e^{i\tau_1 k_1\cdot k_2}$$

$$= e^{ik_2\cdot x}\, e^{ik_1\cdot p\tau_1}\, (y_1)^{k_1\cdot k_2}$$

∴ define :②⑤: to be

$$\frac{1}{y_1 y_2}\, e^{ik_1\cdot x}\, e^{ik_2\cdot x}\, e^{ik_1 p_1\tau_1}\, e^{ik_2\cdot p\tau_2}$$

So that

$$\textcircled{2}\textcircled{5} = :\textcircled{2}\textcircled{5}: \, y_1^{k_1 \cdot k_2}$$

then $\textcircled{1}\textcircled{2}\textcircled{3}\textcircled{4}\textcircled{5}\textcircled{6}$

$$= \textcircled{1}\textcircled{2}\textcircled{5}\textcircled{3}\textcircled{4}\textcircled{6}$$

$$= \textcircled{1}\left(:\textcircled{2}\textcircled{5}: \, y_1^{k_1 \cdot k_2}\right) :\textcircled{3}\textcircled{4}: e^{\tilde{\textcircled{5}}\tilde{\textcircled{4}}} \textcircled{6}$$

$$= \textcircled{1}:\textcircled{2}\textcircled{5}:\textcircled{3}\textcircled{4}:\textcircled{6} \quad y_1^{k_1 \cdot k_2} e^{k_1 \cdot k_2 \log(1 - \frac{y_2}{y_1})}$$

$$= :\textcircled{1}\textcircled{2}\textcircled{3}\textcircled{4}\textcircled{5}\textcircled{6}: \, y_1^{k_1 \cdot k_2} \left(1 - \frac{y_2}{y_1}\right)^{k_1 \cdot k_2}.$$

$$= :\textcircled{1}\textcircled{2}\textcircled{3}\textcircled{4}\textcircled{5}\textcircled{6}: \, (y_1 - y_2)^{k_1 \cdot k_2}$$

$$y_1 \to e^{i\tau_1} \qquad y_2 \to e^{i\tau_2}$$

$$\Rightarrow V(k_1, \tau_1) V(k_2, \tau_2)$$

$$= :V(k_1, \tau_1) V(k_2, \tau_2): \, (e^{i\tau_1} - e^{i\tau_2})^{k_1 \cdot k_2}$$

2.

$$V(k_1, k_2, k_3; k_4) = g_0^2 \int_{-\infty}^{0} dt \, \langle 0; k_1 | V(k_2, 0) \, V(k_3, \tau = -it) \, | 0; k_4 \rangle$$

$\tau = -it \qquad t = i\tau$

$$z = e^{i\tau} = e^{t} \Rightarrow t = \ln z \qquad dt = \frac{dz}{z}$$

$\tau = 0 \Rightarrow z = 1$

$\tau = -\infty \Rightarrow z = 0$

$\therefore \quad V = g_0^2 \int_0^1 \frac{dz}{z} \langle 0; k_1 | V(k_2, 1)\, V(k_3, x) | 0; k_4 \rangle$

If $z = e^{i\tau}$

$e^{ik\cdot x + ik\cdot p\tau} = e^{(ik\cdot x + k\cdot p \ln z)}$.

$= ~~e^{ik\cdot p}~~ e^{ik\cdot x} z^{k\cdot p+1} = z^{k\cdot p - 1} e^{ik\cdot x}$

$k\cdot k = 2$ ✓ $k\cdot k = 2$.

~~$\therefore$ Zero mode contribution in V is~~

~~$V(z) = \int \frac{dz}{z} \langle 0; k_1 |$~~

$V = \int_0^1 \frac{dz}{z} \langle 0; k_1 | ~~e^{ik_2\cdot x}~~ e^{ik_2\cdot x} e^{-k_2\cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{n}}$

$e^{k_3\cdot \sum_{m=1}^{\infty} \frac{\alpha_{-m}}{m} e^{+im\tau}} e^{ik_3\cdot x} z^{k_3\cdot p+1} | 0; k_4 \rangle$.

$= \int_0^1 \frac{dz}{z} z^{k_3\cdot k_4 + 1} \langle 0; k_1 - k_2 | e^{-k_2\cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{n}}$

$e^{k_3\cdot \sum_{m=1}^{\infty} \frac{\alpha_{-m}}{m} e^{+im\tau}} | 0; k_3 + k_4 \rangle$. ✓

$s = -(k_1 + k_2)^2$

$\left. \begin{array}{l} k_3\cdot k_4 + 1 \\ = -\frac{s}{2} - 1 \end{array} \right)$

$= \int_0^1 dz\, z^{-\frac{1}{2}s - 2} \langle 0; k_1 - k_2 | e^{-k_2\cdot \sum \frac{\alpha_n}{n}} e^{k_3\cdot \sum \frac{\alpha_{-m}}{m} z^m} | 0; k_3 + k_4 \rangle$

$$= \int_0^1 dz\, z^{-\frac{1}{2}s-2}\, \delta(k_1-k_2-k_3-k_4)$$

(this gives 1) $\to$

$$\langle 0| : e^{(-k_2\cdot\sum \frac{\alpha_n}{n} + k_3\sum \frac{\alpha_m}{m} z^m)} : |0\rangle .$$

(a relation proven before: $:e^A::e^B: = :e^{A+B}: \times e^{\langle AB\rangle}$) $\to$

$$\times\, e^{\langle(-k_2\cdot\sum\frac{\alpha_n}{n})(k_3\cdot\sum\frac{\alpha_m}{m}z^m)\rangle}$$

$$= \langle -k_2\cdot k_3 \sum \tfrac{1}{n} z^n\rangle$$

$$= k_2\cdot k_3 \log(1-z)$$

~~$V = \int_0^1 \delta(k_1-k_2-k_3-k_4)$~~

Dropping $\delta(k_1-k_2-k_3-k_4)$

$$V = g^2\int_0^1 dz\, z^{-\frac{1}{2}s-2}\, e^{k_2\cdot k_3\log(1-z)}$$

$$= g^2\int_0^1 dz\, z^{-\frac{1}{2}s-2}\,(1-z)^{k_2\cdot k_3}$$

$$\left|\; k_2\cdot k_3 = -\tfrac{1}{2}t-2 \quad (t=-(k_1+k_3)^2) \;\right|.$$

$$= g^2\int_0^1 dz\, z^{-\frac{1}{2}s-2}\,(1-z)^{-\frac{1}{2}t-2}$$

Use $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

and define $\alpha(s) = 1+\frac{s}{2}$, $\alpha(t) = 1+\frac{t}{2}$

$$V = g^2 B(-\alpha(s), -\alpha(t))$$

$$= g^2 \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

$\alpha(x) = 1 + \frac{1}{2}x$, $s = -(k_1+k_2)^2$

$t = -(k_1+k_3)^2$

$\Rightarrow$ Veneziano Amplitude

Very good!