General Relativity II

Problem Set 3

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Fri WK 7 15:00 - 16:30

Q	
1	A-
2	A+
4	B+
5	A-
6	B
7	A

$\boxed{1}$

$K^0=1 \qquad K^1=0 \qquad K^2=0 \qquad K^3=1$

use η to raise and lower indices $\eta=(-1,1,1,1)$

$\therefore K_0=-1 \qquad K_1=0 \qquad K_2=0 \qquad K_3=1$

e_{ab} is symmetric $e_{ab}=e_{ba}$

use η, $\cancel{e_{ab}=}$ $e_{ij}=e^i{}_j=e^{ij}$,

$e^{0i}=-e^i{}_0=e^0{}_i=-e_{0i}$, $e^{00}=e_{00}=-e^0{}_0$

Harmonic constraint : $(e_{ab}-\frac{1}{2}\eta_{ab}e)K^b=0$

$\therefore K_b e^b{}_a=\frac{1}{2}K_a e$

→ $a=0$ ⇒ $K_0e^0{}_0+K_3e^3{}_0=K_0(e^i{}_i+e^0{}_0)\frac{1}{2}$

⇒ $-(e_{00}+e_{30})=(e_{ii}-e_{00})\frac{1}{2}$ (1)

→ $a=j$ (spatial index)

⇒ $K_0e^0{}_j+K_3e^3{}_j=K_j(e_{ii}-e_{00})\frac{1}{2}$

if $j=1$: $e_{01}+e_{31}=0$ (2)

$j=2$: $e_{02}+e_{32}=0$ (3)

$j=3$: $e_{03}+e_{33}=(e_{ii}-e_{00})\frac{1}{2}$ (4)

$=-(e_{00}+e_{03})$ (5)

use (1)

(2), (3), ~~(4)~~ (5) $\Rightarrow$

$$e_{01} = -e_{31}, \quad e_{02} = -e_{32}, \quad e_{03} = -\frac{e_{00} + e_{33}}{2}$$

use (4), $\quad e_{03} + e_{33} = \frac{1}{2}(e_{11} + e_{22} + e_{33} - e_{00})$.

$$\Rightarrow \quad -\frac{1}{2}(e_{00} + e_{33}) + e_{33} = \frac{1}{2}(e_{11} + e_{22}) + \frac{1}{2}(e_{33} - e_{00})$$

$$\Rightarrow \quad \frac{1}{2}(\cancel{e_{33}} - \cancel{e_{00}}) = \frac{1}{2}(\cancel{e_{33}} + \cancel{e_{00}}) + \frac{1}{2}(e_{11} + e_{12})$$

$$\Rightarrow \quad e_{11} = -e_{22}$$

The four underlined ~~equations~~ equations enable us to write e_{0i} and e_{22} ~~in terms~~ in terms of e_{3i}, e_{00} and e_{11}

So e_{3i}, e_{00}, e_{11}, and $e_{12} = e_{21}$, six components remain unconstrained

Now consider $\tilde{e}_{ab} = e_{ab} + K_a \lambda_b + \cancel{K} \lambda_a K_b$

~~Choose~~ $\tilde{e}_{11} = e_{11}$, $\tilde{e}_{12} = e_{12}$ $\because K_1 = K_2 = 0$

$$\tilde{e}_{13} = e_{13} + \lambda_1 \quad (K_3 = 1)$$
$$\tilde{e}_{23} = e_{23} + \lambda_2 \quad (K_3 = 1)$$
$$\tilde{e}_{33} = e_{33} + 2\lambda_3 \quad (K_3 = 1)$$
$$\tilde{e}_{00} = e_{00} - 2\lambda_0 \quad (K_0 = -1)$$

choose $\lambda_1 = -e_{13}$, $\lambda_2 = -e_{23}$, $\lambda_3 = -\frac{1}{2} e_{33}$, $\lambda_0 = \frac{1}{2} e_{00}$ then

$$\tilde{e}_{13} = \tilde{e}_{23} = \tilde{e}_{33} = \tilde{e}_{00} = 0 \Rightarrow \underbrace{\tilde{e}_{31} = \tilde{e}_{32} = 0}_{\text{symmetric}}$$

$\therefore$ Gauge transformation $e \to \tilde{e}$ also satisfies the harmonic gauge

$\therefore$ the ~~modes~~ underlined 4-equations also works for $\tilde{e}$

i.e. $\tilde{e}_{01} = -\tilde{e}_{31} = 0$, $\tilde{e}_{02} = -\tilde{e}_{32} = 0$, $\tilde{e}_{03} = -\frac{\tilde{e}_{00} + \tilde{e}_{33}}{2} = 0$

And $\tilde{e}_{11} = -\tilde{e}_{22}$

Now for $\tilde{e}$, only non-zero components are

$\tilde{e}_{12} = \tilde{e}_{21}$, $\tilde{e}_{11} = -\tilde{e}_{22}$

let $\tilde{e}_{12} = B$, $\tilde{e}_{11} = A$, then

$$\therefore \tilde{e}_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & B & -A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boxed{2}$ massive particle $\Rightarrow$ time-like geodesic

$$-1 = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

when $r < 2m$, $g_{tt} > 0$, $g_{rr} < 0$, $g_{\theta\theta} > 0$
$g_{\phi\phi} > 0$

~~we~~ see $\dot{t}^2 \neq 0$ $\therefore$ to ~~maxis~~ minimise $\dot{r}^2$,
we need $\dot{\theta} = 0$, $\dot{\phi} = 0$ ~~so~~ $\because g_{rr}$ ~~and~~ has

opposite sign to $g_{\theta\theta}$ and $g_{\phi\phi}$

To maximize ~~proper time~~ proper time ~~we~~ we need the particle to follow geodesic path (principle of maximum proper time)

Thus, we require the particle to follow incoming radial geodesic to maximise $\Delta S = \Delta \tau$

$$\rightarrow \quad -1 = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2$$

$\because$ Geodesic $\therefore$ conserved quantity

$$E = \left(1-\frac{2M}{r}\right)\dot{t}$$

$$\Rightarrow E^2 = \left(1-\frac{2M}{r}\right)^2 \dot{t}^2 \Rightarrow \dot{t}^2 = \frac{E^2}{\left(1-\frac{2M}{r}\right)}$$

$$\therefore -1 = -\frac{E^2}{1-\frac{2M}{r}} + \frac{\dot{r}^2}{1-\frac{2M}{r}}$$

$$\therefore \dot{r}^2 = \cancel{\frac{E^2}{1-2M}} \; E^2 - \left(1-\frac{2M}{r}\right)$$

$\because$ incoming $\therefore \dot{r} < 0 \Rightarrow \dot{r} = -\sqrt{\frac{2M}{r} - 1 + E^2} = \frac{dr}{d\tau}$

$$\therefore \frac{d\tau}{dr} = \frac{-1}{\sqrt{\frac{2M}{r} - 1 + E^2}}$$ for maximum proper time path.

$$\Rightarrow \Delta\tau = \Delta S = \int_{2M=r}^{0=r} d\tau = \int_{2M=r}^{0=r} \frac{d\tau}{dr}\, dr$$

$$= -\int_{2m}^{0} \frac{dr}{\sqrt{\frac{2m}{r} - 1 + E^2}} = \int_{0}^{2m} \frac{dr}{\sqrt{\frac{2m}{r} - 1 + E^2}}$$

this is maximised when $E^2 = 0$

$$\therefore \Delta S = \Delta \tau_{max} = \int_{0}^{2m} \frac{dr}{\sqrt{\frac{2m}{r} - 1}}$$

$$= \pi M$$

[4] Hypersurface $\Sigma (S=0)$ has normal

~~$n^\mu = h(x)$~~ $n^\mu = h(x) g^{\mu\nu} \nabla_\nu S(x)$

where h is a normalising function ~~and~~
so that $n^\mu n_\mu$ = 1 → time-like Σ
0 → null Σ
−1 → spacelike Σ

S is scalar function → $\nabla_\nu S = \partial_\nu S$

$\therefore n^\mu = h g^{\mu\nu} \partial_\nu S$, ~~$n^\mu n_\mu = h^2 g^{\mu\nu}$~~

→ $n^\mu n_\mu = h^2 g^{\mu\nu} \partial_\mu S \partial_\nu S$

choosing coordinates such that (x^0, x^1, x^2, x^3) and $x^0 = S$, then ∵ coordinates are independent , for $i = 1,2,3$

$$\partial_0 S = 1 = \frac{\partial S}{\partial S} , \quad \partial_i S = 0 = \frac{\partial S}{\partial x^i}$$

Null - hypersurface $n^\mu n_\mu = 0$

$$\Rightarrow \quad 0 = n^\mu n_\nu = h^2 g^{\mu\nu} \partial_\mu S \partial_\nu S$$

$$= h^2 (g^{00} \underbrace{(\partial_0 S)^2}_{=1} + g^{0i} \partial_0 S \underbrace{\partial_i S}_{=0} + g^{i0} \underbrace{\partial_i S}_{=0} \partial_0 S$$

$$+ g^{ij} \underbrace{\partial_i S}_{=0} \underbrace{\partial_j S}_{=0})$$

$$= h^2 g^{00} \quad \because h \text{ is non-zero function}$$

$$\Rightarrow \therefore \; \underline{\underline{g^{00} = 0}} \quad \square$$

consider $g_{ab} = \begin{pmatrix} V & Y^t \\ Y & A \end{pmatrix}$ and

$S=0$ null hypersurface $\rightarrow$ $g^{ab} = \begin{pmatrix} 0 & X^t \\ X & B \end{pmatrix}$ and note that

$$g^{ac} g_{cb} = \delta^a{}_b \quad \Rightarrow \quad \cancel{g_{ab} g^{ab}}$$

$$\begin{pmatrix} 0 & X^t \\ X & B \end{pmatrix} \begin{pmatrix} V & Y^t \\ Y & A \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$A_{(1)} \quad A_{(2)} \quad A_{(3)}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ & & & \\ & & & \end{pmatrix} = \begin{pmatrix} X^tY & X^tA_{(1)} & X^tA_{(2)} & X^tA_{(3)} \\ & & & \\ & & & \end{pmatrix}$$

where we've only calculated the first row. of the product of matrices.

$\therefore \quad X^tY = 1 \Rightarrow X, Y$ are non-zero ~~vecto~~ 3 vectors.

and,

$$X^t A_{(1)} = X^t A_{(2)} = X^t A_{(3)} = 0$$

$\Rightarrow$ 3-vectors $A_{(1)}, A_{(2)}, A_{(3)}$ are all orthogonal to a non-zero 3-vector X

$\therefore A_{(1)}, A_{(2)}, A_{(3)}$ all lies in the 2D plane perpendicular to X

$\Rightarrow A_{(1)}, A_{(2)}, A_{(3)}$ are linearly dependent.

$\Rightarrow \quad \det(A) = 0$ $\square$

$\rightarrow \quad S = r - 2M = 0 \qquad dr = dS \quad , \quad r = S + 2M$

$\rightarrow$ The ~~Eddington~~ Advanced Eddington-Finkelstein coordinates

$$-d\tau^2 = 2dp\,dr - \left(1 - \frac{2M}{r}\right)dp^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= 2dp\,dS - \left(1 - \frac{2M}{S+2M}\right)dp^2 - (S+2M)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$S = x^0, \ \rho = x^1, \ \theta = x^2, \ \phi = x^3$

$$\therefore g_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1+\frac{2M}{S+2M} & 0 & 0 \\ 0 & 0 & (S+2M)^2 & 0 \\ 0 & 0 & 0 & (S+2M)^2\sin^2\theta \end{pmatrix}$$

$$\underbrace{\qquad}_{A}$$

$$\det(A) = -\left(1 - \frac{2M}{S+2M}\right)(S+2M)^2(S+2M)^2\sin^2\theta.$$

If $S = 0$ $\quad \det(A) = 0 \quad \because 1 - \frac{2M}{2M} = 0.$

$\Rightarrow \quad S = r - 2M = 0 \quad$ a null hyperserface.

$\rightarrow$ Retarded Eddington - Finkelstein metrics.

$S = 0 = r - 2M \qquad dS = dr. \qquad r = S+2M.$

$$\cancel{ds^2} \ -d\tau^2 = -2dq\,dr - \left(1 - \frac{2M}{r}\right)dq^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -2dq\,dS - \left(1 - \frac{2M}{2M+S}\right)dq^2 + (2M+S)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$g_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -1+\frac{2M}{S+2M} & 0 & 0 \\ 0 & 0 & (S+2M)^2 & 0 \\ 0 & 0 & 0 & (S+2M)^2\sin^2\theta \end{pmatrix}$$

$$\underbrace{\qquad}_{A}$$

$$\det(A) = -\left(1 - \frac{2M}{S+2M}\right)(S+2M)^2(S+2M)^2\sin^2\theta.$$

when $S = 0$, $\quad \det(A) = 0.$

$\Rightarrow$ null hypesurface.

$\boxed{5}$ $ds^2 = -(1-\frac{2mr}{\Sigma})dt^2 - \frac{4mar}{\Sigma}\sin^2\theta \, d\phi \, dt$.

$+\frac{1}{\Sigma}\sin^2\theta(\Delta\Sigma + 2Mr(r^2+a^2))d\phi^2$

$+\Sigma(\frac{1}{\Delta}dr^2 + d\theta^2)$.

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2mr + a^2$.

$\therefore g_{tt} = -(1-\frac{2mr}{\Sigma})$, $g_{t\phi} = -\frac{2mar}{\Sigma}\sin^2\theta$.

$g_{rr} = \frac{\Sigma}{\Delta}$, $g_{\theta\theta} = \Sigma$, $\cancel{g_{rr}}$

$g_{\phi\phi} = \frac{1}{\Sigma}\sin^2\theta(\Delta\Sigma + 2Mr(r^2+a^2))$.

time-like Killing vector of Kerr solution is

$K^a = (1,0,0,0)$

$\therefore K_a = g_{ab}K^b = g_{a0}K^0 = g_{at}K^t$.

$\therefore K_t = g_{tt}K^t = g_{tt}$.

$K_\phi = g_{\phi t}K^t = g_{\phi t}$.

$K_r = K_\theta = 0$.

6! $K_{[a}\nabla_b K_{c]} = 0 \Rightarrow$.

$$\rightarrow 0 = K_r \underset{\approx 0}{\nabla_\theta} K_\phi - K_r \nabla_\phi \underset{\approx 0}{K_\theta} + K_\theta \underset{\approx 0}{\nabla_\phi K_r} - K_\theta \nabla_r K_\phi$$
$$+ K_\phi \underbrace{\nabla_r K_\theta}_{=0} - K_\phi \underbrace{\nabla_\theta K_r}_{=0}.$$
true always.

$\rightarrow K_{[t} \nabla_\theta K_{r]} = 0$ always. (similarly).

$$\rightarrow 6!\, K_{[t} \nabla_\theta K_{\phi]} = K_t \nabla_\theta K_\phi - K_t \nabla_\phi K_\theta + \cancel{K_\theta \nabla_\phi K_t}$$
$$- \cancel{K_\theta \nabla_t K_\phi} + K_\phi \nabla_t K_\theta - K_\phi \nabla_\theta \nabla_t.$$

$$= K_t (\nabla_\theta K_\phi - \nabla_\phi K_\theta) + K_\phi (\nabla_t K_\theta - \nabla_\theta K_t).$$

$\because \Gamma^d{}_{ab} = \Gamma^d{}_{bc}$

$$= K_t (\partial_\theta K_\phi - \partial_\phi \underbrace{K_\theta}_{=0}) + K_\phi (\partial_t \underbrace{K_\theta}_{=0} - \partial_\theta K_t).$$

$$= g_{tt} \partial_\theta g_{\phi t} - g_{\phi t} \partial_\theta g_{tt}.$$

$$= \left(1 - \frac{2mr}{\Sigma}\right) \partial_\theta \left(\frac{2mar}{\Sigma} \sin^2\theta \right)$$
$$- \left(\frac{2mar}{\Sigma} \sin^2\theta \right) \partial_\theta \left(1 - \frac{2mr}{\Sigma}\right)$$

$$= \left(1 - \frac{2mr}{\Sigma}\right)\left(-\frac{2mar}{\Sigma^2} \frac{\partial \Sigma}{\partial \theta} + \frac{2mar}{\Sigma} (2 \sin\theta \cos\theta) \right)$$
$$- \left(\frac{2mar}{\Sigma} \sin^2\theta \, \frac{2mr}{\Sigma} \frac{\partial \Sigma}{\partial \theta} \right). \quad \left(\frac{\partial \Sigma}{\partial \theta} = 2a^2 \sin\theta \cos\theta \right)$$

~~unless a =~~

$$= 2\cos\theta \sin\theta \left(1 - \frac{2mr}{\Sigma}\right)\left(-\frac{2ma^3 r}{\Sigma^2} + \frac{2mar}{\Sigma} - \frac{4ma^3 r}{\Sigma^2} \sin^2\theta \right)$$

$\neq 0$ unless $a = 0 \rightarrow J = 0$

Similarly $K_{[t} \nabla_r K_{\phi]} = g_{tt} \partial_r g_{\phi t} - g_{\phi t} \partial_r g_{tt}$

$$= \left(1 - \frac{2Mr}{\Sigma}\right) \partial_r \left(\frac{2Mar}{\Sigma} \sin\theta\right)$$
$$- \left(\frac{2Mar}{\Sigma} \sin^2\theta\right) \partial_r \left(1 - \frac{2Mr}{\Sigma}\right)$$

$= 0$ when $a = 0 \rightarrow J = 0$

$\Rightarrow K_{[a} \nabla_b K_{c]} \neq 0$ unless $J = Ma = 0$

$\rightarrow$ Not static unless $J = 0$ $\square$

[6] $\vec{K} \cdot \vec{\nabla} \times \vec{K} = 0 \Leftrightarrow K_k \varepsilon_{kij} \partial_i K_j = 0 = \varepsilon_{kij} K_k \partial_i K_j$
$\Leftrightarrow K_{[k} \partial_i K_{j]} = 0 \Leftrightarrow K_{[a} \nabla_b K_{c]} = 0 \Leftrightarrow K$ is HSO $\square$

Formal Proof $\rightarrow$ Claim, $\varepsilon_{kij} K_k \partial_i K_j = 0 \Leftrightarrow \partial_{[i} K_{j]} = \alpha_{[i} K_{j]}$

for some vector $\vec{\alpha} \in \mathbb{R}^3$

Proof:

" $\Leftarrow$ " Direct calculation,

$\varepsilon_{kij} K_k \partial_i K_j = \varepsilon_{kij} K_k \partial_{[i} K_{j]} = \varepsilon_{kij} K_k \alpha_{[i} K_{j]}$
$= \underbrace{\varepsilon_{kij} K_k \alpha_i K_j}_{j \leftrightarrow k} = 0$ $\square$

" $\Rightarrow$ " Assume $\partial_{[i} K_{j]} = \alpha_{[i} K_{j]} + \beta_{[i} X_{j]}$ for

$\vec{\alpha}, \vec{\beta} \in \mathbb{R}^3$ and $\vec{X} \cdot \vec{K} = 0$

this is a general form since if $\vec{x}$ has component $\parallel \vec{K}$ then this component can be absorbed into $\alpha_{[i} K_{j]}$

Consider $\vec{x} \cdot \vec{K} = 0$, $\vec{y} \cdot \vec{K} = 0$, $\vec{x} \cdot \vec{y} = 0$ ~~independent~~, $\vec{x}, \vec{y}, \vec{K}$ forms an orthonormal basis of $\mathbb{R}^3$

let $\vec{\beta} = m\vec{x} + n\vec{y} + p\vec{K}$

then the $p\vec{K}$ component ~~can be absor~~ of $\beta_{[i} X_{j]}$ can be absorbed into $\alpha_{[i} K_{j]}$

$\therefore \quad \vec{\beta} = m\vec{x} + n\vec{y}$

$$\varepsilon_{kij} K_k \partial_i K_j = \varepsilon_{kij} K_k \partial_{[i} K_{j]} = \underbrace{\varepsilon_{kij} K_k \alpha_{[i} K_{j]}}_{=0} + \varepsilon_{kij} K_k \beta_{[i} X_{j]}$$

$$= \varepsilon_{ijk} K_k \beta_i X_j = m \underbrace{\varepsilon_{kij} K_k X_i X_j}_{=0} + n \varepsilon_{kij} K_k X_i Y_j$$

$$= n \vec{K} \cdot (\vec{X} \times \vec{Y}) = \pm n |X||Y||K| = 0 \text{ only}$$

if $\quad n = 0 \quad \Rightarrow$ require $\vec{\beta} = m\vec{x}$ only

But then $\beta_{[i} X_{j]} = m X_{[i} X_{j]} = 0$

Hence the only term remains is $\alpha_{[i} K_{j]}$

and $\varepsilon_{kij} K_k \partial_i K_j = 0 \quad \Rightarrow \quad \partial_{[i} K_{j]} = \alpha_{[i} K_{j]}$

$\square$

→ Now use this claim:

Want to ~~proov~~ prove

$\vec{K} \cdot \nabla \times \vec{K} = 0 \iff \vec{K}$ is HSO $\iff \vec{K} = \psi \nabla \phi$

for some functions $\psi(\vec{x})$ and $\phi(\vec{x})$

Proof!

"$\Rightarrow$" use claim $\vec{K}\cdot\vec{\nabla}\times\vec{K}=0 \Rightarrow \partial_i K_{jj} = \alpha_{ij} K_{ij}$

for some $\vec{\alpha} \in \mathbb{R}^3$

Consider $\vec{x}\cdot\vec{K}=0$, $\vec{y}\cdot\vec{K}=0$, so that

$\vec{x}, \vec{y} \in \text{span}\{\vec{K}\}^{\perp} \equiv K^{\perp}$

Now the Lie Bracket $[X, Y]$ satisfies.

$$[\vec{x},\vec{y}]_j K_j = (x_i(\partial_i y_j) - y_i(\partial_i x_j))K_j$$

$$= x_i(\underbrace{\partial_i(y_j K_j)}_{=0} - y_j \partial_i K_j) - y_i(\underbrace{\partial_i(x_j K_j)}_{=0} - x_j \partial_i K_j)$$

$$= -(x_i y_j - x_j y_i)(\partial_i K_j)$$

$$= -2 x_i y_j \partial_{ii} K_{jj} = -2 x_i y_j \alpha_{ii} K_{ij}$$

$$= -2(\alpha_i x_i \underbrace{y_j K_j}_{=0} - \alpha_j y_j \underbrace{x_i K_i}_{=0}) = 0$$

$$\Rightarrow [\vec{x},\vec{y}] \in K^{\perp} \quad \forall\ \vec{x},\vec{y} \in K^{\perp}$$

So By Frobenius Theorem, $\vec{K}$ is a HSO

$\Leftrightarrow$ $\vec{K} = \psi\vec{\nabla}\phi$ for some ψ, ϕ. $\square$

"$\Leftarrow$" if $\vec{K} = \psi\vec{\nabla}\phi \Leftrightarrow K_i = \psi\partial_i\phi$

then By direct calculation

$$\vec{K}\cdot\vec{\nabla}\times\vec{K} = \psi\vec{\nabla}\phi\cdot\vec{\nabla}\times(\psi\vec{\nabla}\phi)$$

$$= \psi\vec{\nabla}\phi\cdot\underbrace{(\vec{\nabla}\psi\times\vec{\nabla}\phi)}_{=0} + \psi\vec{\nabla}\phi\cdot(\psi\underbrace{\vec{\nabla}\times\vec{\nabla}\phi}_{=0})$$

$\vec{a}\cdot(\vec{b}\times\vec{a})=0$ curl of grad $=0$

Lowy: But works, ish
see class

$$=0.$$

$$\therefore \vec{K} \text{ is HSO} \Rightarrow \vec{K}\cdot\vec{\nabla}\times\vec{K}=0$$

Hence $\vec{K}\cdot\vec{\nabla}\times\vec{K}=0 \iff \vec{K}$ is HSO

$\rightarrow$ If $\vec{K}=(y,-x,f(r))$ $r^2=x^2+y^2$.

$$\vec{\nabla}\times\vec{K} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & f \end{vmatrix} = (\partial_y f, -\partial_x f, -2)$$

$$\therefore \frac{\partial r}{\partial x}=2x, \quad \frac{\partial r}{\partial y}=2y, \quad f'=\frac{df}{dr}$$

$$\therefore \partial_x f = 2xf', \quad \partial_y f = 2yf'$$

$$\Rightarrow \vec{\nabla}\times\vec{K} = (2yf', -2xf', -2)$$

$$\vec{K}\cdot\vec{\nabla}\times\vec{K} = (y,-x,f)\cdot(2yf',-2xf',-2)$$

$$= 2y^2f' + 2x^2f' - 2f = 2r^2f'-2f \overset{!}{=} 0$$

$$\Rightarrow f(r) = r^2 f'(r)$$

$$\therefore r^2\frac{df}{dr}=f \Rightarrow \frac{df}{f}=\frac{dr}{r^2} \Rightarrow \ln f = -\frac{1}{r}+C'$$

$\times$

$\therefore \quad f = e^{-\frac{1}{r} + c'} \Rightarrow \underline{f(r) = Ce^{-\frac{1}{r}}}$ b.

integral curves $\quad \frac{d\vec{x}}{dt} = \vec{K}$

$\frac{dx}{dt} = y$, $\frac{dy}{dt} = -x$, $\frac{dz}{dt} = Ce^{-\frac{1}{\sqrt{x^2+y^2}}}$ not quite

$\Rightarrow \frac{d^2x}{dt^2} = \frac{dy}{dt} = -x \Rightarrow \frac{d^2x}{dt^2} + x = 0$

$\Rightarrow x = A\cos(t+\phi)$; $y = \frac{dx}{dt} = -A\sin(t+\phi)$.

$r = \sqrt{x^2+y^2} = A$.

see class $\quad \therefore \frac{dz}{dt} = f = \underbrace{Ce^{-\frac{1}{A}}}_{B} = B = $ const. $\Rightarrow z = Bt$.

$\therefore \begin{cases} x(t) = A\cos(t+\phi) \\ y(t) = -A\sin(t+\phi) \\ z(t) = Bt \end{cases}$

$\rightarrow \vec{K} = (\cos g(z), \sin g(z), 0)$, $\underline{g' \equiv \frac{dg}{dz}}$.

$\vec{\nabla}\times\vec{K} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \cos g(z) & \sin g(z) & 0 \end{vmatrix} = (-\partial_z(\sin g(z)), \partial_z(\cos g(z)), 0)$

$= \cancel{(-\cos(g(z))\frac{dg}{dz}} = (-g'\cos g(z)), -g'\sin(g(z)), 0)$

$\vec{K}\cdot\vec{\nabla}\times\vec{K} = -g'\cos^2 g - g'\sin^2 g = -g'(\underbrace{\cos^2 g + \sin^2 g}_{1})$

$= -g' \neq 0 \Rightarrow K$ never HSO $\square$

$$\frac{d\vec{x}}{dt} = \vec{K}$$

$$\therefore \quad \frac{dx}{dt} = \cos g(z) \qquad \frac{dy}{dt} = \sin g(z), \quad \frac{dz}{dt} = 0$$

$$\frac{dz}{dt} = 0 \quad \Rightarrow \quad z = z_0 = \text{const.}$$

$$\Rightarrow \quad \frac{dx}{dt} = \cos g(z_0), \quad \frac{dy}{dt} = \sin g(z_0)$$

$$\Rightarrow \quad \begin{cases} x(t) = (\cos g(z_0))t \\ y(t) = (\sin g(z_0))t \\ z(t) = z_0 \end{cases}$$

$\rightarrow$ straight lines.

$\boxed{7}$

$$ds^2 = -dT^2 + dr^2 - 2a\sin^2\theta\, dr\, d\Phi + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta\, d\Phi^2 + \frac{2Mr}{\Sigma}(dT - a\sin^2\theta\, d\Phi + dr)^2$$

$$= -dT^2 + dr^2 - 2a\sin^2\theta\, dr\, d\Phi + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta\, d\Phi^2$$

$$+ \frac{2Mr}{\Sigma} dT^2 - \frac{4Mar}{\Sigma}\sin^2\theta\, dT\, d\Phi + \frac{2Mra^2\sin^4\theta}{\Sigma} d\Phi^2.$$

$$+ (\quad)\, dr\, dT + (\quad)\, dr\, d\Phi + (\quad)\, dr^2$$

$$= \left(\frac{2Mr}{\Sigma} - 1\right) dT^2 - \frac{4Mar}{\Sigma}\sin^2\theta\, dT\, d\Phi \cancel{+ 2Ma^2}$$

$$+ \left(\frac{2Ma^2 r\sin^4\theta}{\Sigma} + (r^2+a^2)\sin^2\theta\right) d\Phi^2$$

$+ \Sigma\, d\theta^2 \quad + \quad$ (things ~~in tot~~ involving dr)

set $\quad r = x^0 \quad$ (hypersurfaces $r = r_\pm$).

and $\quad T = x^1, \quad \cancel{\Phi = x^2, \theta =} \quad \Phi = x^2, \; \theta = x^3.$

then matrix A is

$$A = \begin{pmatrix} \frac{2mr}{\Sigma} - 1 & -\frac{2mar}{\Sigma}\sin^2\theta & 0 \\ -\frac{2mar}{\Sigma}\sin^2\theta & \frac{2ma^2 r\sin^4\theta}{\Sigma} + (r^2+a^2)\sin^2\theta & 0 \\ 0 & 0 & \Sigma \end{pmatrix}.$$

$\det(A) = 0$

$$\Rightarrow \Sigma\left[\left(\frac{2mr}{\Sigma} - 1\right)\left(\frac{2ma^2 r \sin^4\theta}{\Sigma} + (r^2+a^2)\sin^2\theta\right) - \frac{4m^2a^2r^2}{\Sigma^2}\sin^4\theta\right]$$

$$= \Sigma\left[\cancel{\frac{4m^2a^2r^2}{\Sigma^2}}\sin^4\theta - \frac{2ma^2 r\sin^4\theta}{\Sigma} + \frac{2mr(r^2+a^2)\sin^2\theta}{\Sigma} - (r^2+a^2)\sin^2\theta - \cancel{\frac{4m^2a^2r^2}{\Sigma^2}}\sin^4\theta\right]$$

$$= -2ma^2 r\sin^4\theta + 2mr(r^2+a^2)\sin^2\theta - (r^2+a^2)\sin^2\theta\,(r^2 + a^2\underbrace{\cos^2\theta}_{(1-\sin^2\theta)})$$

$$= -2Ma^2 r \sin^4\theta + 2Mr(r^2+a^2)\sin^2\theta$$

$$-(r^2+a^2)^2 \sin^2\theta + (r^2+a^2)a^2\sin^4\theta$$

$$= (r^2-2Mr+a^2)a^2\sin^4\theta - (r^2-2Mr+a^2)(r^2+a^2)\sin^2\theta$$

$$= \Delta(a^2\sin^4\theta + (r^2+a^2)\sin^2\theta)$$

At $r = r_t$, $\Delta = 0 \Rightarrow \det(A) = 0$

$\Rightarrow$ $r = r_t$ null hypersurfaces. R

*Note: For Q4 $\det(A) \Rightarrow g^{00} = 0 \Leftrightarrow S = 0$

this is shown by using Crammer's Rule.

$$g = g_{ab} = \begin{pmatrix} V & Y^t \\ Y & A \end{pmatrix} \quad g^{-1} = g^{ab} = \begin{pmatrix} S & X^t \\ X & B \end{pmatrix}$$

$$gg^{-1} = \mathbb{1}_4 \quad \therefore \det(g) \neq 0 \quad \det(g^{-1}) \neq 0$$

$$\underbrace{\begin{pmatrix} V & Y^t \\ Y & A \end{pmatrix}}_{g} \underbrace{\begin{pmatrix} S \\ X \end{pmatrix}}_{\vec{a}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{b} \quad \Rightarrow \quad g\vec{a} = \vec{b}$$

then
$$a_i = \frac{\det(g_{(1)}, \cdots, g_{(i-1)}, \vec{b}, g_{(i+1)}, \cdots g_{(n)})}{\det(g)}$$

is Crammer's Rule.

$$g^{00} = \delta = a_0 = \frac{\det\left(\begin{array}{c|c} 1 & y^t \\ \hline 0 & \\ 0 & A \\ 0 & \end{array}\right)}{\det(g)} = \frac{\det(A)}{\det(g)}$$

$$= 0 \quad \text{if} \quad \det(A) = 0$$

$\square$

$\boxed{3}$ Schwarzschild Geodesics ($\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$)

$$\left(1 - \frac{2M}{r}\right)\dot{t} = E$$

$$-\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -\sigma$$

$$r^2\dot{\phi} = J$$

E, J constants, and $(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\sigma)$

where $\sigma = 1 \Rightarrow$ time like

$\sigma = 0 \rightarrow$ null

$\sigma = -1 \rightarrow$ spacelike

$\therefore$ result is $\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2$

and $V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\sigma + \frac{h^2}{r^2}\right)$

We deal with radial geodesics $\Rightarrow h = 0$

$\therefore$ $\dot{r}^2 + \left(1 - \frac{2M}{r}\right)\sigma = E^2$

$\therefore$ $\dot{r}^2 - \frac{2M}{r}\sigma = E^2 - \sigma$

$\rightarrow$ space-like radial geodesic $\sigma = -1$: $\dot{r}^2 + \frac{2M}{r} = E^2 + 1$

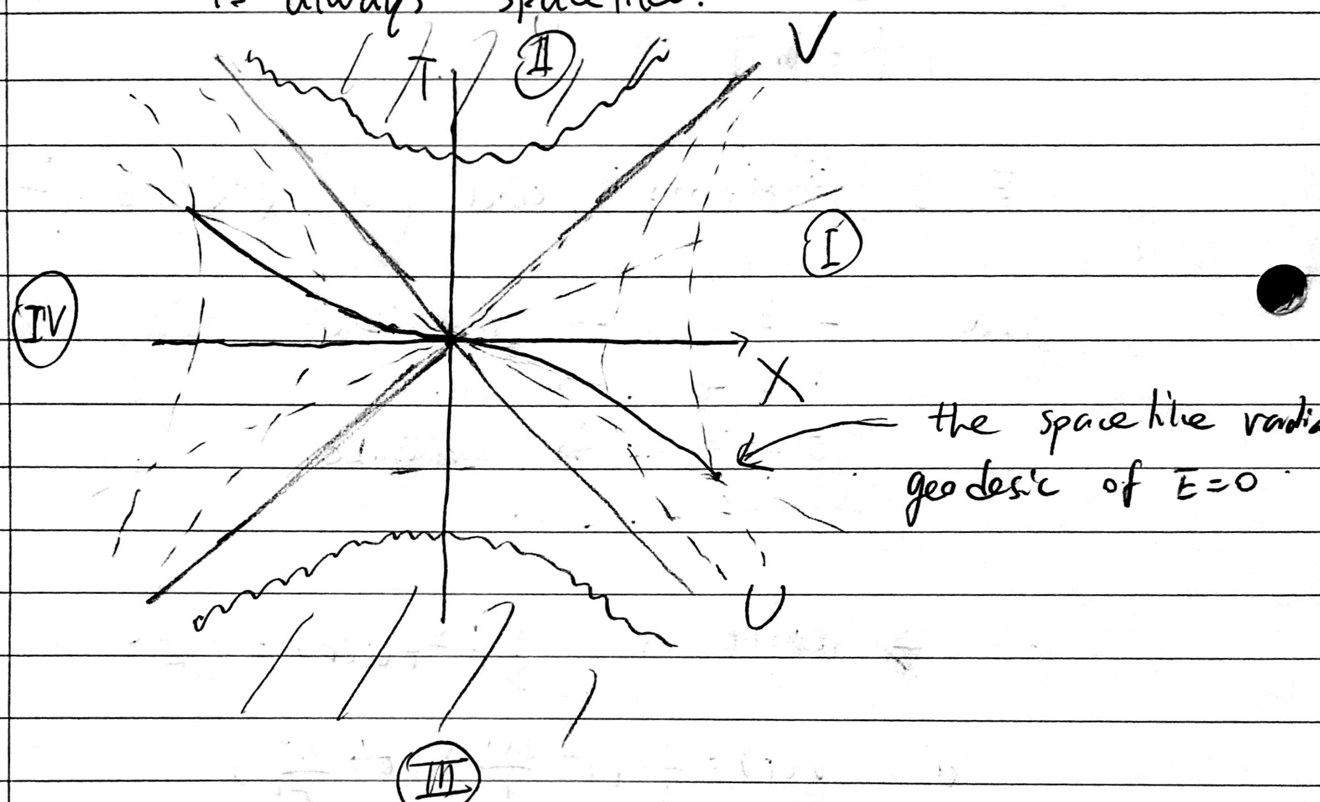
further require $E = 0 \Rightarrow$ $\dot{r}^2 + \frac{2M}{r} = 1$

$\Rightarrow 1 - \frac{2M}{r} \geqslant 0 \Rightarrow$ $\underline{r \geqslant 2M}$

and White Hole).

this means that this geodesic does not go into the Black Hole of both universes in Kruskal ~~space~~ spacetime.

∴ to ensure the geodesic is always spacelike.



$E=0$ means $r \geq 2M$ $\Longrightarrow$ geodesic passes through the ~~wor~~ worm hole to the other universe IV

→ time-like geodesic $\sigma = 1$

∴ $\dot{r}^2 - \frac{2M}{r} = E^2 - 1$ ∵ $0 < E < 1$ ∴ ~~E^2~~

∴ $0 < E^2 < 1 \Rightarrow 1 - E^2 > 0$

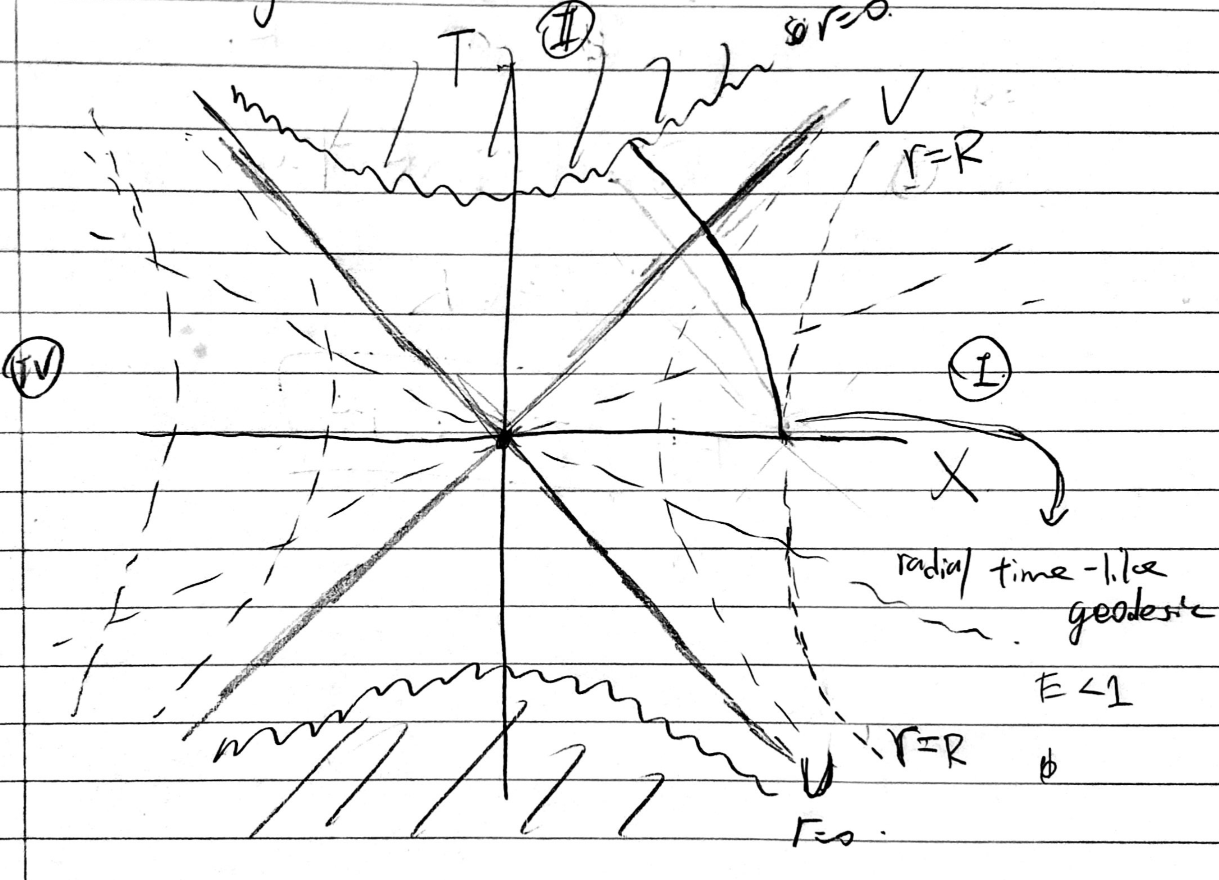
∴ $0 \leq \dot{r}^2 = \frac{2M}{r} - (1 - E^2)$

∴ $\frac{2M}{r} \geq (1 - E^2)$ $r \leq \frac{2M}{1 - E^2} \equiv R$

∴ r is bounded by some maximum

value $R \equiv \frac{2M}{1-E^2}$,

→ this means that the particle will fall into the Black Hole ~~and~~ and eventually reach the singularity following this time like geodesic.



In above diagram, particle dropped from rest at r=R and falls into the Black Hole.

→ Now for time-like geodesic with $E=0$, the maximum ~~radius~~ r value for radial geodesics is $R_{(E=0)} = \frac{2M}{1-0} = \underline{2M}$

and thus particle geodesics starts and end both within the Black Hole i.e. region II

$\cancel{r^2 = x^2+y^2}$

$\boxed{\text{corrections}}$.

$\cancel{R=r^2}$

$$f(r^2) = r^2 f'(r^2) = r^2 \frac{df}{dr^2} \Rightarrow R = r^2$$

$$\frac{f}{R} = \frac{df}{dR}$$

$$\therefore \frac{df}{f} = \frac{dR}{R}.$$

$$\ln f = \ln R + C'.$$

$$\therefore f = CR.$$

$$\Rightarrow \cancel{f=} \boxed{f = Cr^2}.$$