Oxford MMathPhys

String Theory I

Problem Sheet 4 Solutions

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Q1

convenient symbol $D_\mu \equiv \nabla_\mu$

$$\beta^G_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho}$$

$$\beta^B_{\mu\nu} = -\frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} + \nabla^\lambda \Phi H_{\lambda\mu\nu}$$

$$\beta^\Phi = \frac{D-26}{6} + \alpha'\left(-\frac{1}{2}\nabla^2\Phi + \nabla_\lambda\Phi\nabla^\lambda\Phi - \frac{1}{24} H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right)$$

$$H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$

1) the action

$$S_{26} = \frac{1}{2\kappa_0^2}\int d^Dx \sqrt{-G}\, e^{-2\Phi}\left[-\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi + O(\alpha')\right]$$

To find δS_{26}, we individually vary G, B and Φ and then

$$\delta S_{26} = \frac{\delta S_{26}}{\delta G_{\mu\nu}}\delta G_{\mu\nu} + \frac{\delta S_{26}}{\delta B_{\mu\nu}}\delta B_{\mu\nu} + \frac{\delta S_{26}}{\delta \Phi}\delta\Phi$$

First we vary $B_{\mu\nu}$:

$$\frac{\delta S_{26}}{\delta B_{\mu\nu}} \delta B_{\mu\nu} = \delta S \Big|_{G, \Phi}$$

$$\delta S \Big|_{G,\Phi} = \frac{1}{2\kappa_0^2} \int d^D x \, \delta\left(\sqrt{-G}\, e^{-2\Phi} \left(-\frac{1}{12}\right) H_{\mu\nu\rho} H^{\mu\nu\rho} \right)$$

$$= \frac{1}{2\kappa_0^2} \int d^D x \left(-\frac{1}{12}\right) \sqrt{-G}\, e^{-2\Phi}\, \delta\left(H_{\mu\nu\rho} H^{\mu\nu\rho} \right)$$

$$= \frac{1}{2\kappa_0^2} \int d^D x \left(-\frac{1}{12}\right) \sqrt{-G}\, e^{-2\Phi}\, \delta\left(G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} H_{\mu\nu\rho} H_{\mu'\nu'\rho'} \right)$$

$$= \frac{1}{2\kappa_0^2} \int d^D x \left(-\frac{1}{12}\right) \sqrt{-G}\, e^{-2\Phi}\, G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} \delta\left(H_{\mu\nu\rho} H_{\mu'\nu'\rho'} \right)$$

$$= \frac{1}{2\kappa_0^2} \int d^D x \left(-\frac{1}{12}\right) \sqrt{-G}\, e^{-2\Phi}\, G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} \Big[\delta H_{\mu\nu\rho} \cdot H_{\mu'\nu'\rho'}$$

$$+ H_{\mu\nu\rho} \cdot \delta_{\mu'\nu'\rho'} \Big]$$

$$\underbrace{=}_{G \text{ is symmetric}} \frac{1}{2\kappa_0^2} \int d^D x \left(-\frac{1}{12}\right) \sqrt{-G}\, e^{-2\Phi}\, G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} \Big[2 \times \delta H_{\mu\nu\rho} \cdot H_{\mu'\nu'\rho'} \Big]$$

$$\Rightarrow \quad \delta H_{\mu\nu\rho} = \delta\left(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \right)$$

$$= \partial_\mu \delta B_{\nu\rho} + \partial_\nu \delta B_{\rho\mu} + \partial_\rho \delta B_{\mu\nu}$$

&

we know $\delta B_{\mu\nu}$ is antisymmetric, just like $B_{\mu\nu}$

$\Rightarrow$

$$\therefore\ \nabla_\mu \delta B_{\nu\rho} + \nabla_\nu \delta B_{\rho\mu} + \nabla_\rho \delta B_{\mu\nu}$$

$$\begin{aligned}
= \cancel{\partial_\mu B}\ \partial_\mu \delta B_{\nu\rho} &- \cancel{\Gamma^\lambda{}_{\mu\nu} \delta B_{\lambda\rho}} - \Gamma^\lambda{}_{\mu\rho} \delta B_{\nu\lambda} \\
+ \partial_\nu \delta B_{\rho\mu} &- \Gamma^\lambda{}_{\nu\rho} \delta B_{\lambda\mu} - \cancel{\Gamma^\lambda{}_{\nu\mu} \delta B_{\rho\lambda}} \\
+ \partial_\rho \delta B_{\mu\nu} &- \Gamma^\lambda{}_{\rho\mu} \delta B_{\lambda\nu} - \Gamma^\lambda{}_{\rho\nu} \delta B_{\mu\lambda}
\end{aligned}$$

cancellation due to $\Gamma^\lambda{}_{\mu\nu}$ symmetric in $\mu\nu$ and $\delta B_{\mu\nu}$ antisymmetric.

$$= \partial_\mu \delta B_{\nu\rho} + \partial_\nu \delta B_{\rho\mu} + \cancel{\delta_\rho}\ \partial_\rho \delta B_{\mu\nu}$$

$$= \delta H_{\mu\nu\rho}$$

$$\therefore\ \delta S\big|_{G,\Phi} = \frac{1}{2\kappa_0^2}\int d^Dx \left(-\frac{1}{6}\right)\sqrt{-G}\, e^{-2\Phi}\, \delta H_{\mu\nu\rho} \cdot H^{\mu\nu\rho}$$

$$= \frac{1}{2\kappa_0^2}\int d^Dx \left(-\frac{1}{6}\right)\sqrt{-G}\, e^{-2\Phi} H^{\mu\nu\rho} \left(\nabla_\mu \delta B_{\nu\rho} + \nabla_\nu \delta B_{\rho\mu} + \nabla_\rho \delta B_{\mu\nu}\right)$$

$$\underbrace{=}_{H^{\mu\nu\rho}\text{ is totally antisymmetric}} \frac{1}{2\kappa_0^2}\int d^Dx \left(-\frac{1}{6}\times 3\right)\sqrt{-G}\, e^{-2\Phi} H^{\mu\nu\rho}\, \nabla_\mu \delta B_{\nu\rho}$$

$$= \frac{1}{2\kappa_0^2}\int d^Dx \left(-\frac{1}{2}\right)\sqrt{-G}\Big[\nabla_\mu \left(e^{-2\Phi} H^{\mu\nu\rho}\, \delta B_{\nu\rho}\right) - \nabla_\mu \left(e^{-2\Phi} H^{\mu\nu\rho}\right)\delta B_{\nu\rho}\Big]$$

$\because \int d^D x \sqrt{-G}\, \nabla_\mu (\qquad)^\mu = 0$ (divergence theorem)

$$\therefore \delta S|_{G,\Phi} = \frac{1}{2\kappa_0^2}\int d^D x \sqrt{-G}\,(+\tfrac{1}{2})\, \nabla_\mu (e^{-2\Phi} H^{\mu\nu\rho})\, \delta B_{\nu\rho}$$

$$= \frac{1}{2\kappa_0^2}\int d^D x \sqrt{-G}\,(\tfrac{1}{2})\,(-2e^{-2\Phi}\nabla_\mu \Phi \cdot H^{\mu\nu\rho} + e^{-2\Phi}\nabla_\mu H^{\mu\nu\rho})\,\delta B_{\mu\rho}$$

$$= -\frac{1}{2\kappa_0^2}\int d^D x \sqrt{-G}\, e^{-2\Phi}\,\underbrace{(-\tfrac{1}{2}\nabla_\mu H^{\mu\nu\rho} + \nabla_\mu \Phi H^{\mu\nu\rho})}_{\equiv \beta^{B,\nu\rho}}\,\delta B_{\nu\rho}$$

$$= -\frac{1}{2\kappa_0^2}\int d^D x \sqrt{-G}\, e^{-2\Phi}\, \beta^{B,\mu\nu}\, \delta B_{\mu\nu}$$

$$\frac{\delta S}{\delta \Phi}\delta\Phi = \delta G|_{G,B}$$

$$= \frac{1}{2\kappa_0^2}\int d^D x \sqrt{-G}\; \delta(e^{-2\Phi})\Big[-\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}$$
$$+ 4\partial_\mu\Phi\,\partial^\mu\Phi\Big] + \sqrt{-G}\,e^{-2\Phi}\big[\delta(4\partial_\mu\Phi\,\partial^\mu\Phi)\big]$$

$$\because \delta(e^{-2\Phi}) = -2e^{-2\Phi}\,\delta\Phi = e^{-2\Phi}(-2\delta\Phi)$$

$$\therefore \delta S|_{G,B} = \frac{1}{2\kappa_0^2}\int d^D x \sqrt{-G}\, e^{-2\Phi}\Big[\Big(-\frac{2(D-26)}{3\alpha'}$$

$$+ R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \cancel{\Phi}\, 4\partial_\mu \Phi \partial^\mu \Phi \Big) (-2)\,\delta\Phi + \delta(4\partial_\mu\Phi\partial^\mu\Phi)\Big]$$

$$\delta(4\partial_\mu\Phi\partial^\mu\Phi) = 4\,\delta(\partial_\mu\Phi\partial^\mu\Phi)$$

$$= 4G^{\mu\nu}\left(\partial_\mu\delta\Phi\,\partial_\nu\Phi + \partial_\mu\Phi\,\partial_\nu\delta\Phi\right)$$

$$\underbrace{=}_{G^{\mu\nu}\ \text{symmetric}} \; 8G^{\mu\nu}\,\partial_\mu\delta\Phi\,\partial_\nu\Phi = 8G^{\mu\nu}\,\nabla_\mu\Phi\,\nabla_\nu\delta\Phi$$

$$= \cancel{8G^{\mu\cdot}}\; 8\,\nabla^\mu\Phi\,\nabla_\mu\delta\Phi$$

$$\Rightarrow \delta S\big|_{G,B} = \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\Big[-2\Big(-\frac{2(D-26)}{3\alpha'}$$

$$+ R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\nabla_\mu\Phi\nabla^\mu\Phi\Big]\delta\Phi$$

$$+ 8\,\nabla^\mu\Phi\,\nabla_\mu\delta\Phi\Big]$$

$$\Rightarrow \int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\,8\,\nabla^\mu\Phi\,\nabla_\mu\delta\Phi$$

$$= \int d^Dx\,\sqrt{-G}\,\nabla_\mu\left(e^{-2\Phi}\,8\,\nabla^\mu\Phi\cdot\delta\Phi\right) \quad 0$$

$$- \int d^Dx\,\sqrt{-G}\,\nabla_\mu\left(e^{-2\Phi}\,8\,\nabla^\mu\Phi\right)\delta\Phi.$$

$$= -\int d^Dx\,\sqrt{-G}\,\Big(-2e^{-2\Phi}\,\nabla_\mu\Phi\cdot 8\,\nabla^\mu\Phi$$

$$+ e^{-2\Phi}\cdot 8\,\nabla^2\Phi\Big)\delta\Phi$$

$$= -\int d^D x \sqrt{-G} e^{-2\Phi} (-16 \nabla^\mu \Phi \nabla_\mu \Phi + 8 \nabla^2 \Phi) \delta \Phi$$

$$\therefore \quad \delta S \Big|_{G,B} = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G} e^{-2\Phi} \Big\{ \frac{4(D-26)}{3\alpha'} - 2R$$

$$+ \frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} - 8 \nabla_\mu \Phi \nabla^\mu \Phi + 16 \nabla^\mu \Phi \nabla_\mu \Phi - 8 \nabla^2 \Phi \Big\} \delta \Phi$$

$$= -\frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G} e^{-2\Phi} \Big\{ -\frac{4(D-26)}{3\alpha'} + 2R$$

$$- \frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} - 8 \nabla_\mu \Phi \nabla^\mu \Phi + 8 \nabla^2 \Phi \Big\} \delta \Phi$$

$$\Longrightarrow \quad 2 \beta^{G}{}^\lambda{}_\lambda - 8 \frac{1}{\alpha'} \beta^\Phi$$

$$= 2R + 4\nabla^2 \Phi - \frac{1}{2} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{4}{3\alpha'} (D-26)$$

$$+ 4 \nabla^2 \Phi - 8 \nabla_\mu \Phi \nabla^\mu \Phi + \frac{1}{3} H_{\mu\nu\rho} H^{\mu\nu\rho}$$

$$= -\frac{4(D-26)}{3\alpha'} + 2R - \frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} + 8 \nabla^2 \Phi$$

$$- 8 \nabla^\mu \Phi \nabla_\mu \Phi$$

$$\Longrightarrow \quad \delta S \Big|_{G,B} = -\frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G} e^{-2\Phi} \Big(2\delta\Phi \Big($$

$$\beta^{G}{}^\lambda{}_\lambda - \frac{4}{\alpha'} \beta^\Phi \Big) \Big)$$

$$\frac{\delta S}{\delta G_{\mu\nu}} \delta G_{\mu\nu} = \delta S \Big|_{B,\Phi}$$

$$= \frac{1}{2\kappa_0^2} \int d^D x \, \delta(\sqrt{-G} e^{-2\Phi}) \Big[-\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}$$

$$+ 4 \partial_\mu \Phi \partial^\mu \Phi \Big] + \sqrt{-G} e^{-2\Phi} \Big[\delta R - \frac{1}{12} \delta(H_{\mu\nu\rho} H^{\mu\nu\rho})$$

$$+ 4 \delta(\partial_\mu \Phi \partial^\mu \Phi) \Big]$$

This calculation is terribly complicated and people (David Tong, Joseph Polchinski, Christopher Beem, ...) rarely bother to do a full calculation. Hopefully mine is detailed enough.

Consider metric $G_{\mu\nu}$,

$G = \det(G_{\mu\nu}) = \sum_\nu G_{\mu\nu} \Delta^{\mu\nu}$ where $\Delta^{\mu\nu}$ is the cofactor matrix of $G_{\mu\nu}$ whose $\mu\nu$ element is $(-1)^{\mu+\nu}$ times the determinant of the matrix obtained by deleting μ th row and ν th column of $G_{\mu\nu}$ $\Rightarrow$ $\cancel{\frac{\partial \Delta^{\mu\nu}}{\partial G_{\mu\nu}} = 0}$ $\frac{\partial \Delta^{\mu\nu}}{\partial G^{\mu\nu}} = 0$ (NO SUM)

$$\underbrace{\Delta^{\mu\nu} = G G^{\mu\nu}}_{\text{linear algebra.}} = \frac{\partial G}{\partial G_{\mu\nu}}$$

$$\therefore \delta G = \frac{\partial G}{\partial G_{\mu\nu}} \delta G_{\mu\nu} = G\, G^{\mu\nu} \delta G_{\mu\nu}$$

$$\therefore \delta \sqrt{-G} = \frac{1}{2} \frac{-G}{\sqrt{-G}} G^{\mu\nu} \delta G_{\mu\nu} = \frac{1}{2} \sqrt{-G}\, G^{\mu\nu} \delta G_{\mu\nu}$$

first note that

$$\Gamma'^{a}{}_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^f}{\partial x'^b} \frac{\partial x^g}{\partial x'^c} \Gamma^{d}{}_{fg} - \frac{\partial x^d}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x^d \partial x^f}$$

$$\Gamma'^{a}{}_{bc} + \delta \Gamma'^{a}{}_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^f}{\partial x'^b} \frac{\partial x^g}{\partial x'^c} \left(\Gamma^{d}{}_{fg} + \delta \Gamma^{d}{}_{fg} \right)$$
$$- \frac{\partial x^d}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x^d \partial x^f}$$

$$\Rightarrow \quad \delta \Gamma'^{a}{}_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^f}{\partial x'^b} \frac{\partial x^g}{\partial x'^c} \delta \Gamma^{d}{}_{fg}$$

$\Rightarrow$ $\delta \Gamma^{a}{}_{bc}$ is a tensor.

$$\therefore \cancel{\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\rho G_{\sigma\nu} + \partial_\nu G_{\sigma\rho} - \partial_\sigma G_{\nu\rho} \right)}$$

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} G^{\mu\sigma} \left(\partial_\rho G_{\sigma\nu} + \partial_\nu G_{\sigma\rho} - \partial_\sigma G_{\nu\rho} \right)$$

Easy to evaluate if we introduce normal coordinates at p for the unperturbed connection: at p $\partial_\rho G_{\mu\nu} = 0$ and $\Gamma^{\mu}{}_{\nu\rho} = 0$

(see ~~past~~ Cambridge Part III GR Notes)

$$\Rightarrow \quad \delta \Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} G^{\mu\sigma} \left(\partial_\rho \delta G_{\sigma\nu} + \partial_\nu \delta G_{\sigma\rho} - \partial_\sigma \delta G_{\nu\rho} \right)$$

at p.

$\because \Gamma = 0 \quad \therefore \delta\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} G^{\mu\sigma} (\nabla_\rho \delta G_{\sigma\nu} + \nabla_\nu \delta G_{\sigma\rho} - \nabla_\sigma \delta G_{\nu\rho})$

at p $\because$ Both sides are tensors $\therefore$ this equation is true everywhere

$$\Rightarrow \delta\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} G^{\mu\sigma} (\nabla_\rho \delta G_{\sigma\nu} + \nabla_\nu \delta G_{\sigma\rho} - \nabla_\sigma \delta G_{\nu\rho})$$

$$\because R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\tau}{}_{\nu\sigma}\Gamma^{\mu}{}_{\tau\rho} - \Gamma^{\tau}{}_{\nu\rho}\Gamma^{\mu}{}_{\tau\sigma}$$

At p, $\Gamma = 0$ $\therefore$ $\delta R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \delta\Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \delta\Gamma^{\mu}{}_{\nu\rho}$

$$= \nabla_\rho \delta\Gamma^{\mu}{}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^{\mu}{}_{\nu\rho}$$

Both sides are tensors $\Rightarrow$ $\delta R^{\mu}{}_{\nu\rho\sigma} = \nabla_\rho \delta\Gamma^{\mu}{}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^{\mu}{}_{\nu\rho}$ always.

$$\therefore \delta R = \delta(G^{\mu\nu} R_{\mu\nu}) = \delta G^{\mu\nu} R_{\mu\nu} + G^{\mu\nu} \delta R_{\mu\nu}$$

$\cancel{\delta R_{\mu\nu}} =$ $\delta R_{\nu\sigma} = \nabla_\rho \delta\Gamma^{\rho}{}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^{\rho}{}_{\nu\rho}$

$$\therefore \delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^{\rho}{}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\rho}{}_{\mu\rho}$$

$$\therefore G^{\mu\nu}\delta R_{\mu\nu} = G^{\mu\nu}\nabla_\rho \delta\Gamma^{\rho}{}_{\mu\nu} - G^{\mu\nu}\nabla_\nu \delta\Gamma^{\rho}{}_{\mu\rho}$$

$$= \nabla_\rho (G^{\mu\nu}\delta\Gamma^{\rho}{}_{\mu\nu}) - \nabla_\nu (G^{\mu\nu}\delta\Gamma^{\rho}{}_{\mu\rho})$$

$(\nu \leftrightarrow \rho)$ $= \nabla_\rho (G^{\mu\nu}\delta\Gamma^{\rho}{}_{\mu\nu} - G^{\mu\rho}\delta\Gamma^{\nu}{}_{\mu\nu})$

$$\because \delta(G^{\mu\nu} G_{\nu\rho}) = \delta(\delta^{\mu}{}_{\rho}) = 0$$

$$\therefore G^{\mu\nu}\cdot\delta G_{\nu\rho} + \delta G^{\mu\nu}\cdot G_{\nu\rho} = 0$$

$$\therefore G^{\mu\nu} G^{\rho\sigma}\delta G_{\nu\rho} = -\delta G^{\mu\nu}\cdot G_{\nu\rho} G^{\rho\sigma} = -\delta G^{\mu\nu}\delta_{\nu}{}^{\sigma}$$

$$= -\delta G^{\mu\sigma}$$

$$\therefore\ \delta G^{\nu\sigma} = -G^{\mu\nu} G^{\rho\sigma} \delta G_{\nu\rho}$$

$$\therefore\ \delta G^{\mu\nu} = -G^{\mu\rho} G^{\nu\sigma} \delta G_{\rho\sigma}$$

$$\therefore\ \delta R = -G^{\mu\rho} G^{\nu\sigma} \delta G_{\rho\sigma} R_{\mu\nu} + \nabla_\rho \left(G^{\mu\nu} \delta\Gamma^{\rho}_{\mu\nu} - G^{\mu\rho} \delta\Gamma^{\nu}_{\mu\nu} \right)$$

$$= -R^{\mu\nu} \delta G_{\mu\nu} + \nabla_\rho \left(G^{\mu\nu} \delta\Gamma^{\rho}_{\mu\nu} - G^{\mu\rho} \delta\Gamma^{\nu}_{\mu\nu} \right)$$

$$\rightarrow\ \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \delta R$$

$$= \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \left(-R^{\mu\nu} \delta G_{\mu\nu} + \nabla_\rho \left(G^{\mu\nu} \delta\Gamma^{\rho}_{\mu\nu} - G^{\mu\rho} \delta\Gamma^{\nu}_{\mu\nu} \right) \right)$$

$$\Rightarrow\ \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \nabla_\rho \left(G^{\mu\nu} \delta\Gamma^{\rho}_{\mu\nu} - G^{\mu\rho} \delta\Gamma^{\nu}_{\mu\nu} \right)$$

$$\Rightarrow\ \nabla_\rho \left(G^{\mu\nu} \delta\Gamma^{\rho}_{\mu\nu} - G^{\mu\rho} \delta\Gamma^{\nu}_{\mu\nu} \right)$$

$$= \cancel{G_\rho}\ \nabla_\mu \left(G^{bc} \delta\Gamma^{\mu}_{bc} - G^{\mu b} \delta\Gamma^{c}_{bc} \right)$$

$$= \nabla_\mu \Big[\frac{1}{2} G^{bc} G^{\mu d} \left(\nabla_c \delta G_{db} + \nabla_b \delta G_{dc} - \nabla_d \delta G_{bc} \right) - \frac{1}{2} G^{\mu b} G^{cd} \left(\nabla_c \delta G_{db} + \nabla_b \delta G_{dc} - \nabla_d \delta G_{bc} \right)$$

$$= \nabla_\mu \Big[\frac{1}{2} G^{bc} G^{\mu d} \big(\cancel{\nabla_c \delta G_{db}} + \nabla_b \delta G_{dc} - \nabla_d \delta G_{bc} - \cancel{\nabla_c \delta G_{bd}} - \nabla_d \delta G_{bc} + \nabla_b \delta G_{dc} \big) \Big]$$

$$= \nabla_\mu G^{bc} G^{\mu d} \left(\nabla_b \delta G_{cd} - \nabla_d \delta G_{bc} \right)$$

$$\therefore \frac{1}{2\kappa_0^2}\int d^Dx \sqrt{-G}\, e^{-2\Phi}\, \delta R$$

$$= \frac{1}{2\kappa_0^2}\int d^Dx \sqrt{-G}\, e^{-2\Phi}\Big(-R^{\mu\nu}\delta G_{\mu\nu} + \nabla_\mu\big(G^{bc}G^{\mu d}(\nabla_b \delta G_{cd} - \nabla_d \delta G_{bc})\big)\Big)$$

$$\Rightarrow \frac{1}{2\kappa_0^2}\int d^Dx \sqrt{-G}\, e^{-2\Phi}\, \nabla_\mu\big(G^{bc}G^{\mu d}(\nabla_b \delta G_{cd} - \nabla_d \delta G_{bc})\big).$$

$$= \frac{1}{2\kappa_0^2}\int d^Dx \sqrt{-G}\, \nabla_\mu\big(e^{-2\Phi} G^{bc}G^{\mu d}(\nabla_b \delta G_{cd} - \nabla_d \delta G_{bc})\big). \rightarrow = 0$$

$$- \frac{1}{2\kappa_0^2}\int d^Dx \sqrt{-G}\, \underbrace{\nabla_\mu(e^{-2\Phi})}_{-2e^{-2\Phi}\nabla_\mu\Phi}\big(G^{bc}G^{\mu d}\big)\big(\nabla_b \delta G_{cd} - \nabla_d \delta G_{bc}\big)$$

$$= \frac{1}{2\kappa_0^2}\int d^Dx \sqrt{-G}\, e^{-2\Phi}\, 2\nabla_\mu\Phi\big(G^{bc}G^{\mu d}\big)\big(\nabla_b \delta G_{cd} - \nabla_d \delta G_{bc}\big)$$

$$= \frac{1}{2\kappa_0^2}\int d^Dx\, 2\sqrt{-G}\Big\{\nabla_b\big(e^{-2\Phi} G^{bc}G^{\mu d}\delta G_{cd}\nabla_\mu\Phi\big) \rightarrow = 0$$

$$- G^{bc}G^{\mu d}\nabla_b\big(e^{-2\Phi}\nabla_\mu\Phi\big)\delta G_{cd}$$

$$- \nabla_d\big(e^{-2\Phi} G^{bc}G^{\mu d}\delta G_{bc}\nabla_\mu\Phi\big) \rightarrow = 0$$

$$+ G^{bc}G^{\mu d}\nabla_d\big(e^{-2\Phi}\nabla_\mu\Phi\big)\delta G_{bc}.\Big\}$$

$$= \frac{1}{2\kappa_0^2}\int d^Dx \Big\{ -2\sqrt{-G}\, G^{bc} G^{\nu d}\left(-2e^{-2\Phi}\nabla_b\Phi\nabla_\nu\Phi + e^{-2\Phi}\nabla_b\nabla_\nu\right)\delta G_{cd}$$

$$+ 2\sqrt{-G}\, G^{bc} G^{\nu d}\left(-2e^{-2\Phi}\nabla_d\Phi\nabla_\nu\Phi + e^{-2\Phi}\nabla_d\nabla_\nu\Phi\right)\delta G_{bc}$$

$$= \frac{1}{2\kappa_0^2}\int d^Dx \sqrt{-G}\, e^{-2\Phi}\Big\{ 4\nabla^c\Phi\nabla^d\Phi\,\delta G_{cd}$$

$$-2\nabla^c\nabla^d\Phi\,\delta G_{cd} - 4\nabla_\nu\Phi\nabla^\nu\Phi\, G^{bc}\delta G_{bc}$$

$$+ 2\nabla^2\Phi\, G^{bc}\delta G_{bc}\Big\}$$

$$= \frac{1}{2\kappa_0^2}\int d^Dx \sqrt{-G}\, e^{-2\Phi}\Big\{\left(4\nabla^\mu\Phi\nabla^\nu\Phi - 2\nabla^\mu\nabla^\nu\Phi\right)\delta G_{\mu\nu}$$

$$+\left(-4\nabla_\rho\Phi\nabla^\rho\Phi + 2\nabla^2\Phi\right)G^{\mu\nu}\delta G_{\mu\nu}\Big\}$$

$$\delta(\sqrt{-G}\, e^{-2\Phi}) = e^{-2\Phi}\,\delta(\sqrt{-G})$$

$$= \sqrt{-G}\, e^{-2\Phi}\,\frac{1}{2} G^{\mu\nu}\delta G_{\mu\nu}$$

$$\delta(H_{\mu\nu\rho}H^{\mu\nu\rho}) = \delta(G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}H_{\mu\nu\rho}H_{\mu'\nu'\rho'})$$

$$= 3\,\delta G^{\mu\mu'}\, G^{\nu\nu'}G^{\rho\rho'}H_{\mu\nu\rho}H_{\mu'\nu'\rho'}$$

$H_{\mu\nu\rho} = H_{\nu\rho\mu} = H_{\rho\mu\nu}$

$$= 3\,\delta G^{\mu\mu'} H_{\mu\nu\rho} H_{\mu'}{}^{\nu\rho}$$

$$= -3\, G^{\mu a} G^{\mu' b} \delta G_{ab}\, H_{\mu\nu\rho} H_{\mu'}{}^{\nu\rho}$$

$$= -3\, H^{a}{}_{\nu\rho} H^{b\nu\rho}\, \delta G_{ab}$$

$$= -3 H^{\mu}{}_{\lambda\rho} H^{\nu\lambda\rho}\, \delta G_{\mu\nu}$$

$$\delta(\partial_\mu \Phi \partial^\mu \Phi) = \delta(\nabla_\mu \Phi \nabla^\mu \Phi)$$

$$= \delta(G^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi) = \delta G^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi$$

$$= -G^{\mu a} G^{\nu b} \delta G_{ab} \nabla_\mu \Phi \nabla_\nu \Phi$$

$$= -\nabla^a \Phi \nabla^b \Phi\, \delta G_{ab} = -\nabla^\mu \Phi \nabla^\nu \Phi\, \delta G_{\mu\nu}$$

$$\therefore\ \delta S\Big|_{B,\Phi} = -\frac{1}{2\kappa_0^2} \int d^D X \sqrt{-G}\, e^{-2\Phi} \Big[-\frac{1}{2} G^{\mu\nu} \delta G_{\mu\nu} \Big(-\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12} H_{\mu'\nu'\rho} H^{\mu'\nu'\rho} + 4\nabla_{\mu'}\Phi \nabla^{\mu'}\Phi$$

$$-8 \nabla_{\mu'}\Phi \nabla^{\mu'}\Phi + 4\nabla^2 \Phi \Big) + \delta G_{\mu\nu} \Big(\cancel{-4\nabla^\mu \Phi \nabla^\nu \Phi}$$

$$+ 2\nabla^\mu \nabla^\nu \Phi - \underbrace{\left(-\frac{1}{12}\right)(-3)}_{\frac{1}{4}} H^{\mu}{}_{\lambda\rho} H^{\nu\lambda\rho} + R^{\mu\nu} \cancel{+ 4\nabla^\mu \Phi \nabla^\nu \Phi} \Big) \Big]$$

$$= -\frac{1}{2\kappa_0} \int d^D x \sqrt{-G}\, e^{-2\Phi} \Big[-\frac{1}{2} G^{\mu\nu} \delta G_{\mu\nu} \Big(-\frac{2(D-26)}{3\alpha'} + R.$$

$$-\frac{1}{12} H_{\mu'\nu'\rho} H^{\mu'\nu'\rho} - 4 \nabla_{\mu'}\Phi \nabla^{\mu'}\Phi + 4\nabla^2\Phi \Big)$$

$$+ \delta G_{\mu\nu} \left(R^{\mu\nu} + 2\nabla^\mu \nabla^\nu \Phi - \frac{1}{4} H^\mu{}_{\lambda\rho} H^{\nu\lambda\rho} \right) \Big]$$

$$= -\frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \Big[-\frac{1}{2} G^{\mu\nu} \delta G_{\mu\nu} \left(\beta^{G\lambda}{}_{\lambda} - \frac{1}{\alpha'} 4 \beta^{\Phi} \right)$$

$$+ \delta G_{\mu\nu} \beta^{G\mu\nu} \Big]$$

$\Rightarrow$ Finally ...

$$\delta S = \delta S \big|_{B,\Phi} + \delta S \big|_{G,\Phi} + \delta S \big|_{B,G}$$

$$= -\frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \Big[\delta G_{\mu\nu} \beta^{G,\mu\nu} + \delta B_{\mu\nu} \beta^{B,\mu\nu}$$

$$+ \left(2\delta\Phi - \frac{1}{2} G^{\mu\nu} \delta G_{\mu\nu} \right) \left(\beta^{G\lambda}{}_{\lambda} - \frac{1}{\alpha'} 4 \beta^{\Phi} \right) \Big]$$

$\Rightarrow$ for $\delta S = 0$, we need

$$\beta^{G\mu\nu} - \frac{1}{2} G^{\mu\nu} \left(\beta^{G\lambda}{}_{\lambda} - \frac{1}{\alpha'} 4 \beta^{\Phi} \right) = 0 \qquad (1)$$

$$\beta^{B\mu\nu} = 0 \qquad (2)$$

$$\beta^{G\lambda}{}_{\lambda} - \frac{1}{\alpha'} 4 \beta^{\Phi} = 0 \qquad (3)$$

(3) $\to$ (1) $\Rightarrow$ $\underline{\beta^{G\mu\nu} = 0}$ $\quad \therefore \beta^{G\lambda}{}_{\lambda} = 0$

$\Rightarrow \underline{\beta^{\Phi} = 0}$

$$\Rightarrow \quad \beta^{G\mu\nu} = 0 \,, \quad \beta^{B\mu\nu} = 0 \,, \quad \beta^{\Phi} = 0$$

if $\quad \delta S = 0$

$$\square$$

2) $\quad \beta^{G}_{\mu\nu} = 0 \,, \quad \beta^{B}_{\mu\nu} = 0 \quad$ (Polchinski 3.12)

$$\rightarrow \quad R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} = 0$$

$$-\frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} + \nabla^\lambda \Phi H_{\lambda\mu\nu} = 0$$

$$\rightarrow \frac{1}{\alpha'}\partial_\mu \beta^\Phi = \frac{1}{\alpha'}\nabla_\mu \beta^\Phi = -\frac{1}{2}\nabla_\mu \nabla^2 \Phi + \nabla_\mu \nabla_\lambda \Phi \nabla^\lambda \Phi$$

$$- \frac{1}{24} \nabla_\mu (H_{abc} H^{abc})$$

$$\therefore \quad \nabla^\mu \beta^G_{\mu\nu} = 0$$

$$\left.\begin{aligned} &\Rightarrow \quad \nabla^\mu R_{\mu\nu} + 2\nabla^\mu \nabla_\mu \nabla_\nu \Phi - \frac{1}{4}\nabla^\mu (H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho}) = 0 \\ &\Rightarrow \quad \underbrace{\frac{1}{2}\nabla_\nu R}_{\text{Bianchi identity}} + 2\nabla^\mu \nabla_\mu \nabla_\nu \Phi - \frac{1}{4}\nabla^\mu (H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho}) = 0 \end{aligned}\right\} (*)$$

$$\nabla^\mu \nabla_\mu \nabla_\nu \Phi = \nabla_\mu \nabla_\nu \nabla^\mu \Phi = \nabla_\nu \nabla_\mu \nabla^\mu \Phi + R^\mu{}_{\lambda\mu\nu} \nabla^\lambda \Phi$$

$$= \nabla_\nu \nabla^2 \Phi + R_{\lambda\nu} \nabla^\lambda \Phi$$

$$\therefore \quad \nabla_\mu \nabla^2 \Phi = \nabla^2 \nabla_\mu \Phi - R_{\mu\nu} \nabla^\nu \Phi$$

$$(*) \quad \Rightarrow \quad \nabla^2 \nabla_\nu \Phi = \frac{1}{2}\left(- \nabla^\mu R_{\mu\nu} + \frac{1}{4}\nabla^\mu (H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho}) \right)$$

$$= \frac{1}{2}\left(-\frac{1}{2}\nabla_\nu R + \frac{1}{4}\nabla^\mu (H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho}) \right)$$

$$\therefore -\frac{1}{2}\nabla_\mu \nabla^2 \Phi = -\frac{1}{2}(\nabla^2 \nabla_\mu \Phi - R_{\mu\nu}\nabla^\nu \Phi)$$

$$= -\frac{1}{2}\left(\frac{1}{2}\left(-\nabla^\mu R_{\mu\nu} + \frac{1}{4}\nabla^\mu (H_{\mu\lambda\rho}H_\mu{}^{\lambda\rho})\right) - R_{\mu\nu}\nabla^\nu \Phi\right)$$

$$\rightarrow \nabla_\mu(\nabla_\lambda \Phi \nabla^\lambda \Phi) = 2\nabla^\lambda \Phi \nabla_\mu \nabla_\lambda \Phi = 2\nabla^\nu \Phi \nabla_\mu \nabla_\nu \Phi$$

$$\cancel{= 2\nabla^\nu \Phi \nabla_\mu \Phi}$$

$$\because \beta^G_{\mu\nu} = 0 \quad \therefore \quad \nabla_\mu \nabla_\nu \Phi = \frac{1}{2}\left(-R_{\mu\nu} + \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho}\right)$$

$$\therefore \nabla_\mu(\nabla_\lambda \Phi \nabla^\lambda \Phi) = \nabla^\nu \Phi\left(-R_{\mu\nu} + \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho}\right)$$

$$\therefore \frac{1}{\alpha'}\partial_\mu \beta^\Phi = -\frac{1}{2}\nabla_\mu \nabla^2 \Phi + \nabla_\mu(\nabla_\nu \Phi \nabla^\nu \Phi) - \frac{1}{24}\nabla_\mu(H_{abc}H^{abc})$$

$$= \cancel{\tfrac{1}{2}} + \frac{1}{4}\nabla^\nu R_{\mu\nu} - \frac{1}{16}\nabla^\nu(H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho}) + \frac{1}{2}R_{\mu\nu}\nabla^\nu \Phi$$

$$- \nabla^\nu \Phi R_{\mu\nu} + \frac{1}{4}\nabla^\nu \Phi H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} - \frac{1}{24}\nabla_\mu(H_{abc}H^{abc})$$

trace of $\beta^G_{\mu\nu} = \beta^{G\lambda}{}_\lambda = R + 2\nabla^2 \Phi - \frac{1}{4}(H_{abc}H^{abc}) = 0$

$$\nabla_\mu \beta^{G\lambda}{}_\lambda = \nabla_\mu R + \cancel{2\nabla^2\Phi}\, 2\nabla_\mu \nabla^2 \Phi - \frac{1}{4}\nabla_\mu(H_{abc}H^{abc}) = 0$$

$$\Rightarrow \nabla_\mu R = -2\nabla_\mu \nabla^2 \Phi + \frac{1}{4}\nabla_\mu(H_{abc}H^{abc})$$

$$(*) \Rightarrow \quad \nabla_\mu R = -4\nabla^2 \nabla_\mu \Phi + \frac{1}{2}\nabla^\nu(H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho})$$

$$= -4\nabla_\mu \nabla^2 \Phi - 4R_{\mu\nu}\nabla^\nu \Phi + \frac{1}{2}\nabla^\nu(H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho})$$

$$\therefore -2\nabla_\mu \nabla^2 \Phi + \frac{1}{4}\nabla_\mu(H_{abc}H^{abc}) =$$

$$-4\nabla_\mu \nabla^2 \Phi - 4R_{\mu\nu}\nabla^\nu \Phi + \frac{1}{2}\nabla^\nu(H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho})$$

$$\therefore\ 2\nabla_\mu \nabla^2 \Phi + 4 R_{\mu\nu}\nabla^\nu \Phi = \frac{1}{2}\nabla^\nu (H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho}) - \frac{1}{4}\nabla_\mu (H_{abc}H^{abc})$$

$$\therefore\ 2(\nabla^2\nabla_\mu \Phi - R_{\mu\nu}\nabla^\nu\Phi) + 4R_{\mu\nu}\nabla^\nu\Phi = \frac{1}{2}\nabla^\nu (H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho}) - \frac{1}{4}\nabla_\mu (H_{abc}H^{abc})$$

$$\therefore\ R_{\mu\nu}\nabla^\nu\Phi = -\nabla^2\nabla_\mu\Phi + \frac{1}{4}\nabla^\nu (H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho}) - \frac{1}{8}\nabla_\mu (H_{abc}H^{abc})$$

$$\therefore \Longrightarrow \frac{1}{\alpha'}\partial_\mu \beta^\Phi = +\frac{1}{4}\nabla^\nu R_{\mu\nu} + \frac{1}{2}R_{\mu\nu}\nabla^\nu\Phi - \frac{1}{16}\nabla^\nu (H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho}) + \frac{1}{4}\nabla^\nu\Phi H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} - \frac{1}{24}\nabla_\mu (H_{abc}H^{abc})$$

$$= \frac{1}{4}\left(\nabla^\nu R_{\mu\nu} + 2\nabla^\nu\nabla_\nu\nabla_\mu\Phi - \frac{1}{4}\nabla^\nu (H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho})\right)$$

use $\beta^B_{\mu\nu}=0$

$$+ \frac{1}{16}\cancel{\nabla^\nu (H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho})} - \frac{1}{16}\cancel{\nabla^\nu (H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho})}$$

$$+ \underline{\frac{1}{4}\nabla^\nu\Phi H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho}} - \frac{1}{24}\nabla_\mu (H_{abc}H^{abc})$$

$$- \frac{1}{8}\nabla^\nu (H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho}) + \frac{1}{16}\nabla_\mu (H_{abc}H^{abc}).$$

$$= \frac{1}{4}\nabla^\nu \beta^G_{\mu\nu} + \frac{1}{48}\nabla_\mu (H_{abc}H^{abc})$$

$$+ \frac{1}{16}\nabla^\nu \cancel{(H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho})} - \frac{1}{16}\nabla^\nu \cancel{(H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho})}$$

$$+ \frac{1}{8}H_{\mu\lambda\rho}\nabla^\nu H_\nu{}^{\lambda\rho} - \frac{1}{8}\nabla^\nu (H_{\nu\lambda\rho}H_\mu{}^{\lambda\rho}).$$

$$= \underbrace{\frac{1}{4}\nabla^\nu \beta^G_{\mu\nu}}_{=0} + \frac{1}{48}\nabla_\mu (H_{abc}H^{abc}) - \frac{1}{8}(\nabla^\nu H_{\mu\lambda\rho})\nabla_\nu{}^{\lambda\rho}.$$

$$= \frac{1}{48}\nabla_\mu (H_{abc}H^{abc}) - \frac{1}{8}(\nabla^\nu H_{\mu\lambda\rho}) H_\nu{}^{\lambda\rho}$$

$$= \frac{1}{48}\left(\nabla_\mu (H_{abc}H^{abc}) - 6(\nabla^\nu H_{\mu\lambda\rho}) H_\nu{}^{\lambda\rho} \right)$$

Recall

$$H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab}$$
$$= \nabla_a B_{bc} + \nabla_b B_{ca} + \nabla_c B_{ab}$$

Bianchi ~~d~~ identity for H is

$$\partial_a H_{bcd} - \partial_b H_{cda} + \partial_c H_{dab} - \partial_d H_{abc}$$

$$\left.\begin{aligned}
= {} & \partial_a \partial_b B_{cd} + \partial_a \partial_c B_{db} + \partial_a \partial_d B_{bc} \\
& - \partial_b \partial_c B_{da} - \partial_b \partial_d B_{ac} - \partial_b \partial_a B_{cd} \\
& + \partial_c \partial_d B_{ab} + \partial_c \partial_a B_{bd} + \partial_c \partial_b B_{da} \\
& - \partial_d \partial_a B_{bc} - \partial_d \partial_b B_{ca} - \partial_d \partial_c B_{ab}
\end{aligned}\right\} \text{used. } B_{ab} = -B_{ba}$$

$$= 0$$

And $\nabla_a H_{bcd} - \nabla_b H_{cda} + \nabla_c H_{dab} - \nabla_d H_{abc}$

$$\begin{aligned}
= {} & \partial_a H_{bcd} - \Gamma^e_{ab} H_{ecd} - \Gamma^e_{ac} H_{bed} - \Gamma^e_{ad} H_{bce}. \\
& - \partial_b H_{cda} + \Gamma^e_{bc} H_{eda} + \Gamma^e_{bd} H_{cea} + \Gamma^e_{ba} H_{cde} \\
& + \partial_c H_{dab} - \Gamma^e_{cd} H_{eab} - \Gamma^e_{ca} H_{deb} - \Gamma^e_{cb} H_{dae}. \\
& - \partial_d H_{abc} + \Gamma^e_{da} H_{ebc} + \Gamma^e_{db} H_{aec} + \Gamma^e_{dc} H_{abe}
\end{aligned}$$

$= \partial_a H_{bcd} - \partial_b H_{cda} + \partial_c H_{dab} - \partial_d H_{abc}.$

$= 0$

$\therefore \nabla_\nu (H_{abc} H^{abc}) - 6 (\nabla^a H_{\nu bc}) H_a{}^{bc}.$

$= 2\nabla_\nu (H_{abc}) H^{abc} - 6 (\nabla_a H_{\nu bc}) H^{abc}.$

$= \left(2\nabla_\nu (H_{abc}) - 6\nabla_a (H_{\nu bc})\right) H^{abc}.$

$= 2(\nabla_\nu H_{abc} - 3\nabla_a H_{\nu bc}) H^{abc}$

$= \cancel{2(\nabla_\nu H_{abc} - \nabla_a H}$

$= -2(\nabla_a H_{bc\nu} - \nabla_b H_{c\nu a} + \nabla_c H_{\nu ab} - 3\nabla_a H_{\nu bc})$

$\times H^{abc}.$

Bianchi identity

$= -2(\nabla_a H_{\nu bc} + \nabla_b H_{\nu ca} + \nabla_c H_{\nu ab} - 3\nabla_a H_{\nu bc}) H^{abc}$

$= -2(\nabla_a H_{\nu bc} - \cancel{\nabla_a H_{\nu bc}}) H^{abc}. \to = 0$ trivially

$-2(\nabla_b H_{\nu ca} - \cancel{\nabla_a H_{\nu bc}}) H^{abc}. \to = 0 \because H^{abc} = H^{bca}.$

$-2(\nabla_c H_{\nu ab} - \cancel{\nabla_a H_{\nu bc}}) H^{abc}$

$\to = 0 \because H^{abc} = H^{cab}.$

$= 0$

$\Rightarrow \partial_\nu \beta^\Phi = 0 \Rightarrow \beta^\Phi$ is constant

D.