String Theory II

Problem Set 1

Ziyan Li

TA: Mr. Julius Eckhard

Tutor: Mr. Marc-Antoine Fiset.

Th. 15–16.30 Wk 2

1 1) $[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n+m}$; $\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s}$.

$$\delta_a = \begin{cases} 1 & a=0 \\ 0 & a\neq 0 \end{cases}$$

$$L_n = \underbrace{\frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha_{-m}\cdot\alpha_{m+n}:}_{L_n^{(\alpha)}} + \underbrace{\frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}\left(r+\frac{n}{2}\right):b_{-r}\cdot b_{n+r}:}_{L_n^{(b)}}$$

$$G_r = \sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot b_{r+m}$$

→ Note that ~~$L_m^{(\alpha)} = \frac{1}{2}\sum_{p=-\infty}^{0}\alpha_{-p}\alpha_{p+m,\mu} + \frac{1}{2}\sum_{p=1}^{\infty}\alpha_{p+m}\alpha_{-p,\mu}$~~

$$L_m^{(\alpha)} = \frac{1}{2}\sum_{p=-\infty}^{0}\alpha^\mu_{p+m}\cdot\alpha_{-p,\mu} + \frac{1}{2}\sum_{p=1}^{\infty}\alpha^\mu_{-p}\cdot\alpha_{p+m,\mu}$$

this is because in this case if $m\neq 0$ ~~$L_m^{(\alpha)} = \frac{1}{2}\sum_{p=-\infty}$~~ $L_m^{(\alpha)}$ is automatically normal ordered and if $m=0$, $L_0^{(\alpha)} = \frac{1}{2}\alpha_0^2 + \sum_{p=1}^{\infty}\alpha^\mu_{-p}\alpha_{p,\mu}$ which is also normal ordered.

→ Consider $[\alpha_m^\mu, L_n^{(\alpha)}] = \frac{1}{2}\sum_{p=-\infty}^{\infty}[\alpha_m^\mu, :\alpha^\nu_{-p}\alpha_{p+n,\nu}:]$

$$= \frac{1}{2}\sum_{p=-\infty}^{0}[\alpha_m^\mu, \alpha^\nu_{p+n}\alpha_{-p,\nu}] + \frac{1}{2}\sum_{p=1}^{\infty}[\alpha_m^\mu, \alpha^\nu_{-p}\alpha_{p+n,\nu}]$$

$$= \frac{1}{2}\sum_{p=-\infty}^{0}\Big(\underbrace{[\alpha_m^\mu, \alpha^\nu_{p+n}]}_{m\eta^{\mu\nu}\delta_{m+p+n}}\alpha_{-p,\nu} + \alpha_{p+n,\nu}\underbrace{[\alpha_m^\mu, \alpha^\nu_{-p}]}_{m\eta^{\mu\nu}\delta_{m-p}}\Big)$$

$$+\frac{1}{2}\sum_{p=1}^{\infty}\Big(\underbrace{[\alpha_m^\mu, \alpha^\nu_{-p}]}_{m\eta^{\mu\nu}\delta_{m-p}}\alpha_{p+n,\nu} + \alpha_{-p,\nu}\underbrace{[\alpha_m^\mu, \alpha^\nu_{p+n}]}_{m\eta^{\mu\nu}\delta_{m+p+n}}\Big)$$

$$= \frac{1}{2}\sum_{p=-\infty}^{\infty} m\left(\alpha^\mu_{p+n}\delta_{m-p} + \alpha^\mu_{-p}\delta_{m+p+n}\right)$$

$$= m\cdot\frac{1}{2}\left(\alpha^\mu_{m+n} + \alpha^\mu_{m+n}\right) = \underline{m\alpha^\mu_{m+n}}.$$

Now $[L_m^{(\alpha)}, L_n^{(\alpha)}] = \frac{1}{2}\sum_{p=-\infty}^{\infty}[:\alpha_{-p}^{\mu}\alpha_{p+m,\mu}:, L_n^{(\alpha)}]$

$$= \frac{1}{2}\sum_{p=-\infty}^{0}[\alpha_{p+m}^{\mu}\alpha_{-p,\mu}, L_n^{(\alpha)}] + \frac{1}{2}\sum_{p=1}^{\infty}[\alpha_{-p}^{\mu}\alpha_{p+m,\mu}, L_n^{(\alpha)}]$$

$$= \frac{1}{2}\sum_{p=-\infty}^{0}\left([\alpha_{p+m}^{\mu}, L_n^{(\alpha)}]\alpha_{-p,\mu} + \alpha_{p+m,\mu}[\alpha_{-p}^{\mu}, L_n^{(\alpha)}]\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{\infty}\left([\alpha_{-p}^{\mu}, L_n^{(\alpha)}]\alpha_{p+m,\mu} + \alpha_{-p,\mu}[\alpha_{p+m}^{\mu}, L_n^{(\alpha)}]\right)$$

$$= \frac{1}{2}\sum_{p=-\infty}^{0}(p+m)\alpha_{p+m+n}^{\mu}\alpha_{-p,\mu} + (-p)\alpha_{p+m}^{\mu}\alpha_{-p+n,\mu}$$

$$+ \frac{1}{2}\sum_{p=1}^{\infty}(-p)\alpha_{-p+n}^{\mu}\alpha_{p+m,\mu} + (p+m)\alpha_{-p}^{\mu}\alpha_{p+m+n,\mu}$$

$$= \frac{1}{2}\sum_{p=-\infty}^{0}(p+m)\alpha_{p+m+n}^{\mu}\alpha_{-p,\mu}\ \text{①} + \frac{1}{2}\sum_{p=-\infty}^{-n}(-p-n)\alpha_{p+m+n}^{\mu}\alpha_{-p,\mu}\ \text{②}$$

$$+ \frac{1}{2}\sum_{p=1-n}^{\infty}(-p-n)\alpha_{-p}^{\mu}\alpha_{p+m+n,\mu}\ \text{③} + \frac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^{\mu}\alpha_{p+m+n,\mu}(p+m)\ \text{④}$$

Assume $n>0$ without loss of generality.

if $m+n \neq 0$, everything is already normal ordered.

$$= (m-n)\frac{1}{2}\sum_{p=-\infty}^{\infty}:\alpha_{-p}^{\mu}\alpha_{p+m+n,\mu}: \quad = (m-n)L_{m+n}^{(\alpha)}$$

if $m+n=0$, the 3rd term may not be normal ordered.

$$= \left(\text{①} + \text{②} + \text{④} + \frac{1}{2}\sum_{p=0}^{\infty}(-p-n)\alpha_{-p}^{\mu}\alpha_{p+m+n,\mu}\right) + \frac{1}{2}\sum_{p=1-n}^{-1}(-p-n)\alpha_{-p}^{\mu}\alpha_{p+m+n,\mu}$$

(without loss of generality assume $n>0$), where inside ( ) is normal ordered.

$$= (m-n)L_{m+n}^{(\alpha)} + \frac{1}{2}\sum_{p=1-n}^{-1}(-p-n)\underbrace{[\alpha_{-p}^{\mu}, \alpha_{p+m+n}^{\nu}]}_{(-p)\eta^{\mu\nu}}\eta_{\mu\nu}$$

$$= (m-n)L_{m+n}^{(\alpha)}\ \frac{1}{2}\sum_{p=1-n}^{-1}p(p+n)\underbrace{\eta^{\mu\nu}\eta_{\mu\nu}}_{D}$$

D = spacetime dimension

$$= (m-n)L^{(\alpha)}_{m+n} + \frac{D}{2}\sum_{p=1}^{n-1} p^2 - np$$

$$= (m-n)L^{(\alpha)}_{m+n} + \frac{D}{2}\left(\frac{1}{6}(p-1)(p)(2p-1) - n\right)$$

$$\frac{D}{2}\left(\frac{1}{6}(n-1)(n)(2n-1) - n\,\frac{1}{2}(n-1)(n)\right)$$

$$= (m-n)L^{(\alpha)}_{m+n} + \frac{D}{2}(n-1)(n)\left[\frac{n}{3} - \frac{1}{6} - \frac{n}{2}\right]$$

$$= (m-n)L^{(\alpha)}_{m+n} + \frac{D}{2}\left(-\frac{1}{6}\right) n(n-1)(n+1)$$

$$\underset{m=-n}{=} (m-n)L^{(\alpha)}_{m+n} + \frac{D}{12} m(m^2-1)$$

$\therefore$ overall

$$= (m-n)L^{(\alpha)}_{m+n} + \frac{D}{12} m(m^2-1)\,\delta_{m+n}$$

Now we look at $L_n^{(b)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi} \left(r+\frac{n}{2}\right) :b_{-r}\cdot b_{n+r}:$

→ R sector $\phi=0$: $L_n^{(b)} = \sum_{r=-\infty}^{\infty}\left(r+\frac{n}{2}\right) :b_{-r}\, b_{n+r}:$

→ if n even

$$L_n^{(b)} = \sum_{p=-\infty}^{\infty}\left(r+\frac{n}{2}\right):\left(\sum_{r=-\infty}^{-\frac{n}{2}-1} b_{-r}\, b_{n+r} + b_{\frac{n}{2}}\cdot b_{\frac{n}{2}} + \sum_{r=-\frac{n}{2}+1}^{\infty} b_{-r}\cdot b_{n+r}\right)$$

not normal ordered. normal ordered.

$$= \sum_{r=-\infty}^{-\frac{n}{2}-1} -\left(r+\frac{n}{2}\right) b_{n+r}\cdot b_{-r}$$

$\{b, b\} = 0$.

To ensure manifest normal ordering, we write

$$L_n^{(b)} = -\frac{1}{2}\sum_{r=-\infty-\phi}^{-\phi} \left(r+\frac{n}{2}\right) b_{n+r}\cdot b_{-r} + \frac{1}{2}\sum_{r=1-\phi}^{r=\infty-\phi} b_{-r}\cdot b_{n+r}\left(r+\frac{n}{2}\right)$$

so the anticommutitivity of b operators is taken into account by the "−" sign. In fact as $L_n^{(\alpha)}$, no normal ~~Now use~~ ordering is required ~~for~~ except ~~but~~ when $b_{-x}\cdot b_x$ is present in the expansions.

- Now use $[A, BC] = \{A,B\}C - B\{A,C\}$ so that

$$[b_r^\mu, L_n^{(b)}] = -\frac{1}{2}\sum_{s=-\infty-\phi}^{-\phi} \left(s+\frac{n}{2}\right)\Big(\underbrace{\{b_r^\mu, b_{n+s}^\nu\}}_{\eta^{\mu\nu}\delta_{r+n+s}} b_{-s,\nu} - b_{n+s,\nu}\underbrace{\{b_r^\mu, b_{-s}^\nu\}}_{\eta^{\mu\nu}\delta_{r-s}}\Big)$$

$$+\frac{1}{2}\sum_{r=1-\phi}^{r=\infty-\phi} \left(s+\frac{n}{2}\right)\Big(\underbrace{\{b_r^\mu, b_{-s}^\nu\}}_{\eta^{\mu\nu}\delta_{r-s}} b_{n+s,\nu} - b_{-s,\nu}\underbrace{\{b_r^\mu, b_{n+s}^\nu\}}_{\eta^{\mu\nu}\delta_{r+n+s}}\Big)$$

$$= +\frac{1}{2}\sum_{s=-\infty-\phi}^{-\phi}\left(b^\mu_{n+s}\delta_{r-s} - b^\mu_{-s}\delta_{r+n+s}\right)\left(s+\frac{n}{2}\right)$$

$$+\frac{1}{2}\sum_{s=1-\phi}^{\infty-\phi}\left(s+\frac{n}{2}\right)\left(b^\mu_{n+s}\delta_{r-s} - b^\mu_{-s}\delta_{r+n+s}\right)$$

$$= \frac{1}{2}\sum_{s=-\infty-\phi}^{\infty-\phi}\left(s+\frac{n}{2}\right)\left(b^\mu_{n+s}\delta_{r-s} - b^\mu_{-s}\delta_{r+n-s}\right)$$

$$= \sout{\frac{1}{2}\left(s+\frac{n}{2}\right) b^\mu_{n+r}}\ \frac{1}{2}\left(r+\frac{n}{2}\right) b^\mu_{n+r} - \frac{1}{2}\left(-r-\frac{n}{2}\right) b^\mu_{n+r}$$

$$= \left(r+\frac{n}{2}\right) b^\mu_{r+n}$$

Now, to simplify calculation, assume $[L_m^{(b)}, L_n^{(b)}]$ doesn't need normal ordering if $m+n \neq 0$ and verify a posteriori

so,

$$[L_m^{(b)}, L_n^{(b)}] = \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi} (r+\tfrac{m}{2})\,[b_{-r}\cdot b_{m+r}, L_n^{(b)}]$$

$$= \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi} (r+\tfrac{m}{2})\Big([\underbrace{b_{-r}^{\mu}, L_n^{(b)}]}_{(-r+\frac{n}{2})\,b_{-r+n}} * b_{m+r,\mu} + b_{-r,\mu}\,[\underbrace{b_{m+r}^{\mu}, L_n^{(b)}]}_{(r+m+\frac{n}{2})\,b^{\mu}_{r+m+n}}\Big)$$

$$= \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi} (r+\tfrac{m}{2})\Big[(-r+\tfrac{n}{2})\,b_{-r+n}\cdot b_{m+n+r} + (m+r+\tfrac{n}{2})\,b_{-r}\cdot b_{m+n+r}\Big]$$

$\Rightarrow$ (in the first term redefine $r \to r+n$ so $-r \to -r-n$

$$= \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}\Big((r+n+\tfrac{m}{2})(-r-\tfrac{n}{2}) + (r+\tfrac{m}{2})(m+r+\tfrac{n}{2})\Big)\,b_{-r}\cdot b_{m+n+r}$$

$\Rightarrow$ (we see that if $m+n \neq 0$ then no normal ordering issues here)

$$= \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}\Big(-r^2 - rn - \tfrac{1}{2}rm - \tfrac{1}{2}rn - \tfrac{1}{2}n^2 - \tfrac{1}{4}mn + r^2 + mr + \tfrac{1}{2}rn + \tfrac{1}{2}mr + \tfrac{1}{2}m^2 + \tfrac{1}{4}mn\Big)\,b_{-r}\cdot b_{m+n+r}$$

$$= \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}\Big(r(m-n) + \tfrac{1}{2}(m+n)(m-n)\Big)\,b_{-r}\cdot b_{m+n+r}$$

$$= (m-n)\cdot\underbrace{\frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}(r+\tfrac{m+n}{2})\,b_{-r}\cdot b_{m+n+r}}_{L_{m+n}^{(b)}}$$

$$= (m-n)\,L_{m+n}^{(b)} \qquad \text{if } m+n \neq 0$$

Now consider the case $m+n=0$:

in NS sector : $\phi=0$, $L_m^{(b)} = -\frac{1}{2}\sum_{r=-\infty}^{0}(r+\frac{m}{2})\, b_{m+r}\cdot b_{-r} + \frac{1}{2}\sum_{r=1}^{\infty}(r+\frac{m}{2})\, b_{-r}\cdot b_{m+r}$

$$[L_m^{(b)}, L_{-m}^{(b)}] = -\frac{1}{2}\sum_{r=-\infty}^{0}(r+\frac{m}{2})\,[b_{m+r}\cdot b_{-r}, L_{-m}^{(b)}] + \frac{1}{2}\sum_{r=1}^{\infty}(r+\frac{m}{2})\,[b_{-r}\cdot b_{m+r}, L_{-m}^{(b)}]$$

$$= -\frac{1}{2}\sum_{r=-\infty}^{0}(r+\frac{m}{2})\Big(b_{m+r}\cdot[b_{-r}, L_{-m}^{(b)}] + [b_{m+r}, L_{-m}^{(b)}]\cdot b_{-r}\Big) + \frac{1}{2}\sum_{r=1}^{\infty}(r+\frac{m}{2})\Big(b_{-r}\cdot[b_{m+r}, L_{-m}^{(b)}] + [b_{-r}, L_{-m}^{(b)}]\cdot b_{m+r}\Big)$$

$$= -\frac{1}{2}\sum_{r=-\infty}^{0}(r+\frac{m}{2})\Big((-r-\frac{m}{2})\, b_{m+r}\cdot b_{-m-r} + (r+\frac{m}{2})\, b_r\cdot b_{-r}\Big) + \frac{1}{2}\sum_{r=1}^{\infty}(r+\frac{m}{2})\Big((r+\frac{m}{2})\, b_{-r}\cdot b_r + (-r-\frac{m}{2})\, b_{-m-r}\cdot b_{-m+r}\Big)$$

$\Rightarrow$ ($q=m+r \rightarrow r=q-m$, $-r=-q+m$, $r+\frac{m}{2}=q-\frac{m}{2}$

$r=1 \rightarrow q=m+1$ $\quad r=0 \rightarrow q=m$ $\quad r=\pm\infty \rightarrow q=\pm\infty$

then replace $q\rightarrow r$ for the first and fourth term)

$$= -\frac{1}{2}\sum_{r=-\infty}^{0}(r+\frac{m}{2})(r+\frac{m}{2})\, b_r\cdot b_{-r} + \frac{1}{2}\sum_{r=1}^{\infty}(r+\frac{m}{2})(r+\frac{m}{2})\, b_{-r}\cdot b_r$$
① ②
$$-\frac{1}{2}\sum_{r=-\infty}^{m}(r-\frac{m}{2})(-r+\frac{m}{2})\, b_r\cdot b_{-r} + \frac{1}{2}\sum_{r=m+1}^{\infty}(r-\frac{m}{2})(-r+\frac{m}{2})\, b_{-r}\cdot b_r$$
③ ④

$\Rightarrow$ (use $b_r\cdot b_{-r} = -b_{-r}\cdot b_r + \eta^{\mu\nu}\eta_{\mu\nu} = -b_{-r}\cdot b_r + D$)

$\Rightarrow$ (first, second, fourth terms are already manifestly normal ordered. But 3rd term is not. if we normal order the 3rd term we get $2mL_0^{(b)}$ (as in the equation for the $m+n\neq 0$ case) plus a central charge term)

$$= \Big[①+②+④ + \frac{1}{2}\sum_{r=-\infty}^{m}(r-\frac{m}{2})^2 :b_r\cdot b_{-r}:\Big] + (\text{central charge term})$$

$[2mL_0^{(b)}]$

$$= \Big[-\frac{1}{2}\sum_{r=-\infty}^{0}(r+\tfrac{m}{2})(r+\tfrac{m}{2})\, b_r\cdot b_{-r} + \frac{1}{2}\sum_{r=1}^{\infty}(r+\tfrac{m}{2})(r+\tfrac{m}{2})\, b_{-r}\cdot b_r$$

$$-\frac{1}{2}\sum_{r=-\infty}^{0}(r-\tfrac{m}{2})(-r+\tfrac{m}{2})\, b_r\cdot b_{-r} + \frac{1}{2}\sum_{r=1}^{\infty}(r-\tfrac{m}{2})(-r+\tfrac{m}{2})\, b_{-r}\cdot b_r \Big]$$

$$\left(-\frac{1}{2}\sum_{r=1}^{m}(r-\tfrac{m}{2})(-r+\tfrac{m}{2})\, D \right)$$

$\Leftarrow$ used $b_r\cdot b_{-r} = -b_{-r}\cdot b_r + D$

$$= 2mL_0^{(b)} + \frac{1}{2}\sum_{r=1}^{m}(r-\tfrac{m}{2})^2 D$$

$$= 2mL_0^{(b)} + \frac{1}{2}\left[\left(\sum_{r=1}^{m} r^2 - rm\right) + \left(\sum_{r=1}^{m}\frac{m^2}{4}\right)\right] D. \qquad \left(\sum_{r=1}^{m}\frac{m^2}{4} = \frac{m^3}{4}\right)$$

we've calculated this for $L_m^{(\alpha)}$, $= -\frac{D}{6}m(m^2-1)$

$$= 2mL_0^{(b)} - \frac{D}{12}m(m-1) + \frac{m^3 D}{8}$$

$\therefore$ ~~$[L_m, L_n] = [L_m^{(\alpha)}, L$~~

So $[L_m^{(b)}, L_n^{(b)}] = (m-n)L_{m+n}^{(b)} + \left(-\frac{D}{12}m(m^2-1) + \frac{m^3 D}{8}\right)\delta_{m+n}.$

$$\Rightarrow [L_m, L_n] = [L_m^{(\alpha)}, L_n^{(\alpha)}] + [L_m^{(b)}, L_n^{(b)}]$$

$$= (m-n)(L_{m+n}^{(\alpha)} + L_{m+n}^{(b)}) + \left(\frac{D}{12}m(m^2-1) - \frac{D}{12}m(m^2-1) + \frac{D}{8}m^3\right)\delta_{m+n}$$

$$= (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n}$$ for ~~NS~~ R sector.

For NS sector: By similar arguments,

we have :

$$[L_m^{(b)}, L_{-m}^{(b)}] = -\frac{1}{2}\sum_{r=-\infty-\frac{1}{2}}^{-\frac{1}{2}} (r+\tfrac{m}{2})(r+\tfrac{m}{2})\, b_r \cdot b_{-r} + \frac{1}{2}\sum_{r=\frac{1}{2}}^{\infty-\frac{1}{2}} (r+\tfrac{m}{2})(r+\tfrac{m}{2})\, b_{-r}\cdot b_r$$

① ②

$$-\frac{1}{2}\sum_{r=-\infty-\frac{1}{2}}^{m-\frac{1}{2}} (r-\tfrac{m}{2})(-r+\tfrac{m}{2})\, b_r\cdot b_{-r} + \frac{1}{2}\sum_{r=m+\frac{1}{2}}^{\infty-\frac{1}{2}} (r-\tfrac{m}{2})(-r+\tfrac{m}{2})\, b_{-r}\cdot b_r$$

③ ④

similarly ①, ②, ④ are manifestly normal orded and that $2mL_0^{(b)} = ① + ② + ④ + :③:$

and use $b_r \cdot b_{-r} = -b_{-r}\cdot b_r + D$ we have, similar to the R sector case:

$$[L_m^{(b)}, L_{-m}^{(b)}] = 2mL_0^{(b)} - \frac{1}{2}\sum_{r=\frac{1}{2}}^{m-\frac{1}{2}} (r-\tfrac{m}{2})(-r+\tfrac{m}{2})\,D$$

$$= 2mL_0^{(b)} + \frac{1}{2}\sum_{r=\frac{1}{2}}^{m-\frac{1}{2}} (r-\tfrac{m}{2})^2 D$$

$$= 2mL_0^{(b)} + \frac{1}{2}\sum_{r=1}^{m} (r-\tfrac{m+1}{2})^2 D$$

change of variable $r \to r+\frac{1}{2}$, $r=\frac{1}{2} \to r=1$

$$= 2mL_0^{(b)} + \frac{1}{2}\sum_{r=1}^{m} r^2 - r(m+1) + \frac{(m+1)^2}{4}$$

$$= 2mL_0^{(b)} + \frac{1}{2}\Big(\underbrace{\sum_{r=1}^{m}(r^2-rm)}_{-\frac{D}{6}m(m^2-1)}\Big) - \frac{1}{2}\sum_{r=1}^{m} r + \frac{1}{2}\sum_{r=1}^{m}\frac{(m+1)^2}{4}.$$

$$= 2mL_0^{(b)} - \frac{D}{12}m(m^2-1) - \frac{1}{2}\cdot\frac{D}{2}m(m+1) + \frac{(m+1)^2 mD}{8}$$

$$= 2mL_0^{(b)} - \frac{D}{12}m(m^2-1) + \frac{D}{8}(-2m^2-2m+m^3+2m^2+m)$$

$$= 2mL_0^{(b)} - \frac{D}{12}m(m^2-1) + \frac{D}{8}(m^3-m)$$

So $[L_m^{(b)}, L_n^{(b)}] = (m-n)L_{m+n}^{(b)} + \left(\frac{-D}{12}m(m^2-1) + \frac{D}{8}m(m^2-1)\right)\delta_{m+n}$

$\Rightarrow$ $[L_m, L_n] = (m-n)L^{\bullet}_{m+n} + \left(\frac{D}{12}m(m^2-1) - \frac{D}{12}m(m^2-1) + \frac{D}{8}m(m^2-1)\right)\delta_{m+n}$.

for the NS-sector.

$\Rightarrow$ overall, since in R sector $\phi=0$, NS sector $\phi=\frac{1}{2}$

we have $[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m(m^2-2\phi)$

2) keep in mind that $\{AB, C\} = A\{B,C\} - [A,C]B$
$= \{A,C\}B + A[B,C]$

and that $\{A, BC\} = \{A,B\}C - B[A,C] = \{B,A\}C + B[C,A]$

$$G_r = \sum_{m=-\infty}^{\infty} \alpha_{-m} \cdot b_{r+m} \quad , \quad [\alpha^\nu_m, b^\nu_r] = 0$$

$$\Rightarrow \{b^\mu_r, G_s\} = \sum_{m=-\infty}^{\infty} \{b^\mu_{-r}, \alpha_{-m,\nu} b^\nu_{s+m}\}$$

$$= \sum_{m=-\infty}^{\infty} b^\mu_r \alpha_{-m,\nu} b^\nu_{s+m} + \alpha_{-m,\nu} b^\nu_{s+m} b^\mu_{-r}$$

$$= \sum_{m=-\infty}^{\infty} \alpha_{-m,\nu} \underbrace{\{b^\mu_r, b^\nu_{s+m}\}}_{\eta^{\mu\nu}\delta_{r+s+m}} = \underline{\alpha^\mu_{r+s}}$$

$$\Rightarrow [\alpha^\mu_n, G_s] = \sum_{m=-\infty}^{\infty} [\alpha^\mu_n, \alpha^\nu_{-m} b_{s+m,\nu}]$$

$$= \sum_{m=-\infty}^{\infty} \alpha^\mu_n \alpha^\nu_{-m} b_{s+m,\nu} - \alpha^\nu_{-m} b_{s+m,\nu} \alpha^\mu_n$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{[\alpha^\mu_n, \alpha^\nu_{-m}]}_{n\eta^{\mu\nu}\delta_{n-m}} b_{s+m,\nu} = \underline{n b^\mu_{s+n}}$$

in both cases, the result is a single operator so no normal ordering issues.

Now, $\{G_r, G_s\} = \sum_{m=-\infty}^{\infty} \{:\alpha^\mu_{-m} b_{r+m,\mu}:, G_s\}$

we write $G_r = \sum_{m=-\infty}^{0} b_{r+m} \cdot \alpha_{-m} + \sum_{m\geq 1}^{\infty} \alpha_{-m} b_{r+m}$, which is its manifestly normal ordered form.

so $\{G_r, G_s\} = \sum_{m=-\infty}^{0} \{b_{r+m} \cdot \alpha_{-m}, G_s\} + \sum_{m=1}^{\infty} \{\alpha_{-m} \cdot b_{r+m}, G_s\}$

$$= \sum_{m=-\infty}^{0} \underbrace{\{b_{r+m}, G_s\}}_{\alpha_{r+s+m}} \cdot \alpha_{-m} + b_{r+m} \cdot \underbrace{[\alpha_{-m}, G_s]}_{=(-m)\,b_{s-m}}$$

$$+ \sum_{m=1}^{\infty} \alpha_{-m} \cdot \underbrace{\{b_{r+m}, G_s\}}_{\alpha_{r+s+m}} - \underbrace{[\alpha_{-m}, G_s]}_{(-m)\,b_{s-m}} \cdot b_{r+m}$$

$$= \sum_{m=-\infty}^{0} \alpha_{r+s+m} \cdot \alpha_{-m} - m\, b_{r+m} \cdot b_{s-m}$$

$$+ \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_{r+s+m} + m\, b_{s-m} \cdot b_{r+m}$$

Without loss of generality assume $r > 0$

if $r+s \neq 0$ then:

$$\{G_r, G_s\} = \sum_{m=-\infty}^{\infty} \alpha_{-m} \cdot \alpha_{r+s+m} + \sum_{m=-\infty}^{\infty} m\, b_{s-m} \cdot b_{r+m}$$

$$= \underbrace{\sum_{m=-\infty}^{\infty} \alpha_{-m} \cdot \alpha_{r+s+m}}_{2L^{(\alpha)}_{r+s}} + \sum_{m=-\infty}^{\infty} (m+s)\, b_{-m} \cdot b_{r+s+m}$$

where in second term we change variable $m \to m+s$

~~second term~~ Note that $\sum_{m=-\infty}^{\infty} b_{-m} \cdot b_{r+s+m} = 0$ for $r+s \neq 0$

because the $m=x$ term cancels with the term

$m = -r-s-x$, so we can add to the second term

$\sum_{m=-\infty}^{\infty} \left(\frac{r-s}{2}\right) b_{-m} \cdot b_{r+s+m}$ to make it $\sum_{m=-\infty}^{\infty} \left(m + \frac{r+s}{2}\right) b_{-m} \cdot b_{r+s+m}$

$= 2L^{(b)}_{r+s}$

So $\{G_r, G_s\} = 2(L^{(\alpha)}_{r+s} + L^{(b)}_{r+s}) = 2L_{r+s}$

if $r+s=0$ then:

$$\{G_r, G_{-r}\} = \overset{①}{\sum_{m=-\infty}^{0} \alpha_m \cdot \alpha_{-m}} - \overset{③}{\sum_{m=-\infty}^{0} m\, b_{r+m} \cdot b_{-r-m}}$$

$$+ \overset{②}{\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m} + \overset{④}{\sum_{m=1}^{\infty} m\, b_{-r-m} \cdot b_{r+m}}$$

$\Rightarrow$ (change variable $q = m+r$, $m = q-r$

$m=0 \leftrightarrow q=r$

$m=\pm\infty \leftrightarrow q=\pm\infty$ then rename $q \to m$)

$m=1 \leftrightarrow q=r+1$ for ③, ④ only

$$= \sum_{m=-\infty}^{0} \alpha_m \cdot \alpha_{-m} - \sum_{m=-\infty}^{r} (m-r)\, b_m \cdot b_{-m}$$

$$+ \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{m=r+1}^{\infty} (m-r)\, b_{-m} \cdot b_m$$

$$= 2\Big(\underbrace{\tfrac{1}{2}\sum_{m=-\infty}^{0} \alpha_m \cdot \alpha_{-m} + \tfrac{1}{2}\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m}_{L_0^{(\alpha)}}\Big)$$

$$+ 2\Big(\underbrace{\tfrac{1}{2}\sum_{m=-\infty}^{0} (-m)\, b_m \cdot b_{-m} + \sum_{m=1}^{\infty} m\, b_{-m} \cdot b_m}_{L_0^{(b)}}\Big)$$

Here we assume R sector, $\phi = 0$, $r \in \mathbb{Z}$

$$- \sum_{m=1}^{r} m\, b_m \cdot b_{-m} - \sum_{m=1}^{r} m\, b_{-m} \cdot b_m$$

$$+ r \sum_{m=-\infty}^{r} b_m \cdot b_{-m} - r \sum_{m=r+1}^{\infty} b_{-m} \cdot b_m$$

$$= 2L_0 - \sum_{m=1}^{r} m\, (\underbrace{b_m \cdot b_{-m} + b_{-m} \cdot b_m}_{D})$$

$$+ \underbrace{r \sum_{m=-\infty}^{0} b_m \cdot b_{-m} - r \sum_{m=1}^{\infty} b_{-m} \cdot b_m}$$

$= r\, b_0 \cdot b_0$ since all other terms cancel

and $b_0 \cdot b_0 + b_0 \cdot b_0 = D \Rightarrow b_0 \cdot b_0 = \frac{D}{2}$

$= \frac{1}{2} r D$

$$+ \underbrace{r \sum_{m=1}^{r} b_m \cdot b_{-m} + r \sum_{m=1}^{r} b_{-m} \cdot b_m}$$

$$r \sum_{m=1}^{r} (\underbrace{b_m \cdot b_{-m} + b_{-m} \cdot b_m}_{D}) = r^2 D.$$

$$= 2L_0 - \tfrac{1}{2} r(r+1) D + \tfrac{1}{2} r D + r^2 D$$

$$= 2L_0 - \tfrac{1}{2} r^2 D - \tfrac{1}{2} r D + \tfrac{1}{2} r D + r^2 D = 2L_0 + \frac{D}{2} r^2$$

In the NS sector, $\phi = \frac{1}{2}$. m takes half integer values.

$$\therefore \{G_r, G_{-r}\} = 2L_0 - \sum_{m=\frac{1}{2}}^{r} m\, b_m \cdot b_{-m} - \sum_{m=\frac{1}{2}}^{r} m\, b_{m} \cdot b_m$$

$\longrightarrow -\frac{1}{2}(r+\frac{1}{2})^2 D$

$$+ r \sum_{m=-\infty-\frac{1}{2}}^{-\frac{1}{2}} b_m \cdot b_{-m} \quad r \sum_{m=\frac{1}{2}}^{\infty+\frac{1}{2}} b_{-m} \cdot b_m$$

$=0$ exactly cancel (NO $b_0 \cdot b_0$ term)

$$+ r \sum_{m=\frac{1}{2}}^{r} b_m \cdot b_{-m} + r \sum_{m=\frac{1}{2}}^{r} b_{-m} \cdot b_m$$

$r(r+\frac{1}{2})D$

$$= 2L_0 - \frac{D}{2}(r+\tfrac{1}{2})^2 + r(r+\tfrac{1}{2})D$$

$$= 2L_0 - \frac{1}{2}r^2 D - \frac{1}{2}rD - \frac{D}{8} + r^2 D + \frac{1}{2}rD$$

$$= 2L_0 + \frac{D}{2}r^2 - \frac{D}{8} = 2L_0 + \frac{D}{2}(r^2 - \tfrac{1}{4})$$

So overall, in compact form we have

$$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{\phi}{2}\right)\delta_{r+s}$$

For $m \neq 0$, $L_m = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{-p}\cdot\alpha_{p+m} + \frac{1}{2}\sum_{s\in\mathbb{Z}+\phi}(s+\frac{m}{2})\, b_{-s}\, b_{m+s}$

$$[L_m, G_r] = \frac{1}{2}\sum_{p\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} [\alpha_{-p}\cdot\alpha_{p+m}, \alpha_{-k}]\cdot b_{r+k}$$

$$+ \frac{1}{2}\sum_{s\in\mathbb{Z}+\phi}\sum_{k\in\mathbb{Z}} (s+\tfrac{m}{2})\, \alpha_{-k}\cdot[b_{-s}\cdot b_{m+s}, b_{r+k}]$$

$\Rightarrow$ (use $[AB, C] = A\{B, C\} - \{A, C\}B$)

$$= \frac{1}{2}\sum_{p\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \alpha_{-p,\mu}\underbrace{[\alpha^{\mu}_{p+m}, \alpha^{\nu}_{-k}]}_{(p+m)\eta^{\mu\nu}\delta_{p+m-k}} b_{r+k,\nu} + \underbrace{[\alpha^{\mu}_{-p}, \alpha^{\nu}_{-k}]}_{(-p)\eta^{\mu\nu}\delta_{-p-k}}\alpha_{p+m,\mu}\, b_{r+k,\nu}$$

$$+ \frac{1}{2}\sum_{s\in\mathbb{Z}+\phi}\sum_{k\in\mathbb{Z}} (s+\tfrac{m}{2})\, \alpha_{-k,\nu}\, b^{\mu}_{-s,\mu}\underbrace{\{b^{\mu}_{m+s}, b^{\nu}_{r+k}\}}_{\eta^{\mu\nu}\delta_{m+s+r+k}} - (s+\tfrac{m}{2})\underbrace{\{b^{\mu}_{-s}, b^{\nu}_{r+k}\}}_{\eta^{\mu\nu}\delta_{-s+r+k}}\alpha_{-k,\nu}\, b_{m+s,\mu}$$

$$= \frac{1}{2}\sum_{p\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \Big((p+m)\alpha_{-p}\cdot b_{r+k}\,\delta_{p+m-k} - p\,\alpha_{p+m}\cdot b_{r+k}\,\delta_{-p-k}\Big)$$

$$+ \frac{1}{2}\sum_{s\in\mathbb{Z}+\phi}\sum_{k\in\mathbb{Z}} \Big((s+\tfrac{m}{2})\,\alpha_{-k}\cdot b_{-s}\,\delta_{m+s+r+k} - (s+\tfrac{m}{2})\,\alpha_{-k}\cdot b_{m+s}\,\delta_{-s+r+k}\Big)$$

$$= \frac{1}{2}\sum_{k\in\mathbb{Z}} k\,\alpha_{m-k}\cdot b_{r+k} + \frac{1}{2}\sum_{k\in\mathbb{Z}} k\,\alpha_{-k+m}\cdot b_{r+k}.$$

$$+ \frac{1}{2}\sum_{k\in\mathbb{Z}} (-r-k-\tfrac{m}{2})\,\alpha_{-k}\cdot b_{m+r+k}$$

$$+ \frac{1}{2}\sum_{k\in\mathbb{Z}} (-r-k-\tfrac{m}{2})\,\alpha_{-k}\cdot b_{m+r+k}.$$

$$= \frac{1}{2}\sum_{k\in\mathbb{Z}} (m+k)\alpha_{-k}\cdot b_{r+m+k} + \frac{1}{2}\sum_{k\in\mathbb{Z}} (m+k)\alpha_{-k}\cdot b_{r+m+k}$$

$$+ \sum_{k\in\mathbb{Z}} \left(-r-k-\frac{m}{2}\right)\alpha_{-k}\cdot b_{r+m+k}$$

⇐ (we've ~~redfi~~ redefined $-q = m-k$ then $q \to m$ for the first line)

$$= \sum_{k\in\mathbb{Z}} \left(m+k-r-k-\frac{m}{2}\right)\alpha_{-k}\cdot b_{r+m+k}$$

$$= \left(\frac{m}{2}-r\right)\sum_{k\in\mathbb{Z}} \alpha_{-k}\cdot b_{r+m+k}$$

$$= \left(\frac{m}{2}-r\right) G_{m+r}$$

(No normal ordering ambiguity since LHS and RHS are both normal orded, so no problem in the middle)

Now consider L_0

$$[L_0, G_r] = \Big[\frac{1}{2}\sum_{p=-\infty}^{0} \alpha_p\cdot\alpha_{-p} + \frac{1}{2}\sum_{p=1}^{\infty} \alpha_{-p}\cdot\alpha_p - \frac{1}{2}\sum_{s=-\infty-\phi}^{-\phi} (s)\, b_s\cdot b_{-s}$$

$$+ \frac{1}{2}\sum_{p=1-\phi}^{\infty-\phi} s\, b_{-s}\cdot b_s \, , \, G_r \Big]$$

recall that $[\alpha_n^\mu, G_s] = n\, b^\mu_{n+s}$

$$\{b^\mu_r, G_s\} = \alpha^\mu_{r+s}$$

and that $[AB, C] = A\{B,C\} - \{A,C\}B$

we then have

$$\therefore [L_0, G_r] = \frac{1}{2}\sum_{p=-\infty}^{0} (\alpha_p \cdot \underbrace{[\alpha_{-p}, G_r]}_{(-p) b_{-p+r}} + \underbrace{[\alpha_p, G_r]}_{p b_{p+r}} \cdot \alpha_{-p})$$

$$+ \frac{1}{2}\sum_{p=1}^{\infty} (\alpha_{-p} \cdot \underbrace{[\alpha_p, G_r]}_{p b_{p+r}} + \underbrace{[\alpha_{-p}, G_r]}_{(-p) b_{-p+r}} \cdot \alpha_p)$$

$$- \frac{1}{2}\sum_{s=-\infty-\phi}^{-\phi} (s b_s \cdot \underbrace{\{b_{-s}, G_r\}}_{\alpha_{-s+r}} - s \underbrace{\{b_s, G_r\}}_{\alpha_{s+r}} \cdot b_{-s})$$

$$+ \frac{1}{2}\sum_{s=1-\phi}^{\infty-\phi} (s b_{-s} \underbrace{\{b_s, G_r\}}_{\alpha_{s+r}} - s \underbrace{\{b_{-s}, G_r\}}_{\alpha_{-s+r}} b_s)$$

$$= \frac{1}{2}\sum_{p=-\infty}^{0} -p\alpha_p \cdot b_{-p+r} + \frac{1}{2}\sum_{p=-\infty}^{0} p b_{p+r} \cdot \alpha_{-p}$$

$$+ \frac{1}{2}\sum_{p=1}^{\infty} p\alpha_{-p} \cdot b_{p+r} + \frac{1}{2}\sum_{p=1}^{\infty} (-p) b_{-p+r} \cdot \alpha_p$$

$$- \frac{1}{2}\sum_{s=-\infty-\phi}^{-\phi} s b_s \cdot \alpha_{-s+r} + \frac{1}{2}\sum_{s=-\infty-\phi}^{-\phi} s\alpha_{s+r} \cdot b_{-s}$$

$$+ \frac{1}{2}\sum_{s=1-\phi}^{\infty-\phi} s b_{-s} \cdot \alpha_{s+r} - \frac{1}{2}\sum_{s=1-\phi}^{\infty-\phi} s\alpha_{-s+r} \cdot b_s$$

$$= \sum_{p=-\infty}^{0} p b_{p+r} \cdot \alpha_{-p} + \sum_{p=1}^{\infty} p\alpha_{-p} \cdot b_{p+r}$$

$$\sum_{s=1-\phi}^{\infty-\phi} s b_{-s} \cdot \alpha_{s+r} + \sum_{s=-\infty-\phi}^{-\phi} s\alpha_{s+r} \cdot b_{-s}$$

$$= \sum_{p=-\infty}^{\infty} p b_{p+r} \cdot \alpha_{-p} + \sum_{s \in \mathbb{Z}+\phi} s b_{-s} \cdot \alpha_{s+r}$$

(redefine $s = -p' - r \iff p' = -s - r$, and $\because s, r \in \mathbb{Z}+\phi \therefore p' \in \mathbb{Z}$

$$\Rightarrow = \sum_{p \in \mathbb{Z}} p\alpha_{-p} \cdot b_{p+r} + \sum_{p' \in \mathbb{Z}} (-p' - r)\alpha_{-p'} b_{p'+r}$$

$\underline{p' \mapsto p}$

$$= \sum_{p\in\mathbb{Z}} (p-p-r)\,\alpha_{-p}\cdot b_{p+r} \qquad = \underline{-r\,G_r}$$

$\therefore$ overall $\qquad [L_m, G_r] = (\frac{m}{2} - r)\, G_{m+r}$

And ~~the full~~ No normal ordering central charge in this case because ~~both~~ the equation is normal ordered both at the beginning and at the end.

The full super-Virasoro algebra is:

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m(m^2-2\phi)\,\delta_{m+n} \\ [L_m, G_r] = (\frac{m}{2}-r)\,G_{m+r} \\ \{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}(r^2 - \frac{\phi}{2})\,\delta_{r+s} \end{cases}$$

□

2 $S = S_B + S_F = \frac{1}{2\pi}\int d^2\sigma\left(\frac{2}{\alpha'}\partial_+ X\cdot\partial_- X + i(\psi_+\cdot\partial_-\psi_+ + \psi_-\cdot\partial_+\psi_-)\right)$

supersymmetry transformations:

$$\delta X^\mu = i\sqrt{\frac{\alpha'}{2}}\left(\epsilon^+\psi_+^\mu + \epsilon^-\psi_-^\mu\right)$$

$$\delta\psi_+^\mu = -\sqrt{\frac{2}{\alpha'}}\,\epsilon^+\partial_+ X^\mu$$

$$\delta\psi_-^\mu = -\sqrt{\frac{2}{\alpha'}}\,\epsilon^-\partial_- X^\mu$$

under this, we have:

$$\delta S = \frac{1}{2\pi}\int d^2\sigma\Big(\frac{2}{\alpha'}\partial_+\delta X^\mu\partial_- X_\mu + \frac{2}{\alpha'}\partial_+ X^\mu\partial_-\delta X_\mu$$

$$+ i\,(\delta\psi_+^\mu\partial_-\psi_{+,\mu} + \psi_+^\mu\partial_-\delta\psi_{+,\mu}$$

$$+ \delta\psi_-^\mu\partial_+\psi_{-,\mu} + \psi_-^\mu\partial_+\delta\psi_{-,\mu})\Big)$$

$$= \frac{i}{2\pi}\left(-\frac{1}{2}\right)\sqrt{\frac{2}{\alpha'}}\int d\sigma^+ d\sigma^-\Big(\partial_+(\epsilon^+\psi_+^\mu)\partial_- X_\mu +$$

$$\partial_+(\epsilon^-\psi_-^\mu)\partial_- X_\mu + \partial_+ X^\mu\partial_-(\epsilon^+\psi_{+,\mu})$$

$$+ \partial_+ X^\mu\partial_-(\epsilon^-\psi_{-,\mu}) - \epsilon^+\partial_+ X^\mu\partial_-\psi_{+,\mu}$$

$$- \psi_+^\mu\partial_-(\epsilon^+\partial_+ X_\mu) - \epsilon^-\partial_- X^\mu\partial_+\psi_{-,\mu}$$

$$- \psi_-^\mu\partial_+(\epsilon^-\partial_- X_\mu)\Big)$$

(use $\epsilon\psi = -\psi\epsilon$ and integrate by parts)

$$= \frac{i}{2\pi}\left(-\frac{1}{2}\right)\sqrt{\frac{2}{\alpha'}}\int d\sigma^+ d\sigma^- \Big(-\epsilon^+ \psi_+^\mu \partial_+\partial_- X_\mu$$

$$- \epsilon^- \psi_-^\mu \partial_+\partial_- X_\mu - \epsilon^+ \psi_{+,\mu} \partial_-\partial_+ X_\mu$$

$$- \epsilon^- \psi_-^\mu \partial_-\partial_+ X_\mu$$

$$+ \partial_-(\epsilon^+ \partial_+ X^\mu)\psi_{+,\mu} + \partial_-(\epsilon^+ \partial_+ X_\mu)\psi_+^\mu$$

$$+ \partial_+(\epsilon^- \partial_- X^\mu)\psi_{-,\mu} + \partial_+(\epsilon^- \partial_- X_\mu)\psi_-^\mu \Big)$$

(use $\partial_+ \epsilon^- = \partial_- \epsilon^+ = 0$, $\partial_+\partial_- = \partial_-\partial_+$)

$$= \frac{i}{2\pi}\left(-\frac{1}{2}\right)\sqrt{\frac{2}{\alpha'}}\int d\sigma^+ d\sigma^- \Big(-\epsilon^+ \psi_+^\mu \partial_+\partial_- X_\mu$$

$$- \epsilon^- \psi_-^\mu \partial_+\partial_- X_\mu - \epsilon^+ \psi_+^\mu \partial_+\partial_- X_\mu$$

$$- \epsilon^- \psi_-^\mu \partial_+\partial_- X_\mu$$

$$+ \epsilon^+ \psi_+^\mu \partial_+\partial_- X_\mu + \epsilon^+ \psi_+^\mu \partial_+\partial_- X_\mu$$

$$+ \epsilon^- \psi_-^\mu \partial_+\partial_- X_\mu + \epsilon^- \psi_-^\mu \partial_+\partial_- X_\mu \Big)$$

$= 0 \Rightarrow S$ invariant under <u>SUSY</u>

3

1. we can use reparametrisation and Lorentz.

$$\delta_\xi e^a_\alpha = -\xi^\beta \partial_\beta e^a_\alpha - e^a_\beta \partial_\alpha \xi^\beta ,$$

$$\delta_l e^a_\alpha = l \,\varepsilon^a{}_b e^b_\alpha$$

to bring zweibein e^a_α to the form $e^a_\alpha = e^\phi \delta^a_\alpha$

and weyl ~~...~~ transformation to bring ϕ to 0 so $e^a_\alpha = \delta^a_\alpha$

2. write the gravitino as

$$\chi_\alpha = h_\alpha{}^\beta \chi_\beta = (h_\alpha{}^\beta - \tfrac{1}{2}\rho_\alpha \rho^\beta)\chi_\beta + \tfrac{1}{2}\rho_\alpha \rho^\beta \chi_\beta$$

$\{\rho^\alpha, \rho^\beta\} = 2h^{\alpha\beta}$ →

$$= \tfrac{1}{2}\rho^\beta \rho_\alpha \chi_\beta + \tfrac{1}{2}\rho_\alpha \rho^\beta \chi_\beta$$

$$= \tilde{\chi}_\alpha + \rho_\alpha \lambda \quad \text{where } \tilde{\chi}_\alpha \equiv \tfrac{1}{2}\rho^\beta \rho_\alpha \chi_\beta ,\ \lambda \equiv \tfrac{1}{2}\rho^\alpha \chi_\alpha$$

gravitino SUSY transformation is

$$\delta_\epsilon \chi_\alpha = 2\nabla_\alpha \epsilon \equiv 2(T\epsilon)_\alpha + \rho_\alpha \rho^\beta \nabla_\beta \epsilon$$

where $(T\epsilon)_\alpha = (h_\alpha{}^\beta - \tfrac{1}{2}\rho_\alpha \rho^\beta)\nabla_\beta \epsilon = \tfrac{1}{2}\rho^\beta \rho_\alpha \nabla_\beta \epsilon .$

$\{\rho^\alpha, \rho^\beta\} = h^{\alpha\beta} \cdot 2 .$

Locally $\tilde{\chi}_\alpha = \rho^\beta \rho_\alpha \nabla_\beta \kappa$ for some spinor κ.

using identity $\rho^\alpha \rho_\beta \rho_\alpha = 0$, so SUSY transformation $\delta_\epsilon \chi_\alpha$ eliminates κ and $\tilde{\chi}_\alpha$ term goes away.

so $\chi_\alpha = \rho_\alpha \lambda$

Weyl rescaling can set λ to 0 so we can have $\chi_\alpha = 0$

3. energy momentum tensor

$$T_{\alpha\beta} = \frac{2\pi}{e}\frac{\delta S}{\delta e^\beta_a} e_{\alpha a} = -\frac{1}{\alpha'}(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\partial^\gamma X^\mu \partial_\gamma X_\mu)$$
$$+\frac{i}{4}(\bar\psi^\mu \rho_\alpha \partial_\beta \psi_\mu + \bar\psi^\mu \rho_\beta \partial_\alpha \psi_\mu)$$

where we used $e = \sqrt{-h}$, $h^{\alpha\beta} = \eta^{ab} e^\alpha_a e^\beta_b$,

and $\delta h = -h h_{\alpha\beta}\delta h^{\alpha\beta}$.

equation of motion for e is $\frac{\delta S}{\delta e^\beta_\alpha} = 0 \Rightarrow T_{\alpha\beta} = 0$

so $T_{++} = T_{--} = 0$ ∵ transforming to light-cone coordinates only involves take linear conbinations of $T_{\alpha\beta}$.

- super-current $J_\alpha = \frac{2\pi}{e}\frac{\delta S}{i\delta\bar\chi^\alpha}$,

$$= -\frac{1}{4}\sqrt{\frac{2}{\alpha'}}\rho^\beta\rho_\alpha\psi^\mu\partial_\beta X_\mu$$

Equation of motion for χ is $\frac{\delta S}{\delta\bar\chi^\alpha} = 0 \Rightarrow J_\alpha = 0$.

$\Rightarrow J_\pm = 0$