$\boxed{\text{Ziyan Li}}$

$\alpha+$

Great job!

General Relativity I

TA: Diego Berdeja Suarez

Problem Set 3

Wk. 6 Thu. 16:00 - 17:30

1 ↑↑↑

1. $R_{abc}{}^{d} = -R_{bac}{}^{d}$

2. $R_{abc}{}^{d} + R_{bca}{}^{d} + R_{cab}{}^{d} = 0$

When ∇_a is the Levi-Civita connection we have

$$\nabla_a g_{bc} = 0$$

Now consider

$$\underbrace{0}_{\text{Levi-civita connection}} = (\nabla_a \nabla_b - \nabla_b \nabla_a) g_{cd} \underbrace{=}_{\text{Given in Lectures}} R_{abc}{}^{e} g_{ed} + R_{abd}{}^{e} g_{ce}$$

$$= R_{abcd} + R_{abdc}$$

Great!

$\rightarrow$ $R_{abcd} = -R_{abdc}$ 3 (Antisymmetry of last 2 indices)

is the additional algebraic identity.

In D dimensions, each index of R_{abcd} can run from $1, 2, \ldots, D$. So there are D^4 components. But there are constraints given by 1, 2, 3 so the components are not all independent.

Constraint 1 : (1, 2 can be simply put into covariant form)

$$R_{abcd} = -R_{bacd}$$

~~So far~~ indices c and d now has $D \times D = D^2$ choices.

For a, b we have $\binom{D}{2}$ + number of diagonal elements when $a=b$.

In total this gives $\binom{D}{2} + D = \frac{D(D-1)}{2} + D$

$$= \frac{D(D+1)}{2} \text{ for } a \text{ and } b.$$

With c, d we have $D^2 \frac{D(D+1)}{2}$ constraints by $\underline{\underline{1}}$.

Constraint $\underline{\underline{2}}$:

$$R_{abcd} + R_{bcad} + R_{cabd} = 0$$

To avoid double counting, we require a, b, c distinct. Because if $a=b$, then

$$\underbrace{R_{aacd}}_{=0 \text{ by } \underline{\underline{1}}} + R_{acad} + R_{caad} = 0$$

First term $=0$ by $\underline{\underline{1}}$ and remaining terms are simply a rephrase of $\underline{\underline{1}}$. So we've double counted.

Hence we have $D\binom{D}{3} = \frac{D \cdot D(D-1)(D-2)}{6}$

constraints by $\underline{\underline{2}}$ Good!

The final extra D comes from D choices of index d.

Constraint $\underline{\underline{3}}$:

To count this one, we start by considering R_{2356}, a specific excmple.

By $\underline{\underline{1}}$ $\quad R_{2356} = -R_{3256}$

If we also impose $\underline{\underline{3}}$, then $\quad R_{2356} = -R_{2365}$

Then when we also impose $\underline{\underline{3}}$ to ~~R3265 = R~~ ~~R23~~ $R_{3256} = -R_{3265}$. We see that this is in fact unnecessary, because we already know that

$$R_{3256} = -R_{2356} = -(-R_{2365})$$
$$= R_{2365} = -R_{3265}$$

without having to impose it!

the choice of 2, 3. 6, 5 are general so we conclude that in this case if (a,b) appears as first two indices ~~the~~ then we don't count (b,a). Also if ~~the~~ first and second indices are ~~equt~~ equal, we also don't count because we know for sure this term is 0 already by $\underline{\underline{1}}$.

So extra conditions imposed by $\underline{\underline{3}}$ ~~is~~ includes

~~D^2~~ $\binom{D}{2} = \frac{D(D-1)}{2}$ ~~cases~~ choices of first two indices ~~b~~ (a,b). This replaces the "D^2" in case $\underline{\underline{1}}$.

For the second pair of indices (c, d) the case is identical to $\underline{\underline{1}}$.

We need $\binom{D}{2}$ + ~~D~~ cases when $c = d$ so we have a total of $\frac{D(D+1)}{2}$.

So In the end we get

$$\frac{D(D-1)}{2} \times \frac{D(D+1)}{2} = \frac{1}{4} D^2 (D^2-1) \text{ extra}$$

constraints given by $\underline{\underline{3}}$.

Now number of independent components N_D is.

$$N_D = D^4 - D^2 \frac{D(D+1)}{2} - \frac{D \cdot D(D-1)(D-2)}{6} - \frac{1}{4} D^2 (D^2-1)$$

$$= D^4 - \frac{D^4}{2} - \cancel{\frac{D^3}{2}} - \frac{D^4}{6} \cancel{\frac{D^4}{6}} + \cancel{\frac{D^3}{2}} - \frac{D^2}{3} - \frac{D^4}{4} + \frac{D^2}{4}$$

$$= \frac{D^4}{3} - \frac{D^2}{3} - \frac{D^4}{4} + \frac{D^2}{4} = \frac{D^4}{12} - \frac{D^2}{12}$$

$$= \frac{D^2(D^2-1)}{12}$$

Good!

check when $D = 4$, this indeed gives

$$N_4 = \frac{4^2(4^2-1)}{12} = \frac{16 \times 15}{12} = \cancel{20}\ 20$$

Great!

independent components.

(2) ↑↑↑

1. From (1) we know that number of independent components in 2-D Riemann Curvature is $N_2 = \frac{2^2(2^2-1)}{12} = \frac{4\times 3}{12} = \underline{\underline{1}}$ ✓

Only 1 independent component and it completely determines the Riemann tensor.

By conditions $\underline{\underline{1}}$ and $\underline{\underline{3}}$ in (1),

$R_{1111} = R_{1112} = R_{1121} = R_{1122} = R_{1222} = R_{2122} = R_{2222}$
$= R_{1211} = R_{2111} = R_{2211} = R_{2212} = R_{2221} = 0$ ✓

together gives 12 identically 0 components

Condition $\underline{\underline{2}}$ has absolutely no use because for $D=2$ there is no way to make first 3 indices distinct, so $\underline{\underline{2}}$ will be covered by $\underline{\underline{1}}$ and $\underline{\underline{3}}$

there are $2^4 = 16$ components so non-zero ones are

$$16 - 12 = 4 \text{ components.}$$

If we specify $\underline{\underline{R_{1212}}}$, then the other 3 are

$R_{2112} = -R_{1212}$, $R_{1221} = -R_{1212}$, $R_{2121} = R_{1212}$.

the Ricci tensor $R_{ab} = R_{acb}{}^{c} = R_{acbe}\, g^{ec}$
is also completely determined.

— $R_{11} = R_{121}{}^{2} = R_{1212}\, g^{22}$

$\cancel{R_{12} = R_{122}{}^{1} = R_{1221}\, g^{11}}$ $\cancel{R_{12} = R_{122}{}^{2} + R_{112}{}^{1} = 0}$

$\cancel{R_{21} = R_{211}{}^{1} + R_{221}{}^{2} = 0}$

$\cancel{R_{21} = R_{211}{}^{1}}$ $\cancel{R_{21} = R_{21}}$

— $R_{22} = R_{212}{}^{1} = R_{2121}\, g^{11} = R_{1212}\, g^{11}$

$\cancel{R = R^{1}{}_{1} + R^{2}{}_{2} = g^{11}R_{11} + g^{22}R_{22} = (g^{11} + g^{22})R_{1212}}$

$\cancel{= (g^{11}g^{22} + g^{22}g^{11})R_{1212}}$

$\cancel{= 2g^{11}g^{22}\, R_{1212}}$

— $R_{12} = R_{122}{}^{2} = R_{1221}\, g^{12} = -R_{1212}\, g^{12}$

— $R_{21} = R_{211}{}^{1} = R_{2112}\, g^{21} = -R_{1212}\, g^{21}$

$$R = R^{1}{}_{1} + R^{2}{}_{2} = R_{11}g^{11} + R_{12}g^{21} + R_{21}g^{12} + R_{22}g^{22}$$

$$= R_{1212}\,(g^{11}g^{22} - g^{21}g^{12} - g^{12}g^{21} + g^{11}g^{22})$$

$$= 2R_{1212}\,(g^{11}g^{22} - g^{12}g^{21})$$

Now consider the quantity

$$\tfrac{1}{2} R\,(g_{ac}g_{bd} - g_{ad}g_{bc})$$

Clearly ~~(*) = 0 when~~ (*) is antisymmetric under the exchange of (a,b) and (c,d)
So $\underline{\underline{1}}$, $\underline{\underline{3}}$ satisfied. Also since $D=2$, $\underline{\underline{2}}$ is automatically satisfied by $\underline{\underline{1}}$ and $\underline{\underline{3}}$ ✓

So $\frac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc}) = R_{abcd}$ iff we show $R_{1212} = \frac{1}{2}R(g_{11}g_{22} - g_{12}g_{21})$, because (*) also just have 1 independent component. ✓

$$\Rightarrow \frac{1}{2}R(g_{11}g_{22} - g_{12}g_{21}) = \frac{1}{2}\cancel{\times 2R(g^{11}g^{22} - g^{12}g^{21}) \times (g_{11}g_{12} - g_{12}g_{21})}$$

$$= \cancel{\frac{2}{2}} R_{1212}\underbrace{(g^{11}g^{22} - g^{12}g^{21})}_{\det(g^{-1})}\underbrace{(g_{11}g_{12} - g_{12}g_{21})}_{\det(g)}$$

$$= R_{1212}\det(\underbrace{gg^{-1}}_{I}) = \underline{\underline{R_{1212}}}$$

This concludes the proof, where

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \quad \square$$

2. $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ the Einstein Tensor

In 2D: $R_{ac} = R_{abcd}g^{db} = \frac{1}{2}R(g_{ac}g_{bd}g^{db} - g_{ad}g_{bc}g^{db})$

$$= \frac{1}{2}R(g_{ac}\delta^b{}_b - g_{ad}\delta_c{}^d)$$ ✓

where $\delta^b{}_b$ = dimension of space $= 2$

$\therefore \cancel{G_{ac} = R_{ab}} \quad G_{ac} = R_{ac} - \frac{1}{2} g_{ac} R$

$$= \frac{1}{2} R (g_{ac} \underbrace{\delta^b_b}_{2} - g_{ac}) - \frac{1}{2} g_{ac} R$$

$$= \frac{1}{2} \cancel{R g_{ac}} - \frac{1}{2} \cancel{g_{ac} R} = 0 \quad \square$$

3.

$$ds^2 = -du^2 + \cosh^2 u \, d\phi^2$$

(From PS2, Q4):

$$\Gamma^u_{\phi\phi} = \cosh(u)\sinh(u) \qquad \Gamma^\phi_{u\phi} = \Gamma^\phi_{\phi u} = \tanh(u)$$

$\cancel{R_{abc}{}^d =}$ $\quad g_{uu} = -1 \quad g_{\phi\phi} = \cosh^2 u$

$$g^{uu} = -1 \quad g^{\phi\phi} = \frac{1}{\cosh^2 u}$$

$\cancel{R_{abc}{}^d}$

$$R_{abc}{}^d = \frac{\partial}{\partial x^b} \Gamma^d_{ac} - \frac{\partial}{\partial x^a} \Gamma^d_{bc} + \Gamma^e_{ac} \Gamma^d_{eb} - \Gamma^e_{bc} \Gamma^d_{ea}$$

$$\therefore R_{u\phi u}{}^\phi = \underbrace{\frac{\partial}{\partial \phi} \Gamma^\phi_{uu}}_{=0} - \frac{\partial}{\partial u} \Gamma^\phi_{\phi u}$$

$$+ \Gamma^u_{uu} \underbrace{\Gamma^\phi_{u\phi}}_{=0} + \underbrace{\Gamma^\phi_{uu}}_{=0} \Gamma^\phi_{\phi\phi}$$

$$- \underbrace{\Gamma^u_{\phi u} \Gamma^\phi_{uu}}_{=0} - \Gamma^\phi_{\phi u} \Gamma^\phi_{\phi u}$$

$$= -\frac{\partial}{\partial u} \Gamma^\phi_{\phi u} - \Gamma^\phi_{\phi u} \Gamma^\phi_{\phi u}$$

$$= -\frac{\partial}{\partial u}\left(\tanh(u)\right) - \cancel{2}\tanh^2(u)$$

$$= \cancel{-\operatorname{sec}^2} - \operatorname{sech}^2 u - \tanh^2 u$$

$$= -\left(\frac{1}{\cosh^2 u}\right)\left[1 + \sinh^2 u\right]$$

$$= -\frac{1}{\cosh^2 u}\left[\cosh^2 u\right] = -1$$

~~3.~~ $\cancel{R_{u\varphi u}{}^{\varphi}}$

$\cancel{R\ g_{u\varphi} g_{u\varphi} - g}$

$$- \quad R_{u\varphi u\varphi} = R_{u\varphi u}{}^{\varphi} g_{\varphi\varphi} = (-1)(\cosh^2(u))$$

$$= \underline{\underline{-\cosh^2 u}} \quad \checkmark$$

4. Now $\cancel{g_{u\varphi} g_{u\varphi} - g_{uu}}$

$$g_{uu} g_{\varphi\varphi} - \underbrace{g_{u\varphi} g_{\varphi u}}_{=0} = g_{uu} g_{\varphi\varphi}$$

$$= (-1)(\cosh^2 u) = -\cosh^2 u = \underline{\underline{R_{u\varphi u\varphi}}}$$

indeed.

$$R = 2R_{u\varphi u\varphi}\left(g^{uu} g^{\varphi\varphi} - \underbrace{g^{u\varphi} g^{\varphi u}}_{=0}\right)$$

$$= 2(-\cancel{\cosh^2 u})(-1)\left(\frac{1}{\cancel{\cosh^2 u}}\right) = \underline{\underline{2}} \quad \checkmark$$

Good!

(3) ↑↑↑

$$\frac{D^2 X^d}{D\tau^2} = -R_{abc}{}^d V^a X^b V^c$$

$$V^a = \frac{dx^a}{d\tau} \qquad X^b = \frac{dx^b}{ds}$$

$$\frac{D}{D\tau} = V^a \nabla_a$$

Newtonian limit means : (From Lectures)

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} \quad \text{for } \epsilon \ll 1$$

$$\frac{\partial h_{ab}}{\partial t} \simeq 0$$

$$V^a \simeq (1,0,0,0)$$

$$R_{abc}{}^d = -\frac{\epsilon}{2}\eta^{de}(\partial_a \partial_c h_{be} - \partial_a \partial_e h_{bc} + \partial_e \partial_b h_{ac} - \partial_b \partial_c h_{ae})$$

$$\phi = -\frac{\epsilon}{2} h_{00}$$

$$\nabla_a \simeq \partial_a$$

$$\Rightarrow V^a \nabla_a \simeq \cancel{(1,0,0,0)(\partial_t, \partial_x, \partial_y, \partial_z)}$$

$$(1,0,0,0)\cdot(\partial_t, \partial_x, \partial_y, \partial_z)$$

$$= \partial_t$$

$\therefore$

$$\therefore \quad \frac{D^2 x^d}{D\tau^2} \simeq V^a \nabla_a (V^b \nabla_b \cancel{x} X^d)$$

$$= \partial_t^2 X^d = \frac{\partial^2 X^d}{\partial t^2} \checkmark$$

$$-R_{abc}{}^d V^a X^b V^c$$

$$= \simeq -R_{0b0}{}^d V^0 X^b V^0$$

$$= -R_{0b0}{}^d X^b$$

$$= \frac{\epsilon}{2} \eta^{de} (\partial_t \partial_t h_{be} - \partial_t \partial_e h_{b0} + \partial_e \partial_b h_{00}$$

$$- \partial_b \partial_t h_{0e}) X^b$$

the 1st, 2nd, 4th terms in the above bracket vanish because $\partial_t h_{ab} \simeq 0$ ✓

$$\therefore \quad -R_{abc}{}^d V^a X^b V^c \simeq \frac{\epsilon}{2} \eta^{de} (\partial_e \partial_b h_{00}) X^b$$

$$= -\eta^{de} (\partial_e \partial_b \phi) X^b,$$

where we've used $-\frac{\epsilon}{2} h_{00} = \phi$ ✓

$$\therefore \quad \frac{D^2 X^d}{D\tau^2} = -R_{abc}{}^d V^a X^b V^c$$ Good!

$\Rightarrow$ in Newtonian limit

$$\Rightarrow \quad \frac{\partial^2 x^d}{\partial t^2} = -\eta^{de}(\partial_e \partial_b \phi) x^b$$

take spatial component $d = i = 1, 2, 3$

$$\frac{\partial^2 x^i}{\partial t^2} = -\eta^{ie}(\partial_e \partial_b \phi) x^b$$

when $b = 0$ or $e = 0$. $\partial_e \partial_b \phi$ vanishes

$\rightarrow$ ~~$(\partial_e \partial_b \phi) x^b = (\partial_e \partial_j \phi) x^j$ and~~

$\therefore$ We can take $e, b = i, j \in \{1, 2, 3\}$

$\rightarrow$ $(\partial_e \partial_b \phi) x^b = (\partial_e \partial_j \phi) x^j$ ✓

$\rightarrow$ $\because \eta^{11} = \eta^{22} = \eta^{33} = 1$ and all other η^{ij} vanishes $\Rightarrow$ ~~$\eta^{ie} = \delta^{ie}$ for $e = \{1,2,3\}$~~

~~$$\frac{\partial^2 x^i}{\partial t^2} = -\eta^{ie}(\partial_e \partial_j \phi) x^j$$~~ when $i, j \in \{1, 2, 3\}$

~~$$= -\delta^{ie}$$~~

$$\Rightarrow \eta^{ie} = \delta_i^{\ e} \text{ for } e \in \{1, 2, 3\}$$

$$\therefore \frac{\partial^2 x^i}{\partial t^2} = -\eta^{ie}(\partial_e \partial_j \phi) x^j$$

$$= -(\delta_i^{\ e} \partial_e \partial_j \phi) x^j$$

$$= -\partial_i \partial_j \phi \, x^j$$

$$\Rightarrow \frac{\partial^2 x^i}{\partial t^2} = -\sum_j \frac{\partial^2 \phi}{\partial x^i \partial x^j} x^j \quad \square$$

Great!

$$\Rightarrow \quad \frac{\partial^2 x^d}{\partial t^2} = -\eta^{de}(\partial_e \partial_b \phi) x^b$$

take spatial component $d = i = 1, 2, 3$

$$\frac{\partial^2 x^i}{\partial t^2} = -\eta^{ie}(\partial_e \partial_b \phi) x^b$$

when $b = 0$ or $e = 0$. $\partial_e \partial_b \phi$ vanishes

$\rightarrow$ $\cancel{(\partial_e \partial_b \phi) x^b = (\partial_e \partial_j \phi) x^j}$ ~~and~~

$\therefore$ We can take $e, b = i, j \in \{1, 2, 3\}$

$\rightarrow$ $(\partial_e \partial_b \phi) x^b = (\partial_e \partial_j \phi) x^j$ ✓

$\rightarrow$ $\because$ $\eta^{11} = \eta^{22} = \eta^{33} = 1$ and all other η^{ij} vanishes $\Rightarrow$ $\cancel{\eta^{ie} = \delta^{ie} \text{ for } e \in \{1,2,3\}}$

$$\cancel{\frac{\partial^2 x^i}{\partial t^2} = -\eta^{ie}(\partial_e \partial_j \phi) x^j} \quad \text{when } i, j \in \{1, 2, 3\}$$

$$\cancel{= -\delta^{ie}}$$

$$\Rightarrow \quad \eta^{ie} = \delta_i^{\,e} \quad \text{for } e \in \{1, 2, 3\}$$

$$\therefore \quad \frac{\partial^2 x^i}{\partial t^2} = -\eta^{ie}(\partial_e \partial_j \phi) x^j$$

$$= -\left(\delta_i^{\,e} \partial_e \partial_j \phi\right) x^j$$

$$= -\partial_i \partial_j \phi \, x^j$$

$$\Rightarrow \quad \frac{\partial^2 x^i}{\partial t^2} = -\sum_j \frac{\partial^2 \phi}{\partial x^i \partial x^j} x^j \quad \square$$

Great!

We are left to show that the LHS of required equation is indeed $\frac{\partial^2 x^i}{\partial t^2} \simeq \frac{D^2 x^i}{D\tau^2}$

$$\frac{D^2 x^d}{D\tau^2} \simeq (V^a\nabla_a)(V^b\nabla_b) X^d$$

$$V^b \nabla_b X^d = V^b(\partial_b X^d + \Gamma^d{}_{bc} X^c).$$

$$= V^0(\partial_t X^d + \Gamma^d{}_{0c} X^c)$$

$$+ V^i(\partial_i X^d + \Gamma^d{}_{ic} X^c)$$

Newtonian :

$$\Gamma^d{}_{bc} \simeq \frac{\varepsilon}{2}\eta^{de}(\partial_c h_{eb} + \partial_b h_{ec} - \partial_e h_{bc})$$

$$V^a \simeq (1,0,0,0) + O(\varepsilon)$$

If $V^i \sim O(\varepsilon)$, then $V^a V_a = -1$ gives

$$-(V^0)^2 + \underbrace{(V^i)^2}_{O(\varepsilon^2)} \simeq -1 \Rightarrow V^0 \sim 1 + O(\varepsilon^2)$$

$V^i \Gamma^d{}_{ic} \sim O(\varepsilon^2)$ can be ignored.

$$\therefore (V^a\nabla_a)(V^b\nabla_b) X^d \simeq V^a\nabla_a\left(\partial_t X^d + \Gamma^d{}_{0c} X^c + V^i \partial_i X^d\right)$$

$$= V^a \partial_a\left(\partial_t X^d + \Gamma^d{}_{0c} X^c + V^i \partial_i X^d\right)$$

$$+ \underbrace{V^a \Gamma^d{}_{ae}}_{\sim O(\varepsilon)}\left(\partial_t X^e + \underbrace{\Gamma^e{}_{0c} X^c}_{\sim O(\varepsilon)} + \underbrace{V^i \partial_i X^e}_{\sim O(\varepsilon)}\right)$$

$$\simeq \partial_t^2 X^d + \partial_t(\Gamma^d{}_{0c} X^c) + \partial_t(v^i \partial_i X^d)$$

$$+ \Gamma^d{}_{0e}\,\partial_t X^e + O(\varepsilon^2)$$

$$\because \ \partial_t h_{ab} \simeq 0 + O(\varepsilon)$$

$$\therefore \ \partial_t(\Gamma^d{}_{0c} X^c) \simeq \Gamma^d{}_{0c}\,\partial_t X^c + O(\varepsilon^2)$$

$$\therefore \ \frac{D^2 X^d}{D\tau^2} \simeq \partial_t^2 X^d + \partial_t(v^i \partial_i X^d) + 2\Gamma^d{}_{0c}\,\partial_t X^c$$

$$\cancel{+2\Gamma^d{}_{0e}\,\partial_t X^e}$$

$$\simeq \partial_t^2 X^d + \partial_t(v^i \partial_i X^d) + \eta^{de}(\partial_c h_{e0} + \underbrace{\partial_0 h_{ec}}_{\simeq 0} - \partial_e h_{0c}) \times \partial_t X^c$$

If $X^d = X^i$, $\eta^{de} = \delta_i{}^e$, and use $\partial_t h = 0$.

$$\frac{D^2 X^i}{D\tau^2} \simeq \partial_t^2 X^i + \partial_t(v^j \partial_j X^i) + (\partial_j h_{i0} - \partial_i h_{j0})\,\partial_t X^j$$

$$\cancel{\therefore \ \frac{D^2 X^i}{D\tau^2} \simeq O(\varepsilon) = \text{RHS of required equation}}$$

$$\cancel{\therefore \Rightarrow \partial_t^2 X^i + \partial_t(v^j \partial_j X^i) \sim O(\varepsilon)}.$$

~~Consider Dominant balance~~

$$\cancel{\text{If } \partial_t(v^j \partial_j X^i) \sim O(\varepsilon) \text{ then}}$$

$$\cancel{\partial_t^2 X^i \sim \frac{\partial_t(v^j \partial_j X^i)}{\varepsilon} \gg}$$

$\vdots$

Dominant balance of

$$\partial_t^2 x^i + \partial_t (v^j \partial_j x^i) + \varepsilon (\partial_j h_{i0} - \partial_i h_{j0}) \partial_t x^j$$

$$\simeq -\partial_i \partial_j \phi \, x^j \qquad (\phi \sim \tfrac{\varepsilon}{2} h_{00} \sim O(\varepsilon))$$

the order balance are $(\partial_t \sim \frac{1}{T},\ \partial_i \sim \frac{1}{L})$.

$$O\left(\frac{x}{T^2}\right) + O\left(\frac{\varepsilon x}{LT}\right) + O\left(\frac{\varepsilon x}{LT}\right) \sim O\left(\frac{\varepsilon x}{L^2}\right).$$

If $O\left(\frac{\varepsilon x}{LT}\right) \sim O\left(\frac{\varepsilon x}{L^2}\right)$

get $L \sim T$

$$O\left(\frac{x}{T^2}\right) \sim O\left(\frac{x}{L^2}\right) \gg O\left(\frac{\varepsilon x}{LT}\right) \sim O\left(\frac{\varepsilon x}{L^2}\right)$$

$\Rightarrow$ No dominant balance.

If $O\left(\frac{x}{T^2}\right) \sim O\left(\frac{\varepsilon x}{L^2}\right)$ then $\frac{1}{T^2} \sim \varepsilon \frac{1}{L^2}$

$\therefore\ L^2 \sim \varepsilon T^2 \rightarrow L \sim \sqrt{\varepsilon}\, T$

then $O\left(\frac{\varepsilon x}{LT}\right) \sim O\left(\frac{\varepsilon x}{\sqrt{\varepsilon}\, T^2}\right) \sim O\left(\frac{\sqrt{\varepsilon}\, x}{T^2}\right)$

$$\ll O\left(\frac{x}{T^2}\right) \sim O\left(\frac{\varepsilon x}{L^2}\right)$$

$\rightarrow$ Dominant balance between :

$\partial_t^2 x^i$ and $-\partial_i \partial_j \phi \, x^j$

Dominant balance of

$$\partial_t^2 x^i + \partial_t(V^j \partial_j x^i) + \varepsilon\,(\partial_j h_{i0} - \partial^i h_{j0})\,\partial_t x^j$$

$$\simeq -\partial_i \partial_j \phi\, x^j \qquad (\phi \sim \tfrac{\varepsilon}{2} h_{00} \sim O(\varepsilon))$$

the order balance are $(\partial_t \sim \frac{1}{T},\ \partial_i \sim \frac{1}{L})$

$$O(\tfrac{X}{T^2}) + O(\tfrac{\varepsilon X}{LT}) + O(\tfrac{\varepsilon X}{LT}) \sim O(\tfrac{\varepsilon X}{L^2}).$$

If $O(\frac{\varepsilon X}{LT}) \sim O(\frac{\varepsilon X}{L^2})$

get $L \sim T$

$$O(\tfrac{X}{T^2}) \sim O(\tfrac{X}{L^2}) \gg O(\tfrac{\varepsilon X}{LT}) \sim O(\tfrac{\varepsilon X}{L^2})$$

$\Rightarrow$ No dominant balance.

If $O(\frac{X}{T^2}) \sim O(\frac{\varepsilon X}{L^2})$ then $\frac{1}{T^2} \sim \varepsilon \frac{1}{L^2}$

$\therefore\ L^2 \sim \varepsilon T^2 \rightarrow L \sim \sqrt{\varepsilon}\, T$

then $O(\frac{\varepsilon X}{LT}) \sim O(\frac{\varepsilon X}{\sqrt{\varepsilon} T^2}) \sim O(\frac{\sqrt{\varepsilon} X}{T^2})$

$$\ll O(\tfrac{X}{T^2}) \sim O(\tfrac{\varepsilon X}{L^2})$$

$\rightarrow$ Dominant balance between:

$\partial_t^2 x^i$ and $-\partial_i \partial_j \phi\, x^j$

So we have the only valid balance in this equation is

$$\underline{\partial_t^2 x^i \simeq -\partial_i \partial_j \phi x^j}$$ the required equation

However, this ~~wett~~ would mean that $\frac{L}{T} \sim \sqrt{\varepsilon}$ but $V^i \sim O(\varepsilon)$. I don't know what physically does this mean.

We can assert that $\frac{L}{T}$ are physical scales and have nothing to do with $\frac{dx}{dt}$, the motion along geodesic. Or if we are in the rest frame then we can simply put $V_j = 0$ for $j = 1, 2, 3$,

then the dominant balance argument works if ~~we~~ we ignore the term with V_j

(4) I've checked!

Please believe me just as I believe the Einstein equations! I do ;)

↑↑↑

5 Lagrangian $\mathcal{L} = -(1-\frac{2M}{r})\dot{t}^2 + (1-\frac{2M}{r})^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$ ✓

The Euler-Lagrange equations for t :

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{\partial \mathcal{L}}{\partial t} = 0 \checkmark$$

$$\therefore \frac{d}{d\tau}\left(-2(1-\frac{2M}{r})\dot{t}\right) = 0$$

$$\therefore (1-\frac{2M}{r})\dot{t} = \text{constant} = E \quad \text{conserved}$$

✓

For ϕ :

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\therefore \frac{d}{d\tau}\left(2r^2\sin^2\theta\right) = 0 \quad \theta \checkmark$$

$$\rightarrow r^2\sin^2\theta = \text{constant} = J \quad \text{conserved.}$$

E is symmetry of time $\Rightarrow$ Energy ✓

J is symmetry of ϕ $\Rightarrow$ angular momentum. ✓

$\mathcal{L} = g_{ab}\dot{x}^a\dot{x}^b$ = constant since we've used an affine parameter for parameterisation. ✓

When the affine parameter is the proper time.

$\dot{x}^a = V^a$, $\dot{x}^b = V^b$, where V^a and V^b

are 4 velocities.

$$\therefore \mathcal{L} = g_{ab}V^a V^b = V^a V_a$$

In local Minkowski coordinates, we know from special relativity that $V^a V_a = -1$

$\boxed{\text{time like} \rightarrow \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \text{ is real and positive and finite}}$

And $\mathcal{L} = V^a V_a$ is a scalar

$\therefore \mathcal{L} = V^a V_a = -1$ in any other coordinates if parametrised by proper time τ.

Consider the θ equation :

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\Rightarrow \frac{d}{d\tau}(2r^2\dot{\theta}) = 2\sin\theta\cos\theta\,\dot{\phi}^2 r^2$$

If $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$, then the above RHS = LHS = 0, so $\theta = \frac{\pi}{2}$ is a solution to the θ-equation.

Hence we are allowed to choose our coordinates such that $\theta = \pi/2$ always. (equitorial motion)

Does this obstruct generality?

In this case $E = (1-\frac{2m}{r})\dot{t}$, $J = r^2\dot{\phi}$ are conserved.

Now we have

$$\left(1-\frac{2M}{r}\right)\dot{t} = E \quad ①$$

$$r^2\dot{\phi} = J \quad ②$$

$$-\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \mathcal{L} \quad ③$$

and $E, J, \mathcal{L}$ are all constants. ✓

then. substituting ① and ② into ③ gives

$$-\frac{E^2}{\left(1-\frac{2M}{r}\right)} + \frac{\dot{r}^2}{\left(1-\frac{2M}{r}\right)} + \frac{J^2}{r^2} = \mathcal{L} \quad ④$$

$$\therefore \quad -E^2 + \dot{r}^2 + \frac{J^2}{r^2}\left(1-\frac{2M}{r}\right) = \mathcal{L}\left(1-\frac{2M}{r}\right)$$

$$\therefore \quad \dot{r}^2 + \frac{J^2}{r^2}\left(1-\frac{2M}{r}\right) + \frac{2M\mathcal{L}}{r} = \mathcal{L} + E^2$$

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = \cancel{\frac{J}{r^2}\frac{dr}{d\phi}}\ \frac{J}{r^2}\frac{dr}{d\phi}$$

$$\therefore \left(\frac{J}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{J^2}{r^2}\left(1-\frac{2M}{r}\right) + \frac{2M\mathcal{L}}{r} = \mathcal{L} + E^2$$

If $u = \frac{1}{r}$, $\quad \frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} \Rightarrow \left(\frac{du}{d\phi}\right)^2 = \left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2$

then $J^2\left(\frac{du}{d\phi}\right)^2 + J^2u^2(1-2Mu) + 2M\mathcal{L}u = \mathcal{L} + E^2$

$$\therefore \cancel{\left(\frac{du}{d\phi}\right)^2 +}$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 + u^2 = -\frac{2M\mathcal{L}u}{J^2} + 2Mu^3 + \mathcal{L} + E^2$$

Differentiate the above equation gives.

$$2\left(\frac{du}{d\phi}\right)\frac{d^2u}{d\phi^2} + 2u\frac{du}{d\phi} = -\frac{2m\mathcal{L}}{J^2}\frac{du}{d\phi} + 6mu^2\frac{du}{d\phi}$$

$$\Rightarrow \left[\frac{d^2u}{d\phi^2} + u - \left(-\frac{2m\mathcal{L}}{J^2} + 3Mu^2\right)\right]\frac{du}{d\phi} = 0$$

For circular ~~orbit~~ orbit:

$\frac{du}{d\phi} = 0$ is not enough.

Also need $\frac{d^2u}{d\phi^2} = 0$.

$$\therefore \frac{d^2u}{d\phi^2} + u = -\frac{M\mathcal{L}}{J^2} + 3Mu^2$$

and when $r = R$, $u = \frac{1}{R}$. $\frac{d^2u}{d\phi^2} = 0$

$$\therefore u = -\frac{M\mathcal{L}}{J^2} + 3Mu^2$$

$$\therefore \frac{M\mathcal{L}}{J^2} = -u(1 - 3Mu)$$

$$\Rightarrow \frac{M\mathcal{L}}{J^2} = \frac{-1}{R}\left(1 - \frac{3M}{R}\right)$$

For timelike geodesic

need $\mathcal{L} < 0$

$\Rightarrow \quad 1 - \frac{3M}{R} > 0 \quad \therefore \quad \underline{\underline{R > 3M}}$

For time-like geodesic.

Good!

$$-\frac{M\mathcal{L}}{J^2} = \frac{1}{R^2}\left(1 - \frac{3M}{R}\right) = \frac{1}{R^3}(R - 3M)$$

$$\therefore J^2 = (-\mathcal{L})\frac{MR^2}{R-3M}$$

$$\therefore J = \sqrt{-\mathcal{L}}\left(\frac{MR^2}{R-3M}\right)^{\frac{1}{2}}$$

Good!

~~For~~ ④ :

Consider ④, put $\dot{r} = 0$, $r = R$

$$-\frac{E^2}{1-\frac{2M}{R}} + \frac{J^2}{R^2} = \mathcal{L}$$

$$\therefore \frac{E^2}{1-\frac{2M}{R}} = \frac{J^2}{R^2} - \mathcal{L} = -\mathcal{L}\frac{M}{R-3M} - \mathcal{L}$$

$$= -\mathcal{L}\left(\frac{M + R - 3M}{R-3M}\right).$$

$$= -\mathcal{L}\left(\frac{R-2M}{R-3M}\right)$$

$$\therefore \frac{E^2}{\left(1-\frac{2M}{R}\right)} = -\mathcal{L}\left(\frac{1-\frac{2M}{R}}{1-\frac{3M}{R}}\right)$$

$$E = \sqrt{-\mathcal{L}}\,\frac{1-\frac{2M}{R}}{\left(1-\frac{3M}{R}\right)^{1/2}}$$

If affine parameter is proper time τ, then

$$\sqrt{-\mathcal{L}} = \sqrt{1} = 1$$

$$\therefore \quad J = \left(\frac{MR^2}{R-3M}\right)^{\frac{1}{2}}$$

$$E = \frac{1-\frac{2M}{R}}{\left(1-\frac{3M}{R}\right)^{1/2}}$$

Clear!