$\alpha+$

Fantastic work! Congratulations.

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General Relativity I

Problem Set 4

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wk. 8. Thur. 16:00 - 17:30.

↑↑↑

1

The schwarzschild metric

$ds^2 = -d\tau^2 = \cancel{-(1-\frac{2GM}{r})}$

$-(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ ✓

The coordinate time t is the proper time of an observer at infinity $r \to \infty$, this can be seen from

$$1-\frac{2GM}{r}\bigg|_{r\to\infty} \to 1$$ Good!

As proven in Q5, PS3

can choose $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$, affine parameter $= \tau$ so that

$$\mathcal{L} = -(1-\frac{2M}{r})\dot{t}^2 + (1-\frac{2M}{r})^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -1. \quad ✓ \quad (1)$$

Conserved quantities $E = (1-\frac{2M}{r})\dot{t}$ ✓
$J = r^2\dot{\phi}$

and we ~~fur~~ found for circular orbit

$$J = \left(\frac{MR^2}{R-3M}\right)^{\frac{1}{2}}, \quad E = \frac{1-\frac{2M}{R}}{(1-\frac{3M}{R})^{\frac{1}{2}}} \quad (3) ✓$$

(2)

for orbital radius R.

For circular orbit, $\ddot{r} = 0$ ✓,

so in 1

$$-1 = -(1-\tfrac{2M}{R})\dot{t}^2 + R^2\dot{\phi}^2$$

$$= -(1-\tfrac{2M}{R})\dot{t}^2 + \cancel{R^2}\frac{J^2}{R^{\cancel{4}2}}$$

$$= -(1-\tfrac{2M}{R})\dot{t}^2 + \frac{J^2}{R^2}$$

$\therefore$ ~~$2\dot{t}^2 =$~~

$$\left(\frac{dt}{d\tau}\right)^2 = \dot{t}^2 = \frac{1+\frac{J^2}{R^2}}{1-\frac{2M}{R}} = \frac{1+\frac{M}{R-3M}}{1-\frac{2M}{R}}$$ ✓

$$= \frac{\frac{R-2M}{R-3M}}{1-\frac{2M}{R}} = \frac{\left(\frac{\cancel{1-\frac{2M}{R}}}{1-\frac{3M}{R}}\right)}{\cancel{1-\frac{2M}{R}}} = \frac{1}{1-\frac{3M}{R}}$$ ✓

$\square$

$$\left(\frac{d\phi}{dt}\right)^2 = \left(\frac{d\phi}{d\tau}\right)^2\left(\frac{d\tau}{dt}\right)^2 = \left(\frac{\dot{\phi}}{\dot{t}}\right)^2$$

$$= (1-\tfrac{3M}{R})\left(\frac{J^2}{R^4}\right)$$

$$= (1-\tfrac{3M}{R})\frac{1}{R^2}\frac{M}{R-3M}$$

$$= \cancel{(1-\tfrac{3M}{R})}\frac{M}{R^3}\frac{1}{\cancel{1-\frac{3M}{R}}} = \frac{M}{R^3}$$ Fantastic!

R

(2) ↑↑↑

$r(T) = R + \varepsilon(T)$

$\theta = \frac{\pi}{2}, \ \dot{\theta} = 0$ ✓

~~Recall that in PS3, Q5~~

$\mathcal{L} = -(1-\frac{2m}{r})\dot{t}^2 + (1-\frac{2m}{r})^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -1$

timelike geodesics

$$\frac{1}{2}\dot{r}^2 + \underbrace{\left(-\frac{m}{r} + \frac{J^2}{2r^2} - \frac{mJ^2}{r^3}\right)}_{V(r) \text{ effective potential.}} = \frac{E^2-1}{2} \quad \boxed{1}$$ ✓

Differentiate $\boxed{1}$,

$$2\dot{r}\ddot{r} + V'(r)\dot{r} = 0$$

$$\Rightarrow \ddot{r} + V'(r) = 0 \ ✓$$

$$V'(r) = + \frac{m}{r^2} - \frac{J^2}{r^3} + \frac{3mJ^2}{r^4}$$

$$\therefore \ \ddot{r} + \frac{m}{r^2} - \frac{J^2}{r^3} + \frac{3mJ^2}{r^4} = 0$$

Now use $r = R + \varepsilon$, $\ddot{r} = \ddot{\varepsilon}$ ✓

$\therefore$ ~~$\ddot{r} + m(R$~~

$$0 = \ddot{\varepsilon} + \frac{m}{R^2}\left(1+\frac{\varepsilon}{R}\right)^{-2} - \frac{J^2}{R^3}\left(1+\frac{\varepsilon}{R}\right)^{-3} + \frac{3mJ^2}{R^4}\left(1+\frac{\varepsilon}{R}\right)^{-4}$$

$$= \ddot{\varepsilon} + \frac{m}{R^2}\left(1 - \frac{2\varepsilon}{R} + O(\varepsilon^2)\right) - \frac{J^2}{R^3}\left(1 - 3\frac{\varepsilon}{R} + O(\varepsilon^2)\right)$$

$$+ \frac{3mJ^2}{R^4}\left(1 - \frac{4\varepsilon}{R} + O(\varepsilon^2)\right)$$

$$= \underbrace{\left(\frac{M}{R^2} - \frac{J^2}{R^3} + \frac{3MJ^2}{R^4}\right)}_{\boxed{2}} + \underbrace{\left(-\frac{2M}{R^3} + \frac{3J^2}{R^4} - \frac{12MJ^2}{R^5}\right)}_{\boxed{3}} \varepsilon + \ddot{\varepsilon} + O(\varepsilon^2)$$

$J^2 = \frac{MR^2}{R-3M} + O(\varepsilon)$

But $O(\varepsilon)$ terms cancels between themselves

assume $J^2 = \dfrac{MR^2}{R-3M} + O(\varepsilon^2)$ ✓ Good! / ref. PS3, Q5 for ~~a~~ circular orbit

$$\boxed{2} = \frac{M}{R^2} - \cancel{R}\frac{M}{R(R-3M)} + \frac{3M}{R^2}\frac{M}{R-3M} + O(\varepsilon^2)$$

$$= \frac{M}{R^2} - \left(\frac{1}{R} - \frac{3M}{R^2}\right)\frac{M}{R-3M} + O(\varepsilon^2)$$

$$= \frac{M}{R^2} - \left(\frac{\cancel{R-3M}}{R^2}\right)\frac{M}{\cancel{R-3M}} = \frac{M}{R^2} - \frac{M}{R^2} + O(\varepsilon^2)$$

$$= \cancel{0\ \varepsilon^2}\ 0 + O(\varepsilon^2)$$ ✓

$$\boxed{3} = -\frac{2M}{R^3} + \frac{1}{R^2}\frac{3M}{R-3M} - \frac{12M}{R^3}\frac{M}{R-3M}$$ ✓

$$= \frac{1}{R^3(R-3M)}\left(\cancel{-2M} - 2M(R-3M) + 3MR - 12M^2\right)$$

$$= \frac{(MR - 6M^2)}{R^3(R-3M)} = \frac{M}{R^3}\frac{(R-6M)}{(R-3M)}$$

$$\Rightarrow \ddot{\varepsilon} + \underbrace{\frac{M}{R^3}\frac{(R-6M)}{(R-3M)}}_{f(R)}\varepsilon = 0$$ to first order in ε ✓

$$\Rightarrow \ddot{\varepsilon} + f(R)\varepsilon = 0 \qquad \underline{f(R) = \frac{M}{R^3}\frac{6M-R}{3M-R}}$$ Good!

Assume at $T=0$, ~~ε~~ $\varepsilon(0)=0$, ~~[illegible]~~, then.

for $f(R)>0$, $\ddot{\varepsilon}+f\varepsilon=0 \Rightarrow$ ~~$\varepsilon = A\sin\sqrt{f}\,\varepsilon + B\sin$~~

$$\varepsilon = A\sin\sqrt{f}\,t + \cancel{B} \; B\cos\sqrt{f}\,t \checkmark$$

$\varepsilon(0)=0 \Rightarrow B=0$ ~~$\dot{\varepsilon}(0)=0 \Rightarrow$~~

$$\Rightarrow \quad \underline{\varepsilon = A\sin(\sqrt{f}\cdot t)}$$

→ the orbit is <u>stable</u> as perturbation is oscillatory

for $f(R)<0$, $\ddot{\varepsilon}+f\varepsilon=0 \Rightarrow \ddot{\varepsilon}-|f|\varepsilon=0$

$\therefore$ ~~$\varepsilon = Ce^{\sqrt{|f|}t} + De^{-\sqrt{|f|}}$~~

$$\varepsilon = C\cosh(\sqrt{|f|}\,t) + D\sinh(\sqrt{|f|}\,t) \checkmark$$

$\varepsilon(0)=0 \Rightarrow C=0 \quad \therefore \underline{\varepsilon = D\sinh(\sqrt{|f|}\,t)}$

→ the orbit is <u>unstable</u> as perturbation grows. ✓

⇒ for both cases, solution ~~is~~ ~~exist~~ exists only if

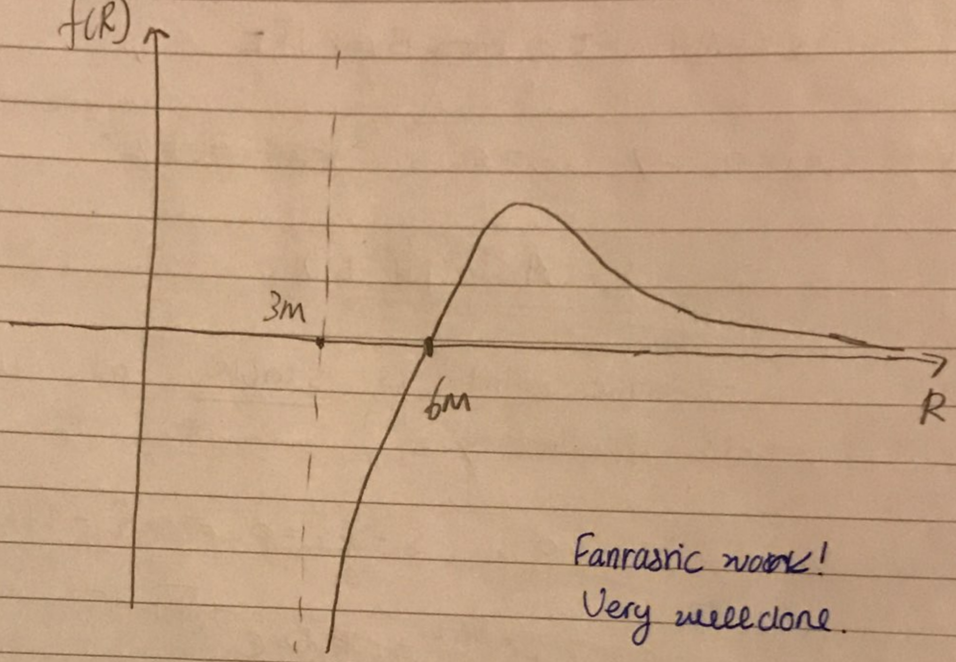
J ~~[illegible]~~ is real. → J^2 ~~[illegible]~~ >0

→ $R-3M>0 \quad \therefore \underline{\underline{R>3M}}$ Good!

and if circular orbit is stable

$f(R) > 0$, $R > 3M$

$\Rightarrow$ $R - 6M > 0$ $\Rightarrow$ $R > 6M$ stable.



Fantastic work!
Very well done.

$$f(R) = \frac{M}{R^3} \frac{R-6M}{R-3M}$$

(↑↑↑)

(3) The energy of a test particle moving with 4-momentum ~~p~~ p^a measured by an observer moving with 4 velocity U^a is given by

$$\tilde{E} = -p^a U_a \qquad p^a = m V^a$$

m: mass of particle. V^a: 4 velocity of test particle.

This equation is true in special relativity, since in the rest frame ~~p~~ $U = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $p = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$

$\eta = \text{diag}(-1,1,1,1)$ and $p^a U_a$ is Lorentz invariant.

✓ Good.

This is also true in general relativity, ~~as~~ since ~~Ua ≥ P~~ U^a, $p^a = mV^a$ are ~~both~~ still 4-vectors in general relativity. ✓

In our case, the observer has $\dot{r} = 0$, $\dot{\theta} = 0$, $\dot{\phi} = 0$. He/she should only have non-trivial $\dot{t} = U^0$

use ~~the that~~ $U^a U_a = -1$ (this observer is not moving along geodesic, but this normalisation should still ~~use~~ work as U^a is a 4 vector and $U^a U_a$ is a scalar and is -1 in local minkowski spacetime)

$$g_{00} U^0 U^0 = -1 \quad \therefore \quad +\left(1 - \frac{2M}{R}\right)(U^0)^2 = 1$$

$$\therefore \quad U^0 = \frac{1}{\sqrt{1 - \frac{2M}{R}}}, \quad U^1 = U^2 = U^3 = 0.$$

Good!

the 4 velocity V^a of ~~p~~ test particle is

$V^a = (\dot{t}, \dot{x}, \dot{y}, \dot{z})$

(big M, not small m).

$\therefore$ ~~E~~ energy per unit mass

$$\mathcal{E} = \frac{\tilde{E}}{m} = -U^a V_a = -g_{ab} U^a V^b$$

$$= -g_{00} U^0 V^0 = -1 - \left(1 - \frac{2M}{R}\right)\left(\frac{1}{\sqrt{1-\frac{2M}{R}}}\right)\dot{t}$$

$U^1 = U^2 = U^3 = 0$

$$= \sqrt{1-\frac{2M}{R}}\,\dot{t}$$ ✓ Great!

$\because$ energy $\tilde{E} = \gamma m$ and $\mathcal{E} = \frac{\tilde{E}}{m}$ $\therefore \mathcal{E} = \gamma$

$$\therefore \gamma = \sqrt{1-\frac{2M}{R}}\,\dot{t}$$ ✓

the conserved quantity E (not $\tilde{E}$).

is in general $E = \left(1 - \frac{2M}{r}\right)\dot{t}$

$$\therefore E = \underbrace{\sqrt{1-\frac{2M}{R}}\,\dot{t}}_{\gamma}\sqrt{1-\frac{2M}{R}} = \gamma\sqrt{1-\frac{2M}{R}}$$ ✓

for small v, small $\frac{2M}{R}$, $(\ll 1)$.

$$\gamma = \frac{1}{\sqrt{1-v^2}} \simeq 1 + \frac{1}{2}v^2 + \dots$$ ✓

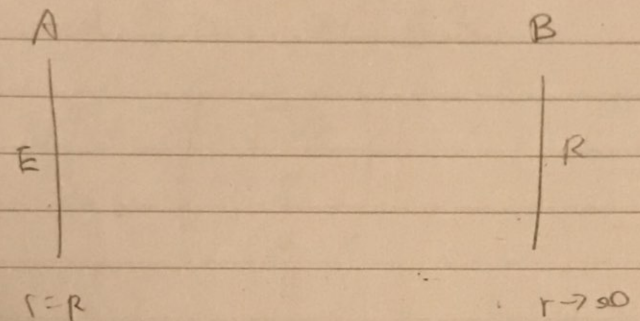
$$\sqrt{1-\frac{2M}{R}} \simeq 1 - \frac{M}{R} + \dots$$

$$\therefore \; E = \gamma \sqrt{1-\frac{2M}{R}} \simeq \left(1+\tfrac{1}{2}v^2+\cdots\right)\left(1-\frac{M}{R}+\cdots\right)$$

$$= 1 + \tfrac{1}{2}v^2 - \frac{M}{R} + \cdots$$

rest mass — kinetic energy — potential energy.

Fantastic!



~~Use~~ Quote the Gravitational shift equation ~~the~~

For a photon emitted at A having energy ν_E and received at B ~~here~~, the energy of the recieved photon, if ~~we~~ the coordinates of emitter and receiver are fixed, is ν_R given by

$$\frac{\nu_R}{\nu_E} = \left(\frac{g_{00}(A)}{g_{00}(B)}\right)^{\frac{1}{2}}$$

Great!

in our case $g_{00}(B) = -1$ $\quad g_{00}(A) = -\left(1-\frac{2M}{R}\right)$

$\cancel{\nu_E = E = \gamma\sqrt{1-\frac{2M}{R}}}$ $\quad \nu_E = E = \sqrt{1-\frac{2m}{R}}\,\dot{t}$

ignore mass m.

then.

$$V_R = V_E \left(\frac{g_{00}(A)}{g_{00}(B)} \right)^{\frac{1}{2}} = \sqrt{1-\frac{2M}{R}}\,\dot{t}\,\sqrt{1-\frac{2M}{R}}$$

$$= \left(1-\frac{2M}{R}\right)\dot{t} = \underline{\underline{E}}$$

$\Rightarrow$ E is precisely the energy of that photon measured at $\infty \sim r$.

Fantastic!

$A^a = \frac{D}{D\tau} u^a = u^b \nabla_b u^a$

$(g_{ab} A^a A^b)^{\frac{1}{2}}$ ∵ Only one component

∴ scalar is the r-component

change coordinates $dt(1-\frac{2M}{r})^{-1} \to dt$

(badly!)

4 ↑↑↑

$$A^a = V^b \nabla_b V^a$$

$$\nabla_b V^a = \partial_b V^a + \Gamma^a{}_{bc} V^c$$ ✓

For a stationary observer at radius R in Schwarzschild geometry the 4-velocity V^a was calculated in Q3

$$V^a = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{2M}{R}}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ ✓

$$\because \quad V^1 = V^2 = V^3 = 0$$

$$\therefore \quad A^a = V^b \nabla_b V^a = V^0 \nabla_0 V^a = V^0 (\partial_t V^a + \Gamma^a{}_{0c} V^c)$$ ✓

~~Also. Now of~~

No V^a components depends on time so

$\partial_t V^a = 0$

$$\therefore \quad A^a = V^0 \Gamma^a{}_{0c} V^c = \frac{1}{\sqrt{1-\frac{2M}{R}}} \Gamma^a{}_{tc} V^c$$ ✓

$t \equiv 0$

$\because V^c$ is ~~on for~~ non-zero only when $c = t$.

$$\therefore \quad A^a = \frac{1}{\sqrt{1-\frac{2M}{R}}} \Gamma^a{}_{tt} V^t = \frac{1}{1-\frac{2M}{R}} \Gamma^a{}_{tt}$$ ✓

~~Consider the Lagrangian~~

$$\cancel{L = -(1-\frac{2M}{r})\dot{t}^2 + (1-\frac{2M}{r})^{-1}\dot{r}^2 + r^2\dot{\theta}^2}$$

$$\cancel{+r^2\sin^2\theta\dot{\phi}^2}$$

the geodesic equations are reduced to.

t:

$$(1-\frac{2M}{r})\dot{t} = \text{const}$$

$$(1-\frac{2M}{r})\ddot{t} + (\frac{2M}{r^2})\dot{r}\dot{t} = 0$$

$$\Rightarrow \Gamma^t{}_{tt} = 0 \quad ✓$$

$$\cancel{\ddot{r} + \frac{M}{r^2}(1-\frac{2M}{r})\dot{t}^2}$$

Similarly it can be shown that $\Gamma^\theta{}_{tt} = \Gamma^\phi{}_{tt} = 0$

Now considering $\Gamma^r{}_{tt}$. ✓

$$\cancel{\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) - \frac{\partial L}{\partial r}}$$

By definition of levi-Civita connection, it can be shown that

$$\Gamma^a{}_{bb} \text{ (No sum)} = -\frac{1}{2g_{aa}}\frac{\partial g_{bb}}{\partial x^a}.$$

$$\cancel{\Gamma^r{}_{tt} = -\frac{1}{2g_{tt}}\frac{\partial}{\partial r}(g}$$

$$\Gamma^r{}_{tt} = -\frac{1}{2g_{rr}}\frac{\partial g_{tt}}{\partial r} \quad ✓$$

$$= -\frac{1}{2}\left(1-\frac{2M}{r}\right)\frac{\partial}{\partial r}\left(-\left(1-\frac{2M}{r}\right).\right)$$

$$= \frac{1}{2}\left(1-\frac{2M}{r}\right)\frac{\partial}{\partial r}\left(1-\frac{2M}{r}\right).$$

$$= \frac{M}{r^2}\left(1-\frac{2M}{r}\right). \checkmark$$

At $r=R$ $\quad \Gamma^r{}_{tt} = \frac{M}{R^2}\left(1-\frac{2M}{R}\right)$

$$A^r = V^b \nabla_b V^a = V^0 \nabla_0 V^r = V^t \Gamma^r{}_{tt} V^t$$

$$= \frac{1}{\cancel{1-\frac{2M}{R}}} \frac{M}{R^2} \cancel{\left(1-\frac{2M}{R}\right)} = \frac{M}{R^2}$$

$\therefore$ ~~A^r~~ and $A^t = A^\theta = A^\phi = 0$

$\therefore \quad A^a = \left(0, \frac{M}{R^2}, 0, 0\right) \checkmark$ $\quad$ ("$-$" sigh different)

Now $\quad a = (g_{ab} A^a A^b)^{1/2}$

$$= (g_{rr} A^r A^r)^{1/2}$$

$$= \left(\left(\frac{M}{R^2}\right)^2 \frac{1}{\left(1-\frac{2M}{R}\right)}\right)^{\frac{1}{2}}$$

$$= \frac{M}{R^2}\left(1-\frac{2M}{R}\right)^{-\frac{1}{2}} \checkmark$$

For $R \gg 2M$.

$$a = \frac{M}{R^2}\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \frac{M}{R^2}\left(1 + \frac{M}{R} + \dots\right) \simeq \frac{M}{R^2}$$

This is the newtonian expectation.

$$a = \frac{F}{m} = \frac{1}{\cancel{m}}\frac{\cancel{m}M}{R^2} = \frac{M}{R^2}.$$

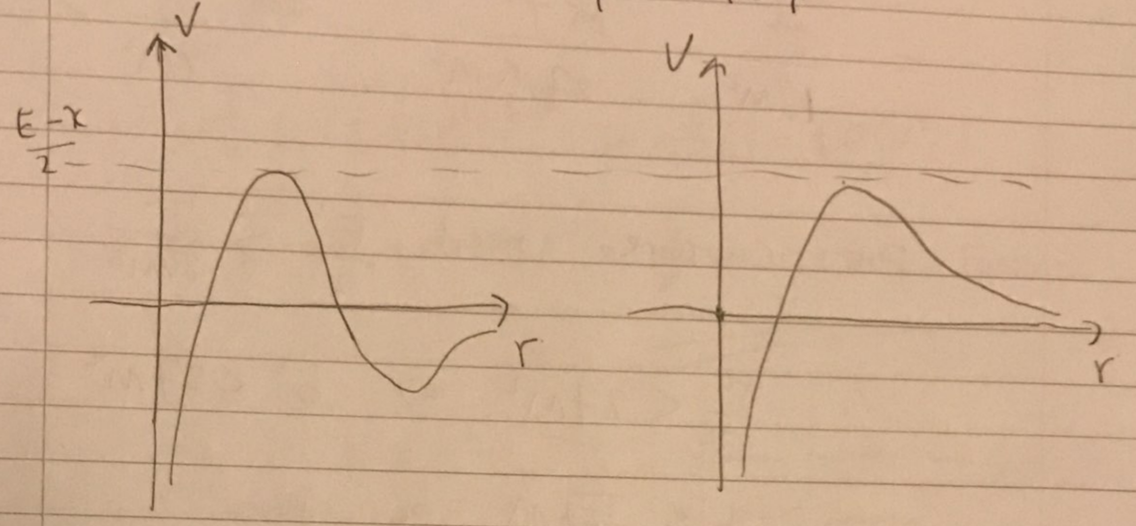
- Stationary observer can only exist if a is real

$$\Rightarrow \quad 1 - \frac{2M}{R} > 0 \quad \therefore \quad R > 2M$$

Correct!

(5)

Capture requires that:



$\kappa=1$ $\kappa=0$

$$\max(V(r)) \leqslant \frac{E^2-\kappa}{2}.$$

a) Null geodesic $\kappa=0$

$$b = \frac{J}{\sqrt{E^2}} = \frac{J}{E} \qquad (E>0 \text{ at capture})$$

$$V(r) = \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}$$

there is only one stationary point which is the maximum. ~~that~~ $(V''<0)$

$$V'(r^*) = 0 \Rightarrow 0 = J^2\left[-\frac{1}{r^{*3}} + \frac{3M}{r^{*4}}\right] = 0$$

$$\Rightarrow 0 = r^{*4} J^2 [3M - r^*] \Rightarrow r^* = 3M$$

$$V_{max} = V(r^*) = \frac{J^2}{2r^{*2}} - \frac{MJ^2}{r^{*3}}$$

$$= \frac{J^2}{18M^2} - \frac{\cancel{M}J^2}{27\cancel{M}\cdot M^2} = \frac{J^2}{54M^2}$$ ✓

For capture, need $\frac{E^2}{2} > \frac{J^2}{54M^2}$

$\therefore \frac{J^2}{E^2} < 27M^2 \Rightarrow b^2 < 27M^2$

$\Rightarrow \underline{\underline{b < \sqrt{27}M = b_c}}$ Clear!

and cross section $\underline{\underline{\sigma = \pi b_c^2 = 27\pi M^2}}$ ✓

b) For the massive case, $\kappa = 1$

$\therefore b = \frac{J}{\sqrt{E^2 - 1}}$

Recall in Q3 we had $\cancel{E = \gamma\sqrt{1 - \frac{2M}{R}}}$

$E = \gamma\sqrt{1 - \frac{2m}{r}}$ and as $r \to \infty$ $\cancel{E = \gamma}$ $\underline{\underline{E = \gamma}}$

At $r \to \infty$ $E = \gamma = \frac{1}{\sqrt{1 - v^2}}$ $(v \ll 1)$

$\therefore$ $b = \frac{J}{\sqrt{E^2 - 1}} = \frac{J}{\sqrt{\underbrace{\frac{1}{1 - v^2}}_{E^2} - 1}} = \frac{J}{\sqrt{(1 + v^2 + O(v^4) + \dots) - 1}}$

$$= \frac{J}{\sqrt{v^2 + O(v^4)}} = \frac{J}{v\sqrt{1 + O(v^2)}}$$

$$= \frac{J}{v}\left(1 + O(v^2)\right)^{-\frac{1}{2}} \simeq \frac{J}{v}\left(1 - \tfrac{1}{2}O(v^2)\right)$$

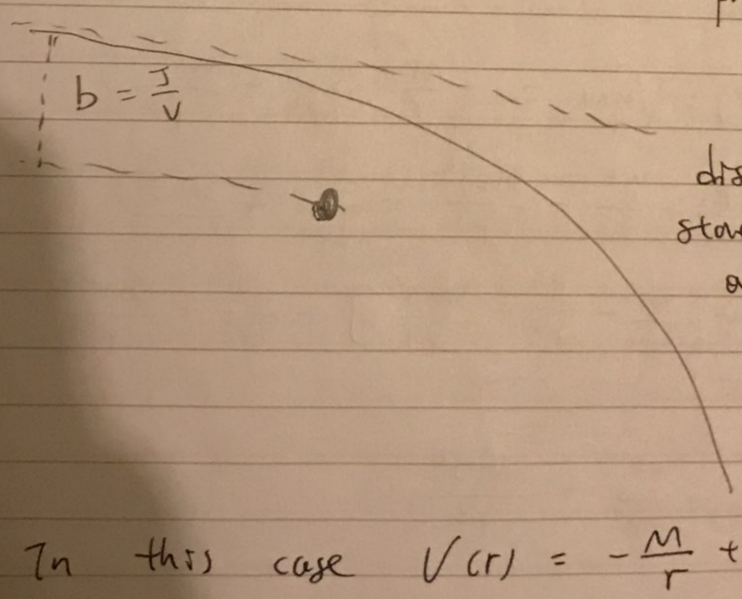
$$= \frac{J}{v}\left(1 + O(v^2)\right) \simeq \frac{J}{v} + O(v)$$ Fantastic!

$\Rightarrow \quad b \simeq \frac{J}{v} \rightarrow$ angular momentum per unit mass.

~~vel~~ velocity at $r \to \infty$

impact parameter.

physical significance:



b is the perpendicular distance from the starting point of trajectory at $r \to \infty$ to the gravitating body.

Clear!

In this case $V(r) = -\frac{M}{r} + \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}$.

$\therefore V'(r) = 0 \Rightarrow$

$$0 = +M\frac{1}{r^2} - \frac{J^2}{r^3} + \frac{3MJ^2}{r^4}$$

$$\Rightarrow \quad Mr^2 - J^2 r + 3MJ^2 = 0$$

$$\Rightarrow \quad r = \frac{J}{2M}\left(J \pm \sqrt{J^2 - 12M^2}\right)$$

the maximum will occur as the smaller one of the above two r's.

$$\therefore \quad r = \frac{J}{2M}\left(J - \sqrt{J^2 - 12M^2}\right) = r_{max} = r^*.$$

~~Now approximate~~ $\cancel{\frac{M}{J} \ll 1}$ ~~and~~

$$\cancel{r^* = \frac{J}{2M}\left(J - J\left(1 - \frac{12M^2}{J^2}\right)^{\frac{1}{2}}\right)}$$

$$\cancel{= \frac{J}{2M}\left(J - J + \frac{6M^2}{J}\right) \simeq 3M}$$

~~then~~ $\cancel{V(r^*) = V_{max} \simeq -\frac{M}{3M} +}$

Now $V(r^*) = V_{max} = -\frac{M}{r} + \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}$

$$= -\frac{M}{r} + \frac{J^2}{r^2} - \frac{3MJ^2}{r^3} - \frac{J^2}{2r^2} + \frac{2MJ^2}{r^3}$$

$$= -r\underbrace{\left[\frac{M}{r^2} - \frac{J^2}{r^3} + \frac{3MJ^2}{r^4}\right]}_{0} - \frac{J^2}{2r^2} + \frac{2MJ^2}{r^3}$$

$$= -\frac{J^2}{2r^2} + \frac{2MJ^2}{r^3}$$

$$\because \quad MJ^2 = \frac{1}{3}\left(J^2 r - Mr^2\right)$$

$$\therefore \quad V_{max} = -\frac{J^2}{2r^2} + \frac{2}{3r^3}\left(J^2 r - Mr^2\right)$$

$$= -\frac{J^2}{2r^2} + \frac{2J^2}{3r^2} - \frac{2M}{3r}$$

$$= \frac{J^2}{6r^2} - \frac{2M}{3r}$$

$$= \frac{1}{6r^2}(J^2 - 4Mr)$$

$$r = \frac{J}{2M}\left(J - \sqrt{J^2 - 12M^2}\right)$$

$$r^2 = \frac{J^2}{4M^2}\left(2J^2 - 12M^2 - 2J\sqrt{J^2-12M^2}\right)$$

$$\therefore V_{max} = \frac{J^2 - 2J\left(J - \sqrt{J^2-12M^2}\right)}{\frac{3J^2}{2M^2}\left(2J^2 - 12M^2 - 2J\sqrt{J^2-12M^2}\right)}$$

$$= \frac{2J\sqrt{J^2-12M^2} - J^2}{\frac{3J^2}{M^2}\left(J^2 - 6M^2 - J\sqrt{J^2-12M^2}\right)}$$

e.g. $J > 5M$
$V_{max} > 0$
No capture.

Capture requires $\frac{E-1}{2} \geq V_{max}$.

$$\frac{E-1}{2} \simeq \frac{1+v^2-1}{2} \simeq 0 + O(v^2) \simeq 0$$

$\therefore$ we require roughly $V_{max} = 0$ for capture

Good!

$$\Rightarrow \quad 2J\sqrt{J^2-12M^2} - J^2 = 0$$

$$\therefore \quad 2\sqrt{J^2-12M^2} = J. \quad \rightarrow 4J^2 - 48M^2 = J^2$$

$$\cancel{4J^2 \geq 48M^2 =} \rightarrow 3J^2 = 48M^2$$

$$\therefore \quad J^2 = 16M^2 \Rightarrow \quad J = 4M.$$ Fantastic!

critical Impact parameter $\quad b_c = \frac{J}{v}\Big|_{J=4m} = \frac{4M}{v}$ ✓

~~So~~ capture cross-section is

$$\sigma = \pi b_c^2 = \pi\left(\frac{4M}{v}\right)^2 = 16\pi M^2/v^2$$

Very good!

$$V(r^*) = \frac{M^3}{r^3}\left(\left(\frac{r^2}{M^2} + \frac{J^2}{M^2}\right) - \frac{J^2 r}{2M^3}\right)$$

$$V(r) = \left(\frac{r}{M} - \frac{J}{M}\,\frac{J/m + \sqrt{(J/m)^2-16}}{4}\right)$$

$$\times\left(\frac{r}{m} - \frac{J}{m}\,\frac{J/m - \sqrt{(J/m)^2-16}}{4}\right)$$

$$\Rightarrow (J/m)^2 \geq 16 \quad \Rightarrow \quad J^2 \geq 16M^2$$

$$b^2 \geq \frac{16M^2}{v^2} \qquad b_c = \frac{4M}{v}$$

$$\sigma = 16\frac{M^2}{v^2}\pi.$$

7

↑↑↑

Modified Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi T_{ab}$$

~~For~~ perfect fluid $\cancel{T_{ab} = P g_{ab} + (\rho + \frac{P}{c^2}) U_a U_b}$

$\cancel{8\pi T_{ab} - \Lambda g_{ab} =}$ $\quad T_{ab} = P g_{ab} + (\rho + P) U_a U_b$ ✓

rescale ρ and P such that

$$\tilde{\rho} = \rho + \rho_\Lambda = \rho + \frac{\Lambda}{8\pi}, \quad \tilde{P} = P + P_\Lambda = P - \frac{\Lambda}{8\pi} \quad \text{then}$$ ✓

$$\tilde{T}_{ab} = \tilde{P} g_{ab} + (\tilde{\rho} + \tilde{P}) U_a U_b$$

$$= \left(P - \frac{\Lambda}{8\pi}\right) g_{ab} + \left(\rho + \cancel{\frac{\Lambda}{8\pi}} + P - \cancel{\frac{\Lambda}{8\pi}}\right) U_a U_b$$ ✓

$$= P g_{ab} + (\rho + P) U_a U_b - \frac{\Lambda}{8\pi} g_{ab} = T_{ab} - \frac{\Lambda}{8\pi} g_{ab}$$

$$\therefore 8\pi \tilde{T}_{ab} = 8\pi T_{ab} - \Lambda g_{ab}$$ ✓

$$\Rightarrow R_{ab} - \frac{1}{2} g_{ab} R = 8\pi \tilde{T}_{ab}$$ ✓ Good!

where $\tilde{T}_{ab}$ is the energy-momentum tensor with pressure $\tilde{P} = P + P_\Lambda$ and density $\tilde{\rho}$ given by $\tilde{\rho} = \rho + \rho_\Lambda$. ✓

$\because$ The solution of $R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$ and T_{ab} being stress-energy tensor (perfect fluid) with pressure P and density ρ is

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho+3P)$$ ✓

Hence the equation $R_{ab} - \frac{1}{2}g_{ab}R = \tilde{T}_{ab}$ has solution

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\tilde{\rho}}{3} - \frac{k}{a^2}, \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\tilde{\rho}+3\tilde{P})$$ ✓

$$\Rightarrow \quad \left(\frac{a'}{a}\right)^2 = \frac{8\pi}{3}\left(\rho+\frac{\Lambda}{8\pi}\right) - \frac{k^2}{a} = \frac{8\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$ ✓

$$\Rightarrow \quad \frac{a''}{a} = -\frac{4\pi}{3}(\tilde{\rho}+3\tilde{P}) = -\frac{4\pi}{3}\left(\rho+\frac{\Lambda}{8\pi}+3P-\frac{3\Lambda}{8\pi}\right)$$

$$= \cancel{-\tfrac{4\pi}{3}} -\frac{4\pi}{3}(\rho+3P) + \frac{\Lambda}{3}$$ ✓

The equation $\rho' + 3\frac{a'}{a}(\rho+P) = 0$ is unaffected by Λ since $\rho' = \tilde{\rho}'$ and $P+\rho = \tilde{P}+\tilde{\rho}$

For pressureless dust: $P = 0$.

$$\cancel{\rho} \quad \rho' + 3\frac{a'}{a}\rho = 0 \Rightarrow \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0$$ ✓

$$\Rightarrow \quad \rho a^3 = \text{const} \rightarrow \rho_m \sim \frac{1}{a^3}$$ ✓

For radiation: ~~From therre theenort theromd~~

From thermodynamics of photon gas.

Good!

$$P = \frac{1}{3}\rho$$

$\rightarrow \quad \rho' + \frac{3a'}{a} \cdot \frac{4}{3}\rho = 0.$

$\Rightarrow \frac{1}{a^4}\frac{d}{d\tau}(\rho a^4) = 0 \Rightarrow \rho a^4 = \text{const.}$

$\therefore \rho_r \sim \frac{1}{a^4}$

expanding universe.

Hence at early times of ~~expan~~ expanding universe, ~~for~~ a is very small $\frac{1}{a^4} \gg \frac{1}{a^3} \gg \Lambda = \text{const.}$

$\therefore$ Dominated by ρ_r, the radiation energy density

At late times, ~~att~~ both ρ_m, ρ_r scales with $\frac{1}{a^k}$ where $k \in \mathbb{Z}^+$ $\therefore \rho_m, \rho_r$ decays away as a becomes large.

$\Rightarrow$ Dominated by cosmological constant.

At very late times, treat $\rho \sim 0$, $p \sim 0$

$\therefore \frac{a''}{a} = \frac{\Lambda}{3} \quad \therefore a'' - \frac{\Lambda}{3}a = 0.$

$\because \frac{\Lambda}{3} > 0 \quad \therefore a(\tau) \simeq Ae^{\sqrt{\frac{\Lambda}{3}}\tau} + Be^{-\sqrt{\frac{\Lambda}{3}}\tau}$

For very large τ the second term can be neglected

$\rightarrow \quad a(\tau) \sim \exp\left(\sqrt{\frac{\Lambda}{3}}\tau\right)$

$$p' + \frac{3a'}{a} \cdot \frac{4}{3} p = 0.$$

$$\Rightarrow \frac{1}{a^4} \frac{d}{d\tau}(\rho a^4) = 0 \Rightarrow \rho a^4 = \text{const.}$$

$$\therefore \rho_r \sim \frac{1}{a^4}$$

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For very large τ the second term can be neglected

$$\rightarrow a(\tau) \sim \exp\left(\sqrt{\frac{\Lambda}{3}}\tau\right)$$

Hence the scale factor grows exponentially.

$\because$ ~~growth is~~ expansion is exponential

$\therefore$ It has future horizon.

$$\left(\int_0^{l} dl' = -\int_{\tau_0}^{\tau} \frac{d\tau'}{a(\tau')} \simeq -\int_{\tau_0}^{\tau} e^{-\sqrt{\frac{\Lambda}{3}}\tau'} d\tau' \right.$$

$$\left. \simeq \text{finite.} \right).$$

Good!.

Late times, $a'^2 = \frac{8\pi \rho a^2}{3} - k + \frac{\Lambda a^2}{3}$

$$\sim \frac{1}{a} \qquad \sim 1$$

$$a(\tau) \approx e^{\sqrt{\Lambda/3}\,\tau}.$$

$d\tau = a(\tau)\,dr$

$d\tau = a(\tau)\,d\eta. \quad \Rightarrow ds^2 = 0 = a(\tau)^2(-d\eta^2 + dr^2)$

$$\left(1 - \frac{2m}{r}\right)^{-1} dr$$

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g_{rr}