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General Relativity I

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Wk 2,4,6,8 Thu 16:00 - 17:30

Problem Set 2

$(\uparrow\uparrow\uparrow)^5 (\uparrow\uparrow)$

α

Great work!

① ⓜⓣ $t = (\frac{1}{g} + z')\sinh(gt')$

$z = (\frac{1}{g} + z')\cosh(gt') - \frac{1}{g}$

$x = x'$

$y = y'$

(1) If $t' \ll \frac{1}{g}$, then $gt' \ll 1$

$\cosh(gt') \simeq 1 + \frac{1}{2}(gt')^2 + \cdots$ ①

~~$z = \frac{1}{g}$~~ $\sinh(gt') \simeq gt' + \frac{(gt')^3}{6} + \cdots$

$\therefore t \simeq (\frac{1}{g} + z')gt' = t' + z'gt' = (1 + z'g)t'$

$z \simeq (\frac{1}{g} + z')(1 + \frac{1}{2}g^2t'^2) - \frac{1}{g}$

$= z' + \frac{1}{2}z'g^2t'^2 + \frac{1}{2}gt'^2 = z' + \frac{1}{2}g(1 + z'g)t'^2$

From the transformations we see that

at $(t, z) = (0, 0) \Leftrightarrow (t', z') = (0, 0)$

$\therefore$ An observer S at origin of O sees an moving observer S' passes by at $t = 0$, $z = 0$.

S' has an non-inertial frame O' attached to itself. At time t', S' passes by S. the position coordinate z' of S' in O' is always $\underline{\underline{z' = 0}}$ since in O', S' is always at origin.

$\therefore$ At time t' in O', the event of S' being at $(t' = t', z' = 0)$ is transformed to O by.

$t \simeq t'$, $z \simeq \frac{1}{2} g t'^2$ $\Rightarrow$ $\underline{\underline{z = \frac{1}{2} g t^2}}$

✓

According to S, S' is ~~go~~ undergoing uniform acceleration. ✓ Good!

(2)

when $z' = 0$,

$$t = \frac{1}{g} \sinh(gt') \quad , \quad z = \frac{1}{g} \cosh(gt') - \frac{1}{g}$$

$$\therefore \quad gt = \sinh(gt') \qquad 1 + gz = \cosh(gt')$$

$$\rightarrow \quad \cosh^2(gt') - \sinh^2(gt') = 1 = (1+gz)^2 - (gt)^2$$

$\Rightarrow$

$$\underline{\underline{(z + \frac{1}{g})^2 - t^2 = \frac{1}{g^2}}} \quad (1)$$

Observations:

$\rightarrow$ $(z, t) = (0, 0)$ solves (1)

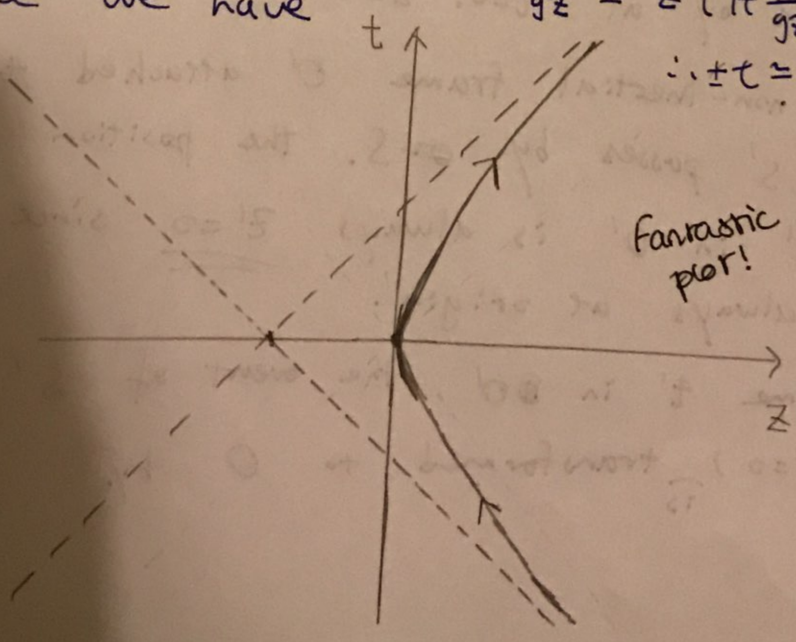
$\rightarrow$ as $z \gg \frac{1}{g}$, $(z + \frac{1}{g})^2 \simeq$ ~~$z^2 + \frac{2z}{g} z^2$~~ $z^2 + \frac{2z}{g} + \frac{1}{g^2}$ ~~$t^2 \ldots \frac{1}{g} = 0$~~

~~$\therefore z^2 = t^2 \qquad z = \pm t$~~

$\therefore t^2 = z^2 + \frac{2z}{g} \Rightarrow \pm t = z\sqrt{1 + \frac{2}{gz}} \simeq z\left(1 + \frac{2/}{gz}\right) \simeq z + \frac{1}{g}$.

Hence we have

$\therefore \pm t \simeq z + \frac{1}{g}$ as $z \to \infty$



(3)

$$dt = \frac{\partial t}{\partial z'} dz' + \frac{\partial t}{\partial t'} dt'$$

$$dz = \frac{\partial z}{\partial z'} dz' + \frac{\partial z}{\partial t'} dt'$$

Good!

$$\therefore dt = \sinh(gt')\, dx' + \left(\frac{1}{g} + z'\right) g \cosh(gt')\, dt'$$

$$dz = \cosh(gt')\, dz' + \left(\frac{1}{g} + z'\right) g \sinh(gt')\, dt'$$

→ the proper time of an observer $d\tau$ is given by

$$-d\tau^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$dt^2 - dz^2 = \sinh^2(gt')\, d^2z' + 2\sinh(gt')\cancel{\cosh(gt')}[1 + gz']\, dz'dt'$$

$$+ (1+gz')^2 \cosh^2(gt')\, d^2t' - \cosh^2(gt')\, d^2z'$$

$$- 2\sinh(gt')\cancel{\cosh(gt')}(1+gz')\, dz'dt'$$

$$- \sinh^2(gt')(1+gz')^2\, dt'^2$$

$$= [\cosh^2(gt') - \sinh^2(gt')](1+gz')^2 dt'$$

$$- [\cosh^2(gt') - \sinh^2(gt')]\, dz'^2$$

$$= (1+gz')^2 dt'^2 - dz'^2$$

Good!

A shorter solution is to compare $\left.\frac{\Delta S}{\Delta S'}\right|_{t'=t}$.

$$dx' = dx, \quad dy' = dy$$

$\Rightarrow$ ~~$-d\tau^2 = dt$~~

$$-d\tau^2 = -(1+gz')^2 dt'^2 + dx'^2 + dy'^2 + dz'^2$$

The proper time of an observer at $z' = h$,

$dx' = dy' = dz' = 0$, is given by

$$d\tau_h = (1+gh)\,dt'$$

The proper time of an observer at $z'=0$, $dx' = dy' = dz' = 0$ is given by

$$d\tau_0 = dt'$$

$$\therefore \quad \frac{d\tau_h}{d\tau_0} = 1+gh$$

clock at $z'=h$ runs faster by this factor than at $z'=0$.

Good!

(4) Equivalent principle states that inertial mass is ~~equal~~ equivalent to gravitational mass. So uniform acceleration is ~~equal to a gr~~ uniform gravitational field. Since time dilation exists in ~~uniform~~ uniformly accelerated frame, it also exists in a uniform ~~grav~~ gravitational field. The clock ~~of~~ ~~ru~~ runs faster as it gets higher Good! altitude (i.e. far away from the gravitational field)

(5) $\because$ line element $ds^2 = -d\tau^2$, use result in (3)

$$\therefore \quad ds^2 = -(1+gz)^2 dt^2 + dx^2 + dy^2 + dz^2$$

Good!
Although this is not a very formal proof ("physicist's intuition" is useful though).

In (1) If we consider fully non-relativistic case, we should also have $gz' \ll c^2 = 1$

then ~~the~~ we can have constant z' in O' ~~and~~ because the situation is non relativistic.

$$\Rightarrow \quad t = t' + \underbrace{z'gt'}_{\simeq 0} \simeq t' \quad \Rightarrow \quad \underline{\underline{t = t'}}$$

$$z \simeq z' + \frac{1}{2}\underbrace{(z'g)}_{\simeq 0}\underbrace{(gt')}_{\simeq 0} + \frac{1}{2}gt'^2 \Rightarrow \underline{\underline{z \simeq z' + \frac{1}{2}gt^2}}$$

Corresponds to the Galileo transformation. of ~~rela~~ accelerating frame.

2 ↑↑↑

(1) line element

$$\cancel{ds^2} \quad ds^2 = \cancel{g_{ab}(x^1)^a} \; g_{ab}\, \cancel{d(x^1)^a\, d(x^2)^b} \; \cancel{ds^2} = g_{ab}\, dx^a dx^b$$

$$= \cancel{g}\,(dx^1)^2 + (dx^2)^2$$

why? what is g_{ab}? $[g_{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\because\ x^1 = r\cos\phi \qquad dx^1 = -r\sin\phi\, d\phi + \cos\phi\, dr$

$x^2 = r\sin\phi \qquad dx^2 = r\cos\phi\, d\phi + \sin\phi\, dr$ ✓

$$ds^2 = \cos^2\phi\, dr^2 - 2r\cancel{\sin\phi\cos\phi}\, dr\, d\phi + r^2\sin^2\phi\, d\phi^2$$

$$+ \sin^2\phi\, dr + 2r\cancel{\sin\phi}\cos\phi\, dr\, d\phi + r^2\cos^2\phi\, d\phi^2$$

$$= dr^2 + r^2 d\phi^2$$

Good!

Hence the new metric $\quad g_{rr} = 1 \quad g_{\phi\phi} = r^2$

$g_{r\phi} = g_{\phi r} = 0$ ✓

(2) (a) Lagrangian $\mathcal{L} = g_{ab}\dot{x}^a\dot{x}^b$ ✓ $= g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2$

$\mathcal{L} = \dot{r}^2 + r^2\dot{\phi}^2$ ✓ , $\dot{r} = \frac{dr}{d\tau}$, $\dot{\phi} = \frac{d\phi}{d\tau}$ (τ is an affine parameter)

Lagrange equation: $\quad \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$

Good!

$\therefore\ r: \quad \frac{d}{d\tau}(2\dot{r}) = 2r\dot{\phi}^2 \Rightarrow \ddot{r} = r\dot{\phi}^2$

$\phi: \quad \frac{d}{d\tau}(2r^2\dot{\phi}) = 0 \Rightarrow r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} = 0$ ✓

$\Rightarrow r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$

$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0$

Geodesic equation

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$$ ✓

$$\Rightarrow \ddot{r} + \Gamma^r_{bc}\dot{x}^b\dot{x}^c = 0 \iff \ddot{r} + (-r)\dot{\phi}\dot{\phi} = 0$$

$$\therefore \underline{\Gamma^r_{\phi\phi} = -r} \quad \underline{\Gamma^r_{rr} = 0} \quad \underline{\Gamma^r_{r\phi} = \Gamma^r_{\phi r} = 0}$$ ✓

$$\Rightarrow \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \Rightarrow \underline{\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}}$$ ✓

$$\underline{\Gamma^\phi_{rr} = 0 \quad \Gamma^\phi_{\phi\phi} = 0}$$ ✓

(b)

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

$$\Gamma^r_{\phi\phi} = \frac{1}{2}g^{rr}(\partial_\phi g_{\phi r} + \partial_\phi g_{\phi r} - \partial_r g_{\phi\phi})$$

$$= \frac{1}{2}\cdot 1\cdot(0 + 0 - 2r) = \underline{\underline{-r}}$$ ✓

$$\Gamma^\phi_{r\phi} = \frac{1}{2}g^{\phi\phi}(\partial_r g_{\phi\phi} + \partial_\phi g_{r\phi} - \partial_\phi g_{r\phi}).$$

$$= \frac{1}{2}\frac{1}{r^2}(2r) = \underline{\underline{\frac{1}{r}}}$$ ✓

[we've used $g_{ab} = \begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix}$ $\therefore$ $g^{ab} = \begin{pmatrix}1 & 0\\ 0 & \frac{1}{r^2}\end{pmatrix}$.]

All other Christoffel symbols vanish.

Good!

(c) Transformation of Christoffel symbol.

$$\Gamma'^{a}_{bc} = \frac{\partial x'^a}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^b}\frac{\partial x^\sigma}{\partial x'^c}\Gamma^\rho_{\tau\sigma} + \frac{\partial x'^a}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^b \partial x'^c}$$

~~$\therefore \Gamma'^{\rho}_{\phi\phi} =$ Origin~~

~~Originally~~ originally $\Gamma^{x_1}_{x_1x_2} = \Gamma^{x_1}_{x_2x_1} = \Gamma^{x_1}_{x_2x_2}$

$= \Gamma^{x_2}_{x_1x_2} = \Gamma^{x_2}_{x_2x_1} = \Gamma^{x_2}_{x_2x_2} = 0$

$\because$ All derivatives ~~vanishes~~ of g vanishes.

$\therefore \Gamma'^{a}_{bc} = \frac{\partial x'^a}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^b \partial x'^c}$

what about, e.g. $\partial_r g_{\phi\phi}$?

$\therefore \Gamma^{r}_{\phi\phi} = \frac{\partial r}{\partial x^1}\frac{\partial^2 x^1}{\partial\phi\partial\phi} + \frac{\partial r}{\partial x^2}\frac{\partial^2 x^2}{\partial\phi\partial\phi}$

$x^1 = r\cos\phi \qquad x^2 = r\sin\phi \qquad \therefore x_1^2 + x_2^2 = r^2$

$\tan\phi = \frac{x_2}{x_1}$

$r\frac{\partial r}{\partial x_1} = x_1 \qquad \therefore \frac{\partial r}{\partial x_1} = \frac{x_1}{r}, \quad \frac{\partial r}{\partial x_2} = \frac{x_2}{r}.$

$\therefore \Gamma^r_{\phi\phi} = \frac{x^1}{r}(-r\cos\phi) - \frac{x^2}{r}(-r\sin\phi).$

$= (-x^1\cos\phi - x^2\sin\phi)$.

$= -r\cos^2\phi - r\sin^2\phi = \underline{\underline{-r}}$

$\Gamma^\phi_{r\phi} = \frac{\partial\phi}{\partial x^1}\frac{\partial^2 x^1}{\partial r\partial\phi} + \frac{\partial\phi}{\partial x^2}\frac{\partial^2 x^2}{\partial r\partial\phi}$

$\tan\phi = \frac{x_2}{x_1} \qquad \sec^2\phi\frac{\partial\phi}{\partial x_1} = -\frac{x_2}{x_1^2} \qquad \frac{\partial\phi}{\partial x_1} = -\cos^2\theta\frac{x_2}{x_1^2}$

$\sec^2\phi\frac{\partial\phi}{\partial x_2} = \frac{1}{x_1} \qquad \therefore \frac{\partial\phi}{\partial x_2} = \frac{1}{x_1}\cos^2\theta.$

$$\Gamma^{\phi}_{r\phi} = -\cos^2\phi\,(-\sin\phi)\,\frac{x_2}{x_1^2} + \cos^2\phi\,\cos\phi\,\frac{1}{x_1}$$

$$= \cancel{\cos^2}\phi\left(\frac{r\sin^2\phi}{r^2\cancel{\cos^2\phi}} + \frac{r\cos^2\phi}{r^2\cancel{\cos^2\phi}}\right)$$

$$= \frac{r}{r^2} = \frac{1}{r} \quad ✓$$

All other Christoffel symbols vanish.

(3) Geodesics: Great!

$$\ddot{r} - r\dot{\phi}^2 = 0 \quad (1)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \quad (2) \quad \Rightarrow \quad r^2\dot{\phi} = k = \text{const}$$

How?

~~A straight in $\mathbb{R}^2$ has $\ddot{x}_1 = 0$, $\ddot{x}_2 = 0$.~~

~~$\Rightarrow r\sin\theta = uT + x_0,\ r\cos\theta = vT \Rightarrow r = \sqrt{u^2+v^2}\,T$~~

~~$\tan\phi = \frac{u}{v}$~~

Consider

$$0 = 2r\dot{r}\dot{\phi}^2 + 2r\dot{r}\dot{\phi}^2 - 4r\dot{r}\dot{\phi}^2$$

$$= 2\dot{r}\underbrace{(r\dot{\phi}^2)}_{\ddot{r}} + 2r\dot{r}\dot{\phi}^2 + 2r^2\dot{\phi}\underbrace{\left(-\frac{2}{r}\dot{r}\dot{\phi}\right)}_{\ddot{\phi}}$$

$$= 2\dot{r}\ddot{r} + 2r\dot{r}\dot{\phi}^2 + 2r^2\dot{\phi}\ddot{\phi}$$

$$= \frac{d}{dT}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \quad ✓$$

$\therefore \quad \dot{r}^2 + r^2\dot{\phi}^2 = m^2 = \text{const} \quad ③$

③ ~~com~~ is obvious since τ is affine parameter, and $m^2 = 1$ if τ is ~~proper~~ differential line element.

$\because \quad r^2\dot{\phi} = k \quad \therefore \quad \dot{\phi} = \frac{k}{r^2}$

sub into ③ $\Rightarrow \quad \dot{r}^2 + r^2 \frac{k^2}{r^4} = \cancel{0}m \quad \therefore \quad \dot{r}^2 = m^2 - \frac{k^2}{r^2}$

$\therefore \quad \dot{r} = \left(m^2 - \frac{k^2}{r^2}\right)^{\frac{1}{2}}$

$\therefore \quad \cancel{\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} =} \quad \frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = \frac{k}{r^2}\left(m^2 - \frac{k^2}{r^2}\right)^{-\frac{1}{2}}$

$\frac{d\phi}{dr} = \frac{(k/m)}{r^2}\left(1 - \frac{(k/m)^2}{r^2}\right)^{-\frac{1}{2}}$

$\cancel{= \ldots}$

Integrate this gives $\phi = \phi_0 + \cos^{-1}\left(\frac{k/m}{r}\right)$.

$\therefore \quad r\cos(\phi - \phi_0) = \frac{k}{m} = \text{const}.$

$\therefore \quad (\underbrace{r\cos\phi}_{x_1})\cos\phi_0 + (\underbrace{r\sin\phi}_{x_2})\sin\phi_0 = \frac{k}{m}$

$\therefore \quad x_1\cos\phi_0 + x_2\sin\phi_0 = \frac{k}{m} = \text{const}.$

$\rightarrow$ ~~The~~ This is a straight line $\square$

✓

Good!

3 (i) ↑↑↑

$$\Gamma'^{a}{}_{bc} = \frac{\partial x^p}{\partial x'^b}\frac{\partial x^q}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^r}\Gamma^r{}_{pq} - \frac{\partial^2 x'^a}{\partial x^p \partial x^q}\right) \quad (1)$$

$$\nabla'_b V'^a = \partial'_b V'^a + \Gamma'^{a}{}_{bc} V'^c$$

$$\partial'_b V'^a = \cancel{\partial}\frac{\partial}{\partial x'^b}\left(\frac{\partial x'^a}{\partial x^\mu} V^\mu\right) = \frac{\partial x'^a}{\partial x^\mu}\frac{\partial V^\mu}{\partial x'^b} + V^\mu \frac{\partial^2 x'^a}{\partial x'^b \partial x^\mu}$$

$$= \frac{\partial x'^a}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^b}\frac{\partial V^\mu}{\partial x^\nu} + \frac{\partial^2 x'^a}{\partial x^\mu \partial x^\nu}\frac{\partial x^\nu}{\partial x'^b} V^\mu \quad (2)$$

$$V'^c = \frac{\partial x'^c}{\partial x^\lambda} V^\lambda \quad (3)$$

(1), (2), (3) $\Rightarrow$

$$\nabla'_b V'^a = \partial'_b V'^a + \Gamma'^a{}_{bc} V'^c = \frac{\partial x'^a}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^b}\frac{\partial V^\mu}{\partial x^\nu} + \frac{\partial^2 x'^a}{\partial x^\mu \partial x^\nu}\frac{\partial x^\nu}{\partial x'^b} V^\mu$$

$$+ \frac{\partial x^p}{\partial x'^b}\frac{\partial x^q}{\partial x'^c}\frac{\partial x'^a}{\partial x^r}\Gamma^r{}_{pq}\frac{\partial x'^c}{\partial x^\lambda} V^\lambda$$

$$- \cancel{\frac{\partial x^p}{\partial x'^b}\frac{\partial x^q}{\partial x'^c}\frac{\partial x'^a}{\partial x^r}\Gamma^r{}_{pq}} \frac{\partial x^p}{\partial x'^b}\frac{\partial x^q}{\partial x'^c}\frac{\partial^2 x'^a}{\partial x^p \partial x^q}\frac{\partial x'^c}{\partial x^\lambda} V^\lambda$$

$$= \frac{\partial x'^a}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^b}\frac{\partial V^\mu}{\partial x^\nu} + \cancel{\frac{\partial^2 x'^a}{\partial x^\mu \partial x^\nu}\frac{\partial x^\nu}{\partial x'^b} V^\mu} + \frac{\partial x^p}{\partial x'^b}\frac{\partial x'^a}{\partial x^r}\Gamma^r{}_{p\lambda} V^\lambda$$

$$- \cancel{\frac{\partial x^p}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^p \partial x^\lambda} V^\lambda}$$

where we used $\frac{\partial x^q}{\partial x'^c}\frac{\partial x'^c}{\partial x^\lambda} = \delta^q_\lambda$

~~and~~ and if we ~~relat~~ relabel dummies

$\nu \to p$

$\mu \to r$

then

$$\nabla'_b V'^a = \frac{\partial x'^a}{\partial x^r}\frac{\partial x^p}{\partial x'^b}\underbrace{\left(\partial_p V^r + \Gamma^r{}_{p\lambda}V^\lambda\right)}_{\nabla_p V^r} \quad ✓$$

$$= \frac{\partial x'^a}{\partial x^r}\frac{\partial x^p}{\partial x'^b}\nabla_p V^r$$

Great!

transforms as a tensor $\square$

(2)

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)$$

$$\Gamma'^a{}_{bc} = \frac{1}{2}g'^{ad}\left(\partial'_b g'_{cd} + \partial'_c g'_{bd} - \partial'_d g'_{bc}\right)$$

$$= \frac{1}{2}\frac{\partial x'^a}{\partial x^\sigma}\frac{\partial x'^d}{\partial x^\rho}g^{\sigma\rho}\left(\frac{\partial x^\beta}{\partial x'^b}\frac{\partial}{\partial x^\beta}\left(\frac{\partial x^\mu}{\partial x'^c}\frac{\partial x^\nu}{\partial x'^d}g_{\mu\nu}\right) + \frac{\partial x^\beta}{\partial x'^c}\frac{\partial}{\partial x^\beta}\left[\frac{\partial x^\mu}{\partial x'^b}\frac{\partial x^\nu}{\partial x'^d}g_{\mu\nu}\right] - \frac{\partial x^\beta}{\partial x'^d}\frac{\partial}{\partial x^\beta}\left(\frac{\partial x^\mu}{\partial x'^b}\frac{\partial x^\nu}{\partial x'^c}g_{\mu\nu}\right)\right)$$

$$= \frac{1}{2}\frac{\partial x'^a}{\partial x^\sigma}\frac{\partial x'^d}{\partial x^\rho}g^{\sigma\rho}\left(\frac{\partial x^\beta}{\partial x'^b}\frac{\partial^2 x^\mu}{\partial x^\beta\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^c}\frac{\partial x^\nu}{\partial x'^d}g_{\mu\nu} + \frac{\partial^\beta}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^c}\frac{\partial^2 x^\nu}{\partial x^\beta\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^d}g_{\mu\nu}\right.$$

$\frac{\partial x'^d}{\partial x^\rho}\frac{\partial x^\nu}{\partial x'^d} = \delta^\nu_\rho$ $\qquad$ $\frac{\partial x'^d}{\partial x^\rho}\frac{\partial x^\alpha}{\partial x'^d} = \delta^\alpha_\rho$

$$+ \frac{\partial x^\beta}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^c}\frac{\partial x^\nu}{\partial x'^d}\frac{\partial g_{\mu\nu}}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^c}\frac{\partial^2 x^\mu}{\partial x^\beta\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^b}\frac{\partial x^\nu}{\partial x'^d}g_{\mu\nu} \quad ✓$$

$\frac{\partial x'^d}{\partial x^\rho}\frac{\partial x^\nu}{\partial x'^d} = \delta^\nu_\rho$ $\qquad$ $\frac{\partial x'^d}{\partial x^\rho}\frac{\partial x^\nu}{\partial x'^d} = \delta^\nu_\rho$

$$+ \frac{\partial x^\beta}{\partial x'^c}\frac{\partial x^\mu}{\partial x'^b}\frac{\partial^2 x^\nu}{\partial x^\beta\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^d}g_{\mu\nu} + \frac{\partial x^\beta}{\partial x'^c}\frac{\partial x^\mu}{\partial x'^b}\frac{\partial x^\nu}{\partial x'^d}\frac{\partial g_{\mu\nu}}{\partial x^\beta}$$

$\frac{\partial x'^d}{\partial x^\rho}\frac{\partial x^\alpha}{\partial x'^d} = \delta^\alpha_\rho$ $\qquad$ $\frac{\partial x'^d}{\partial x^\rho}\frac{\partial x^\nu}{\partial x'^d} = \delta^\nu_\rho$

$$\cancel{- \frac{\partial x^\beta}{\partial x'^d}\frac{\partial^2 x^\mu}{\partial x^\beta\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^b}g} - \frac{\partial x^\beta}{\partial x'^d}\frac{\partial x^\mu}{\partial x'^b}\frac{\partial^2 x^\nu}{\partial x^\beta\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^c}g_{\mu\nu}$$

$\frac{\partial x'^d}{\partial x^\rho}\frac{\partial x^\beta}{\partial x'^d} = \delta^\beta_\rho$ $\quad$ ✓

$$- \frac{\partial x^\beta}{\partial x'^d}\frac{\partial^2 x^\mu}{\partial x^\beta\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^b}\frac{\partial x^\nu}{\partial x'^c}g_{\mu\nu}$$

$\frac{\partial x'^d}{\partial x^\rho}\frac{\partial x^\beta}{\partial x'^d} = \delta^\beta_\rho$

$$\left. - \frac{\partial x^\beta}{\partial x'^d}\frac{\partial x^\mu}{\partial x'^b}\frac{\partial x^\nu}{\partial x'^c}\frac{\partial g_{\mu\nu}}{\partial x^\beta}\right)$$

$\frac{\partial x'^d}{\partial x^\rho}\frac{\partial x^\beta}{\partial x'^d} = \delta^\beta_\rho$

$$= \frac{1}{2}\frac{\partial x'^a}{\partial x^\sigma} g^{\sigma\rho}\Big(\frac{\partial x^\beta}{\partial x'^b}\frac{\partial^2 x^\mu}{\partial x^\beta \partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^c} g_{\mu\rho} + \frac{\partial x^\beta}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^c}\underbrace{\frac{\partial^2 x^\nu}{\partial x^\beta \partial x^\rho}}_{①} g_{\mu\nu}$$

$g^{\sigma\rho} g_{\mu\rho} = \delta^\sigma_\mu$

$$+ \frac{\partial x^\beta}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^c}\frac{\partial g_{\mu\rho}}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^c}\frac{\partial^2 x^\mu}{\partial x^\beta \partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^b} g_{\mu\rho}$$

$g^{\sigma\rho} g_{\mu\rho} = \delta^\sigma_\mu$

$$+ \frac{\partial x^\beta}{\partial x'^c}\frac{\partial x^\mu}{\partial x'^b}\underbrace{\frac{\partial^2 x^\nu}{\partial x^\beta \partial x^\rho}}_{②} g_{\mu\nu} + \frac{\partial x^\beta}{\partial x'^c}\frac{\partial x^\mu}{\partial x'^b}\frac{\partial g_{\mu\rho}}{\partial x^\beta}$$

$$- \frac{\partial x^\mu}{\partial x'^b}\underbrace{\frac{\partial^2 x^\nu}{\partial x^\rho \partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^c}}_{③} g_{\mu\nu} - \frac{\partial^2 x^\mu}{\partial x^\rho \partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^b}\underbrace{\frac{\partial x^\nu}{\partial x'^c}}_{④} g_{\mu\nu}$$

$$- \frac{\partial x^\mu}{\partial x'^b}\frac{\partial x^\nu}{\partial x'^c}\frac{\partial g_{\mu\nu}}{\partial x^\rho} \Big)$$

$\because$ ① = ④ , ② = ③

$= \frac{\partial x^\beta}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^c}\frac{\partial g_{\beta\rho}}{\partial x^\mu}$ by $\mu \leftrightarrow \beta$

$$\therefore = \frac{1}{2}\frac{\partial x'^a}{\partial x^\sigma} g^{\sigma\rho}\Big(\frac{\partial x^\beta}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^c}\frac{\partial g_{\mu\rho}}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^c}\frac{\partial x^\mu}{\partial x'^b}\frac{\partial g_{\mu\rho}}{\partial x^\beta} - \frac{\partial x^\mu}{\partial x'^b}\frac{\partial x^\nu}{\partial x'^c}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\Big) + \cancel{\frac{\partial x^\beta}{\partial x'^b}}$$

can change dummie $\nu \to \beta$

$$+ \frac{1}{2}\frac{\partial x'^a}{\partial x^\sigma}\Big(\frac{\partial x^\beta}{\partial x'^b}\frac{\partial^2 x^\sigma}{\partial x^\beta \partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^c} + \frac{\partial x^\beta}{\partial x'^c}\frac{\partial^2 x^\sigma}{\partial x^\beta \partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^b}\Big).$$

they are equal by swapping $\alpha \leftrightarrow \beta$ in one of them

$$= \frac{\partial x^\beta}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^c}\Big[\frac{\partial x'^a}{\partial x^\sigma}\Big(\underbrace{\frac{1}{2} g^{\sigma\rho}\left(\partial_\beta g_{\mu\rho} + \partial_\mu g_{\beta\rho} - \partial_\rho g_{\mu\beta}\right)}_{\Gamma^\sigma_{\beta\mu}}\Big)$$

$$+ \frac{\partial x'^a}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^\beta \partial x^\mu}\Big]$$

change α to μ
$\alpha \to \mu$

Good!

Now consider $\frac{\partial x'^a}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^b} = \delta^a_b$ ~~(∵ δ^a_b is a tensor)~~

$$\therefore\ 0 = \frac{\partial}{\partial x^\mu}\left(\frac{\partial x'^a}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^b}\right)$$

$$= \frac{\partial^2 x'^a}{\partial x^\mu \partial x^\beta}\frac{\partial x^\beta}{\partial x'^b} + \frac{\partial x'^a}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial x^\mu \partial x'^b}$$

$$= \frac{\partial x^\beta}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^\mu \partial x^\beta} + \frac{\partial x'^a}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^\mu \partial x^\beta}\frac{\partial x^\beta}{\partial x'^b}$$

$$= \frac{\partial x^\beta}{\partial x'^b}\left(\frac{\partial x'^a}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^\mu \partial x^\beta} + \frac{\partial^2 x'^a}{\partial x^\mu \partial x^\beta}\right).$$

$$\therefore\ \frac{\partial x^\beta}{\partial x'^b}\frac{\partial^2 x'^a}{\partial x^\mu \partial x^\beta} = -\frac{\partial x^\beta}{\partial x'^b}\frac{\partial x'^a}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^\mu \partial x^\beta}$$

$$\Rightarrow\ \Gamma'^a{}_{bc} = \frac{\partial x^\beta}{\partial x'^b}\frac{\partial x^\mu}{\partial x'^c}\left[\frac{\partial x'^a}{\partial x^\sigma}\Gamma^\sigma{}_{\beta\mu} - \frac{\partial^2 x'^a}{\partial x^\beta \partial x^\mu}\right]$$

$$\rightarrow\ \Gamma'^a{}_{bc} = \frac{\partial x^p}{\partial x'^b}\frac{\partial x^q}{\partial x'^c}\left[\frac{\partial x'^a}{\partial x^r}\Gamma^r{}_{pq} - \frac{\partial^2 x'^a}{\partial x^p \partial x^q}\right]$$

Good!

□

(3) if ϕ is a scalar then $\nabla_a\phi = \partial_a\phi$ ①

W_aV^a is a scalar

$$\partial_b(W_aV^a) = \nabla_b(W_aV^a) = V^a\nabla_b W_a + W_a\nabla_b V^a$$

?

$$\therefore\ V^a\partial_b W_a + \cancel{W_a\partial_b V^a} = V^a\nabla_b W_a + \cancel{W_a\partial_b V^a} + W_a\Gamma^a{}_{bc}V^c$$

$$= V^a\nabla_b W_a + W_c\Gamma^c{}_{ba}V^a.$$

①

$$\therefore\ V^{a}\nabla_b w_a = \cancel{V^a(\partial_b w_a - \Gamma^c{}_{ba} V^a)}$$

$$V^a(\partial_b w_a - \Gamma^c{}_{ab} w_c)$$

$$\therefore\ V^a\left[\nabla_b w_a - (\partial_b w_a - \Gamma^c{}_{ab} w_c)\right] = 0$$

This is true for any V^a

Be careful when massaging indices!

$$\therefore\ \nabla_b w_a = \partial_b w_a - \Gamma^c{}_{ab} w_c$$

Good!

□

(4) ~~For~~ (p,q) tensor $T^{a_1\cdots a_p}{}_{b_1\cdots b_q}$

$$\nabla_c T^{a_1\cdots a_p}{}_{b_1\cdots b_q} = \partial_c T^{a_1\cdots a_p}{}_{b_1\cdots b_q}$$

$$+ \Gamma^{a_1}{}_{c\rho_1} T^{\rho_1 a_2\cdots a_p}{}_{b_1\cdots b_q} + \cdots + \Gamma^{a_p}{}_{c\rho_p} T^{a_1\cdots a_{p-1}\rho_p}{}_{b_1\cdots b_q}$$

$$- \Gamma^{\sigma_1}{}_{cb_1} T^{a_1\cdots a_p}{}_{\sigma_1 b_2\cdots b_q} + \cdots + \Gamma^{\sigma_q}{}_{cb_q} T^{a_1\cdots a_p}{}_{b_1\cdots b_{q-1}\sigma_q}$$

□.

Great!

4

$$ds^2 = -du^2 + \cosh^2 u \, d\phi^2$$

11

(1) $u = u_c = \text{const}$ $\therefore du = 0$ ✓

proper length $ds^2 = \cosh^2 u_c \, d\phi^2$

$\rightarrow$ $ds = \cosh u_c \, d\phi$ ✓ $\Delta s = \int_0^{2\pi} \cosh u_c \, d\phi$

$= 2\pi \cosh u_c$.

(2) Lagrangian ~~$L = g_{ab}\dot{x}$~~

$\mathcal{L} = g_{ab}\dot{x}^a\dot{x}^b$ ~~with~~ with $x = \begin{pmatrix} u \\ \phi \end{pmatrix}$

$$g = \begin{pmatrix} -1 & 0 \\ 0 & \cosh^2 u \end{pmatrix}$$

$\therefore$ $\mathcal{L} = -\dot{u}^2 + \cosh^2 u \, \dot{\phi}^2$

Euler Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) = \frac{\partial \mathcal{L}}{\partial u} \Rightarrow -2\ddot{u} = \cancel{2\dot{\phi}} \, \dot{\phi}^2 \, 2\cosh u \sinh u.$$

$$\therefore \underline{\ddot{u} + \cosh(u)\sinh(u)\,\dot{\phi}^2 = 0} \quad ①$$

$$\therefore \underline{\Gamma^u_{\phi\phi} = \cosh(u)\sinh(u)} \quad ✓$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\therefore \frac{d}{d\tau}\left(2\cosh^2 u \, \dot{\phi}\right) = 0$$

Good!

$$\rightarrow \ddot{\phi}\cosh^2 u + 2\cosh(u)\sinh(u)\,\dot{u}\dot{\phi} = 0 \quad ②$$

$$\therefore \underline{\ddot{\phi} + 2\tanh(u)\,\dot{u}\dot{\phi} = 0}$$

$\Gamma^\phi_{u\phi} = \Gamma^\phi_{\phi u}$ ✓
$\underline{= \tanh(u)}$

(5)

(3) $J = \cosh^2 u \dot{\phi}$

$$\dot{J} = \frac{dJ}{d\tau} = \frac{d}{d\tau}(\cosh^2 u \dot{\phi}) = 2\cosh(u)\sinh(u)\dot{\phi}\dot{u} + \cosh^2(u)\ddot{\phi}$$

$$= 2\cosh(u)\sinh(u)\dot{\phi}\dot{u} - 2\cosh(u)\sinh(u)\dot{\phi}\dot{u}$$

By geodesic ②

$$= 0$$

ⓙ conserved.

$E = \dot{u}^2 - \cosh^2 u \dot{\phi}^2$

$$\dot{E} = \frac{dE}{d\tau} = \frac{d}{d\tau}(\dot{u}^2 - \cosh^2 u \dot{\phi}^2)$$

$$= 2\dot{u}\ddot{u} - 2\cosh(u)\sinh(u)\dot{u}\dot{\phi}^2$$
$$- 2\cosh^2 u \dot{\phi}\ddot{\phi}$$

$$= -2\dot{u}\cosh(u)\sinh(u)\dot{\phi}^2 - 2\cosh(u)\sinh(u)\dot{u}\dot{\phi}^2$$
$$\cancel{-2\cosh^2 u \dot{\phi}(}$$
$$-2\dot{\phi}(-2\cosh(u)\sinh(u)\dot{u}\dot{\phi})$$

$$= \cosh(u)\sinh(u)\dot{u}\dot{\phi}^2(\underbrace{-2-2+4}_{0})$$

$$= 0$$

conserved.

Good! it is easier to use the fact that $L \neq L(s, \phi)$.

(4)

$\because \dot{\phi}(0) = 0 \quad \therefore J(0) = \cosh^2(u(0))\dot{\phi}(0) = 0$

$\because \dot{J} = 0 \quad \therefore J = J(0) = 0$ □ Good!

$\because E = \dot{u}^2 - \cosh^2(u)\dot{\phi}^2 = \dot{u}^2 - J\dot{\phi}$

and $J = 0$ always $\quad \therefore E = \dot{u}^2 - 0 = \dot{u}^2$ □ Good!

(5)

$v = \tanh u \quad \therefore \quad \frac{dv}{d\phi} = \frac{d}{d\phi}\tanh u = \frac{1}{\cosh^2 u}\frac{du}{d\phi}$

$\therefore \quad \frac{du}{d\phi} = \cosh^2 u \, \frac{dv}{d\phi}$

$\because \quad \cosh^2 u \, \dot{\phi} = J \quad \therefore \quad \dot{\phi} = \frac{J}{\cosh^2 u}$

Divide $E = \dot{u}^2 - \cosh^2 u \, \dot{\phi}^2$ by $\dot{\phi}$ gives.

$\left(\frac{\dot{u}}{\dot{\phi}}\right)^2 - \cosh^2 u = \frac{E}{\dot{\phi}^2}, \quad \because \frac{\dot{u}}{\dot{\phi}} = \frac{du}{d\phi}.$

$\therefore \quad \left(\frac{du}{d\phi}\right)^2 - \cosh^2 u = \frac{E}{J^2}\cosh^4 u$

$\therefore \quad \cosh^4 u \left(\frac{dv}{d\phi}\right)^2 - \cosh^2 u = \frac{E}{J^2}\cosh^4 u$

$\therefore \quad \left(\frac{dv}{d\phi}\right)^2 = \frac{E}{J^2} + (\cosh^2 u)^{-1}$

$\because \quad \operatorname{sech}^2 u = \frac{1}{\cosh^2 u} = 1 - \tanh^2 u = 1 - v^2$

$\therefore \quad \left(\frac{dv}{d\phi}\right)^2 = \left(\frac{E}{J^2} + 1\right) - v^2$ Clear! ① $\square$

Differentiate ①

$$2\cancel{\frac{dv}{d\phi}}\frac{d^2 v}{d\phi^2} + 2v\cancel{\frac{dv}{d\phi}} = 0$$

$$\therefore \quad \frac{d^2 v}{d\phi^2} + v = 0 \Rightarrow v \sim e^{i\phi} ?$$

The general solution to this is given by

$V(\phi) = A\cos(\phi + \phi_0)$ [A, ϕ_0 are constants]

$\therefore$ $\left(\frac{dv}{d\phi}\right)^2 + v^2 = \frac{E}{J^2} + 1$

$\therefore$ $\left(-A\sin(\phi+\phi_0)\right)^2 + \left(A\cos(\phi+\phi_0)\right) = \frac{E}{J^2} + 1$

$\Rightarrow$ $A^2\left(\sin^2(\phi+\phi_0) + \cos^2(\phi+\phi_0)\right) = \frac{E}{J^2} + 1$

$\therefore$ $A^2 = \frac{E}{J^2} + 1$ $\quad$ $\therefore A = \pm\sqrt{\frac{E}{J^2} + 1}$

$\therefore$ $V(\phi) = \pm\sqrt{\frac{E}{J^2}+1}\,\cos(\phi + \phi_0)$

$\frac{E}{J^2}$ ~~is the~~ scales with the amplitude of ~~to~~ the oscillatory solution of $V(\phi)$.

<u>Great!</u>

(5) $\uparrow\uparrow\uparrow$

$$L_X T_{ab} = X^c \partial_c T_{ab} + (\partial_a X^c) T_{cb} + (\partial_b X^c) T_{ac}$$

(1)

$$X^c \nabla_c T_{ab} + (\nabla_a X^c) T_{cb} + (\nabla_b X^c) T_{ac}.$$

$$= X^c \left[\partial_c T_{ab} - \Gamma^{\nu}_{ca} T_{\nu b} - \Gamma^{\nu}_{cb} T_{a\nu} \right]$$

$$+ (\partial_a X^c + \Gamma^{c}_{a\lambda} X^{\lambda}) T_{cb}$$

$$+ (\partial_b X^c + \Gamma^{c}_{b\lambda} X^{\lambda}) T_{ac}$$

$$= X^c \partial_c T_{ab} + (\partial_a X^c) T_{cb} + (\partial_b X^c) T_{ac}$$

$$- \cancel{X^c \Gamma^{\nu}_{ca} T_{\nu b}} - \cancel{X^c \Gamma^{\nu}_{cb} T_{a\nu}}$$

$$+ \cancel{X^{\lambda} \Gamma^{c}_{a\lambda} T_{cb}} + \cancel{X^{\lambda} \Gamma^{c}_{b\lambda} T_{ac}}$$

$$= L_X T_{ab}$$

Good!

$\Rightarrow$ can replace ∂_a by ∇_a.

$\because$ $L_X T_{ab}$ can be expressed as sum of (0,2) tensors with indices a and b. $\therefore$ $L_X T_{ab}$ is a (0,2) tensor.

Great!

(2)

$$S = \int_{s_1}^{s_2} g_{ab} \dot{x}^a \dot{x}^b \, ds$$ transformation

$x'^a = x^a + \delta x^a$
$= x^a + \epsilon K^a$

$$g'_{ab} = g_{ab}(x^{\lambda} + \epsilon K^{\lambda})$$

$$= g_{ab}(x^{\lambda}) + \frac{\partial g_{ab}}{\partial x^{\lambda}} \epsilon x^{\lambda}$$

$\dot{x}'^a = \dot{x}^a + \epsilon \dot{K}^a$ $\qquad$ $\dot{K}^a = \frac{dK^a}{ds}$

$$S' = \int_{s_1}^{s_2} ds\, g'_{ab}\, \dot{x}'^a \dot{x}'^b = \int_{s_1}^{s_2} (g_{ab} + (\partial_\lambda g_{ab}) \epsilon K^\lambda) \times$$

$$(\dot{x}^a + \epsilon \dot{K}^a)(\dot{x}^b + \epsilon \dot{K}^b)\, ds$$

$$= \int_{s_1}^{s_2} ds\, g_{ab} \dot{x}^a \dot{x}^b + \int_{s_1}^{s_2} ds\, \epsilon (\dot{x}^a \dot{x}^b \partial_\lambda g_{ab} K^\lambda + g_{ab} \dot{K}^a \dot{x}^b + g_{ab} \dot{K}^b \dot{x}^a)$$

$$+ \int_{s_1}^{s_2} ds\, \cancel{O(\epsilon^2)} \overset{!}{=} S \quad \text{for } \underline{\underline{\text{any}}} \text{ trajectory}$$

$$\Rightarrow K^\lambda \dot{x}^a \dot{x}^b \partial_\lambda g_{ab} + g_{ab} \dot{K}^a \dot{x}^b + g_{ab} \dot{K}^b \dot{x}^a = 0$$

$$\because \dot{K}^a = \frac{dK^a}{ds} = \frac{dx^\lambda}{ds} \frac{\partial K^a}{\partial x^\lambda} = \dot{x}^\lambda \partial_\lambda K^a$$

$$\therefore K^\lambda \dot{x}^a \dot{x}^b \partial_\lambda g_{ab} + g_{ab} \dot{x}^b \dot{x}^\lambda \partial_\lambda K^a + g_{ab} \dot{x}^a \dot{x}^\lambda \partial_\lambda K^b = 0$$

$\downarrow a \leftrightarrow \lambda$ $\quad$ $\downarrow b \leftrightarrow \lambda$

$$\Rightarrow \dot{x}^a \dot{x}^b (\underbrace{K^\lambda \partial_\lambda g_{ab} + (\partial_a K^\lambda) g_{\lambda b} + (\partial_b K^\lambda) g_{a\lambda}}_{L_K g_{ab}}) = 0$$

true for any $\dot{x}$

$$\therefore \underline{\underline{L_K g_{ab} = 0}}$$

Clear! ~~guaran~~ guarantees the invariance of S.

Fantastic!

...ing to (1)

metric-compatible connection

$$L_K g_{ab} = \cancel{X^c \nabla_c g_{ab}}\; K^c \cancel{\nabla_c} g_{ab} + (\nabla_a K^c) g_{cb} + (\nabla_b K^c) g_{ac} = 0$$

First term vanishes since $\nabla_c g_{ab} = 0$

$$\therefore\ 0 = (\nabla_a K^c) g_{cb} + (\nabla_b K^c) g_{ac}$$

$$= \nabla_a (g_{cb} K^c) + \nabla_b (g_{ac} K^c)$$

$$= \nabla_a K_b + \nabla_b K_a = \nabla_{(a} K_{b)}$$

Clear! $\square$

(3)

$$\frac{d}{ds}(g_{ab} K^a \dot{x}^b) = \frac{d}{ds}(K_b \dot{x}^b)$$

$$\cancel{= \dot{x}^\lambda \partial_\lambda (K_b \dot{x}^b)}$$

$$\cancel{= \dot{x}^\lambda \dot{x}^b \partial_\lambda K_b + K_b \dot{x}^\lambda \partial_\lambda \dot{x}^b}$$

$$= \dot{K}_b \dot{x}^b + K_b \ddot{x}^b$$

Geodesic equation

$$\ddot{x}^b + \Gamma^b_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$$

According to (1)

metric-compatible connection

$$L_K g_{ab} = \cancel{X^c \nabla_c g_{ab}}\ K^c \cancel{\nabla_c g_{ab}} + (\nabla_a K^c) g_{cb} + (\nabla_b K^c) g_{ac} = 0$$

First term vanishes since $\nabla_c g_{ab} = 0$

$\therefore$,

$$\therefore 0 = (\nabla_a K^c) g_{cb} + (\nabla_b K^c) g_{ac}$$

$$= \nabla_a (g_{cb} K^c) + \nabla_b (g_{ac} K^c)$$

$$= \nabla_a K_b + \nabla_b K_a = \nabla_{(a} K_{b)}$$

$\square$

Clear!

(3)

$$\frac{d}{ds}(g_{ab} K^a \dot{x}^b) = \frac{d}{ds}(K_b \dot{x}^b)$$

$$\cancel{= \dot{x}^\lambda \partial_\lambda (K_b \dot{x}^b)}$$

$$\cancel{= \dot{x}^\lambda \dot{x}^b \partial_\lambda K_b + K_b \dot{x}^\lambda \partial_\lambda \dot{x}^b}$$

$$= \dot{K}_b \dot{x}^b + K_b \ddot{x}^b$$

Geodesic equation

$$\ddot{x}^b + \Gamma^b_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$$

Consider $\nabla_\lambda K_b = \partial_\lambda K_b - \Gamma^\mu{}_{\lambda b} K_\mu$

$\nabla_b K_\lambda = \partial_b K_\lambda - \Gamma^\mu{}_{\lambda b} K_\mu$

$$2\frac{d}{d\zeta}(g_{ab}K^a\dot{x}^b) = (\dot{K}_b\dot{x}^b + K_b\ddot{x}^b)\times 2$$

$$= 2\dot{x}^\lambda(\partial_\lambda K_b)\dot{x}^b + 2K_b(-\Gamma^b{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)$$

$$= \cancel{2Q}\ \dot{x}^\lambda\dot{x}^b(\partial_\lambda K_b) + \dot{x}^\lambda\dot{x}^b(\partial_b K_\lambda) - \cancel{2Q}\ 2K_b\Gamma^b{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$= \dot{x}^\lambda\dot{x}^b\left[\nabla_\lambda K_b + \Gamma^\mu{}_{\lambda b}K_\mu + \nabla_b K_\lambda + \Gamma^\mu{}_{\lambda b}K_\mu\right] - 2K_b\Gamma^b{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$= \dot{x}^\lambda\dot{x}^b[\underbrace{\nabla_\lambda K_b + \nabla_b K_\lambda}_{=0 \text{ by def. of Killing vector}}] + 2\dot{x}^\lambda\dot{x}^b\Gamma^\mu{}_{\lambda b}K_\mu - 2\dot{x}^\mu\dot{x}^\nu\Gamma^b{}_{\mu\nu}K_b$$

equal by ~~$(\lambda,b)\leftrightarrow(\mu,\nu)$~~
$\mu \leftrightarrow b$
$\nu \leftrightarrow \lambda$

$= 0$

$\therefore$ $g_{ab}K^a\dot{x}^b$ conserved along geodesic $\Box$

Good!

Another way to do it is to realize
$\nabla_{\dot{x}}\dot{x} = 0$ for a geodesic

$\Rightarrow \nabla_{\dot{x}}(K\cdot\dot{x}) = \dot{x}\nabla_{\dot{x}}K = \dot{x}_\alpha\dot{x}_\beta\nabla^\alpha K^\beta$

$= \dot{x}_\alpha\dot{x}_\beta\nabla^{(\alpha}K^{\beta)}$

$= 0.$

(4)

$$\nabla_u K_\phi = \nabla_u (\underbrace{g_{\phi\phi}}_{\cosh^2(u)} \underbrace{K^\phi}_{1}) \qquad (\because g_{u\phi} = g_{\phi u} = 0)$$

$$= \nabla_u(\cosh^2(u)) = \partial_u(\cosh^2(u)) - \underbrace{\Gamma^\phi{}_{u\phi}}_{\tanh(u)} \underbrace{K_\phi}_{\cosh^2(u)} - \underbrace{\Gamma^u{}_{u\phi} K_u}_{=0}$$

$$= 2\cosh(u)\sinh(u) - \tanh(u)\cosh^2(u)$$

$$\nabla_\phi K_u = \nabla_\phi (g_{uu} K^u)$$

$$= \underbrace{\partial_\phi(0)}_{=0} - \underbrace{\Gamma^u{}_{u\phi} K_u}_{=0} - \Gamma^\phi{}_{u\phi} K_\phi$$

$$= -\tanh(u)\cosh^2(u)$$

$$\therefore \nabla_u K_\phi + \nabla_\phi K_u = 2\cosh(u)\sinh(u) - 2\tanh(u)\cosh^2(u)$$

$$= 2\cosh(u)\,[\sinh(u) - \underbrace{\tanh(u)\cosh(u)}_{\sinh(u)}] = 0 \quad \square$$

$\therefore K^a$ is a Killing vector.

Conserved quantity

$$g_{ab} K^a \dot{X}^b = K_a \dot{X}^a = \underbrace{K_u}_{=0} \dot{u} + K_\phi \dot{\phi}$$

$$= K_\phi \dot{\phi} = \underline{\underline{\cosh^2(u)\,\dot{\phi} = J}} \quad \square$$

Great!

(5) If X^a, Y^a are two Killing vectors.

i.e. $\cancel{\nabla_c X^a}$ $\nabla_c X_a + \nabla_a X_c = 0$ ①

and $\nabla_c Y_a + \nabla_a Y_c = 0$ ②

$\therefore$ adding ① and ②

$$\nabla_c (X_a + Y_a) + \nabla_a (X_c + Y_c) = 0$$

$\Rightarrow$ $X^a + Y^a$ is a Killing vector. ✓

- For the commutator $[X, Y]^a = X^b \nabla_b Y^a - Y^b \nabla_b X^a$

Consider the quantity

$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu$ (V^μ a vector)

$$= \nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu$$

$$= \nabla_\alpha (\partial_\beta V^\mu + \Gamma^\mu_{\beta\nu} V^\nu) - \nabla_\beta (\partial_\alpha V^\mu + \Gamma^\mu_{\alpha\nu} V^\nu)$$

$$= \cancel{\partial_\alpha \partial_\beta V^\mu} + \Gamma^\mu_{\alpha\nu} \partial_\beta V^\nu - \cancel{\Gamma^\nu_{\alpha\beta} \partial_\nu V^\mu}$$
$$+ \partial_\alpha \Gamma^\mu_{\beta\nu} V^\nu + \Gamma^\mu_{\alpha\sigma} \Gamma^\sigma_{\beta\nu} V^\nu - \cancel{\Gamma^\sigma_{\alpha\beta} \Gamma^\mu_{\sigma\nu} V^\nu}$$
$$- \cancel{\partial_\beta \partial_\alpha V^\mu} - \Gamma^\mu_{\beta\nu} \partial_\alpha V^\nu + \cancel{\Gamma^\nu_{\beta\alpha} \partial_\nu V^\mu}$$
$$- \partial_\beta \Gamma^\mu_{\alpha\nu} V^\nu - \Gamma^\mu_{\beta\sigma} \Gamma^\sigma_{\alpha\nu} V^\nu + \cancel{\Gamma^\sigma_{\beta\alpha} \Gamma^\mu_{\sigma\nu} V^\nu}$$

$$= \Gamma^\mu_{\alpha\nu} \partial_\beta V^\nu + \partial_\alpha \Gamma^\mu_{\beta\nu} V^\nu + \Gamma^\mu_{\alpha\sigma} \Gamma^\sigma_{\beta\nu} V^\nu$$
$$- \Gamma^\mu_{\beta\nu} \partial_\alpha V^\nu - \partial_\beta \Gamma^\mu_{\alpha\nu} V^\nu - \Gamma^\mu_{\beta\sigma} \Gamma^\sigma_{\alpha\nu} V^\nu$$

$\sigma \to \epsilon$

$$= \left(\partial_\alpha \Gamma^\mu{}_{\beta\nu} V^\nu - \Gamma^\mu{}_{\beta\nu}\partial_\alpha V^\nu\right) - \left(\partial_\beta \Gamma^\mu{}_{\alpha\nu} V^\nu - \Gamma^\mu{}_{\alpha\nu}\partial_\beta V^\nu\right)$$

$$+ \Gamma^\mu{}_{\alpha\epsilon}\Gamma^\epsilon{}_{\nu\beta} V^\nu - \Gamma^\mu{}_{\epsilon\beta}\Gamma^\epsilon{}_{\nu\alpha} V^\nu$$

$$= (\partial_\alpha \Gamma^\mu{}_{\beta\nu})V^\nu - (\partial_\beta \Gamma^\mu{}_{\alpha\nu})^\nu + (\Gamma^\mu{}_{\alpha\epsilon}\Gamma^\epsilon{}_{\nu\beta} - \Gamma^\mu{}_{\epsilon\beta}\Gamma^\epsilon{}_{\nu\alpha})V^\nu$$

$$= \Big[\underbrace{\partial_\alpha \Gamma^\mu{}_{\beta\nu} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} + \Gamma^\mu{}_{\alpha\epsilon}\Gamma^\epsilon{}_{\nu\beta} - \Gamma^\mu{}_{\epsilon\beta}\Gamma^\epsilon{}_{\nu\alpha}}_{R^\mu{}_{\nu\alpha\beta}}\Big]V^\nu$$

$= R^\mu{}_{\nu\alpha\beta} V^\nu$. The quantity in final bracket, $R^\mu{}_{\nu\alpha\beta}$, is called the Riemann tensor.

We have $\underline{(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)V^\mu = R^\mu{}_{\nu\alpha\beta}V^\nu \quad (*)}$

Now suppose X^μ, Y^μ are Killing vectors.

$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$; $\nabla_\mu Y_\nu + \nabla_\nu Y_\mu = 0$

commutator

$\downarrow$

$W^\mu = [X, Y]^\mu = X^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu X^\mu$

then • lowering μ gives

$W_\mu = X^\nu \nabla_\nu Y_\mu - Y^\nu \nabla_\nu X_\mu$

$$\therefore \nabla_\sigma W_\mu + \nabla_\mu W_\sigma = \nabla_\sigma(X^\nu \nabla_\nu Y_\mu) - \nabla_\sigma(Y^\nu \nabla_\nu X_\mu)$$
$$+ \nabla_\mu(X^\nu \nabla_\nu Y_\sigma) - \nabla_\mu(Y^\nu \nabla_\nu X_\sigma)$$

Now consider (*) and lowering μ

$$\nabla_\alpha \nabla_\beta V_\mu - \nabla_\beta \nabla_\alpha V_\mu = R_{\mu\nu\alpha\beta} V^\nu$$

(Riemann tensor $R^\mu{}_{\nu\alpha\beta}$ is a tensor $\because$ left side is a tensor, and V^ν is a vector $\therefore$ By <u>quotient rule</u> $R_{\mu\nu\alpha\beta}$ is a tensor)

$\therefore \nabla_\sigma W_\mu + \nabla_\mu W_\sigma$ product rule expansion

$$= (\nabla_\sigma X^\nu)(\nabla_\nu Y_\mu) + X^\nu \nabla_\sigma \nabla_\nu Y_\mu$$
$$- (\nabla_\sigma Y^\nu)(\nabla_\nu X_\mu) - Y^\nu \nabla_\sigma \nabla_\nu X_\mu$$
$$+ (\nabla_\mu X^\nu)(\nabla_\nu Y_\sigma) + X^\nu \nabla_\mu \nabla_\nu Y_\sigma$$
$$- (\nabla_\mu Y^\nu)(\nabla_\nu X_\sigma) - Y^\nu \nabla_\mu \nabla_\nu X_\sigma$$

Now $(\nabla_\sigma X^\nu)(\nabla_\nu Y_\mu) - (\nabla_\mu Y^\nu)(\nabla_\nu X_\sigma)$

$$= (\nabla_\sigma X^\nu)(\nabla_\nu Y_\mu) - (\nabla_\mu Y_\nu)(\nabla^\nu X_\sigma)$$
$$= (\nabla_\sigma X^\nu)(\nabla_\nu Y_\mu) - (-\nabla_\nu Y_\mu)(-\nabla_\sigma X^\nu)$$

$\nabla_\nu Y_\mu + \nabla_\mu Y_\nu = 0$

$\nabla_\sigma X_\nu + \nabla_\nu X_\sigma = 0$

$\therefore \nabla_\sigma X^\nu + \nabla^\nu X_\sigma = 0$

$$= 0$$

similarly

$$(\nabla_\mu X^\nu)(\nabla_\nu Y_\sigma) - (\nabla_\sigma Y^\nu)(\nabla_\nu X_\mu)$$

$$= (\nabla_\mu X^\nu)(\nabla_\nu Y_\sigma) - (\nabla_\sigma Y_\nu)(\nabla^\nu X_\mu)$$

$$= (\nabla_\mu X^\nu)(\nabla_\nu Y_\sigma) - (-\nabla_\nu Y_\sigma)(-\nabla_\mu X^\nu)$$

$$= 0$$

Hence

$$\nabla_\sigma W_\mu + \nabla_\mu W_\sigma$$

$$= X^\nu \nabla_\sigma \nabla_\nu Y_\mu - Y^\nu \nabla_\sigma \nabla_\nu X_\mu + X^\nu \nabla_\mu \nabla_\nu Y_\sigma - Y^\nu \nabla_\mu \nabla_\nu X_\sigma$$

use (*)

$$= X^\nu(\nabla_\nu \nabla_\sigma Y_\mu + R_{\mu\epsilon\sigma\nu} Y^\epsilon) - Y^\nu(\nabla_\nu \nabla_\sigma X_\mu + R_{\mu\epsilon\sigma\nu} X^\epsilon)$$

$$+ X^\nu(\nabla_\nu \nabla_\mu Y_\sigma + R_{\sigma\epsilon\mu\nu} Y^\epsilon) - Y^\nu(\nabla_\nu \nabla_\mu X_\sigma + R_{\sigma\epsilon\mu\nu} X^\epsilon)$$

$$= X^\nu \nabla_\nu (\underbrace{\nabla_\sigma Y_\mu + \nabla_\mu Y_\sigma}_{=0}) - Y^\nu \nabla_\nu (\underbrace{\nabla_\sigma X_\mu + \nabla_\mu X_\sigma}_{=0})$$

$$+ R_{\mu\epsilon\sigma\nu} X^\nu Y^\epsilon - R_{\mu\epsilon\sigma\nu} Y^\nu X^\epsilon$$

$$+ R_{\sigma\epsilon\mu\nu} X^\nu Y^\epsilon - R_{\sigma\epsilon\mu\nu} Y^\nu X^\epsilon$$

$$= R_{\mu\tau\sigma\nu} X^\nu Y^\tau - R_{\mu\nu\sigma\tau} X^\nu Y^\tau$$

$$+ R_{\sigma\tau\mu\nu} X^\nu Y^\tau - R_{\sigma\nu\mu\tau} X^\nu Y^\tau$$

exchange $\nu \leftrightarrow \tau$

$$= X^\nu Y^\tau \left(R_{\mu\tau\sigma\nu} - R_{\mu\nu\sigma\tau} + R_{\sigma\tau\mu\nu} - R_{\sigma\nu\mu\tau} \right)$$

Now we examine the symmetry of Riemman tensor.

Recall $R^\sigma{}_{bcd} = \dfrac{\partial \Gamma^\sigma{}_{bd}}{\partial x^c} - \dfrac{\partial \Gamma^\sigma{}_{bc}}{\partial x^d} + \Gamma^\tau{}_{bd}\Gamma^\sigma{}_{c\tau} - \Gamma^\tau{}_{bc}\Gamma^\sigma{}_{d\tau}$

$R_{abcd} = g_{a\sigma} R^\sigma{}_{bcd}$

$\cancel{= g_{a\sigma}\left(\partial_c \Gamma^\sigma{}_{bd} - \partial_d \Gamma^\sigma{}_{bc} + \Gamma^\tau{}_{bd}\Gamma^\sigma{}_{c\tau} - \Gamma^\tau{}_{bc}\Gamma^\sigma{}_{d\tau}\right)}$

It is well known that (proof omitted, too long...)

$$R_{abcd} = -R_{bacd} = -R_{abdc}$$

$$\boxed{R_{abcd} = R_{cdab}}$$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0.$$

Hence

$$\nabla_\sigma W_\mu + \nabla_\mu W_\sigma = X^\nu Y^\tau \left(\cancel{\underbrace{R_{\mu\tau\sigma\nu}}_{=R_{\sigma\nu\mu\tau}}} - \cancel{\underbrace{R_{\mu\nu\sigma\tau}}_{=R_{\sigma\tau\mu\nu}}} + \cancel{R_{\sigma\tau\mu\nu}} - \cancel{R_{\sigma\nu\mu\tau}} \right)$$

Cancel!

$$= 0$$

$\Rightarrow$ $W^a = [X, Y]^a$ is a Killing vector. $\square$

⑥ (???)

$\nabla_a F^{ab} = 0 \quad , \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0$

$F_{ab} = -F_{ba}$

$T_{ab} = \frac{1}{4\pi}(F_{ac}F^c{}_b + \frac{1}{4}F^{cd}F_{cd}\, g_{ab})$

$T^{ab} = \frac{1}{4\pi}(F^{ac}F_c{}^b + \frac{1}{4}F^{cd}F_{cd}\, g^{ab})$

$\cancel{\therefore -4\pi T^{ab} = F^{ac}F}$

(1) $\because F_{cb} = -F_{bc} \quad \therefore F_c{}^b = -F^b{}_c$ (this doesn't mean $F_c{}^b$ is antisymmetric, and its not. this is obtained by ~~multipl~~ $g^{ba}F_{ca} = -g^{ba}F_{ac}$) ⑦

$\therefore -4\pi T^{ab} = F^{ac}F^b{}_c - \frac{1}{4}F^{cd}F_{cd}\, g^{ab}$

$\therefore -4\pi \nabla_a T^{ab} = \nabla_a(F^{ac}F^b{}_c) - \frac{1}{4}g^{ab}\nabla_a(F^{cd}F_{cd})$

where we've used $\nabla_a g^{ab} = 0$

$$\therefore -4\pi \nabla_a T^{ab} = \underbrace{(\nabla_a F^{ac})}_{=0}F^b{}_c + F^{ac}\nabla_a F^b{}_c - \frac{1}{4}g^{ab}(\nabla_a F^{cd})F_{cd} - \frac{1}{4}g^{ab}\underbrace{F^{cd}(\nabla_a F_{cd})}_{=F_{cd}\nabla_a F^{cd}}$$

$$= F^{ac}(\nabla_a F^b{}_c) - \frac{1}{2}g^{ab}F^{cd}(\nabla_a F_{cd})$$

$$= F^{ac}g^{b\beta}(\nabla_a F_{\beta c}) - \frac{1}{2}g^{ab}F^{\beta c}(\nabla_a F_{\beta c})$$

① ②

where in the latter term, we change dummy index d to β and swap β and c. The 2 minus signs generated by the swap cancels.

$\therefore$ ~~$-4\pi T^{ab}$~~ $-4\pi \nabla_a T^{ab} = \overbrace{-F^{ac} g^{b\beta} \nabla_a F_{c\beta}}^{①}$

$+ \frac{1}{2} g^{ab} F^{\beta c} (\nabla_\beta F_{ca} + \nabla_c F_{a\beta})$ ②

we used $F_{\beta c} = -F_{c\beta}$ on ①

and $\nabla_a F_{\beta c} = -(\nabla_\beta F_{ca} + \nabla_c F_{a\beta})$ on ②.

$$-4\pi \nabla_a T^{ab} = \frac{1}{2}(\underbrace{-F^{ac} g^{b\beta} \nabla_a F_{c\beta} + g^{ab} F^{\beta c} \nabla_\beta F_{ca}}_{③})$$

$$+ \frac{1}{2}(\underbrace{-F^{ac} g^{b\beta} \nabla_a F_{c\beta} + g^{ab} F^{\beta c} \nabla_c F_{a\beta}}_{④})$$

where we split ① in half and combine with ② to get ③ and ④ after rearrangement.

~~$-4\pi \nabla_a T^{ab} =$~~

→ For ③ : $g^{ab} F^{\beta c} \underbrace{\nabla_\beta F_{ca}} = F^{ac} g^{b\beta} \nabla_a F_{c\beta}$

swap dummies a, β
and use $g^{ab} = g^{ba}$

Good!

→ ③ $= 0$

→ For ④ :

$\because\ g^{ab} F^{\beta c} \nabla_c F_{a\beta} = F^{\beta c} \underbrace{\nabla_c F^b{}_\beta}_{\substack{\beta \to c \\ c \to a}} = F^{ca} \nabla_a F^b{}_c$

$F^{ac} g^{b\beta} \nabla_a F_{c\beta} = F^{ac} \nabla_a \underbrace{F_c{}^b}_{\substack{F^{ac} = -F^{ca} \\ F_c{}^b = -F^b{}_c}} = (-F^{ca}) \nabla_a (-F^b{}_c)$

$= F^{ca} \nabla_a F^b{}_c$

$\rightarrow g^{ab} F^{\beta c} \nabla_c F_{a\beta} = F^{ac} g^{b\beta} \nabla_a F_{c\beta}$

$\rightarrow$ ④ $= 0$

$\because$ ③ $=$ ④ $= 0$

$\therefore -4\pi \nabla_a T^{ab} = 0 \qquad \therefore \nabla_a T^{ab} = 0 \qquad \therefore \nabla^a T_a{}^b = 0$

$\therefore$ ~~g~~ $\quad g_{bc} \nabla^a T_a{}^c = 0 \quad \Rightarrow \quad \underline{\underline{\nabla^a T_{ab} = 0}}$ ✓

□

(2) consider one definition of the determinant of a matrix A

$\det A = \sum_j A_{ij} (\operatorname{adj} A)_{ji}$, where adj A is the adjugate matrix of A. The sum is only over j, not i.

By definition, $(\operatorname{adj} A)_{ij}$ only depends on entries in A that are <u>NOT</u> from the i th row and j th column.

So certainly $(\mathrm{adj}A)_{ij}$ doesn't depend on A_{ij}.

$$\therefore \quad \frac{\partial \det A}{\partial A_{ij}} = (\mathrm{adj} A)_{ji}$$

Now consider

$$\cancel{\mathrm{d}t} \quad \frac{d}{d\alpha}\det A = \frac{\partial \det A}{\partial A_{ij}} \frac{dA_{ij}}{d\alpha} = (\mathrm{adj} A)_{ji} \frac{dA_{ij}}{d\alpha}$$

By definition $A^{-1} = \frac{1}{\det A} \mathrm{adj} A$ $\quad \therefore \cancel{(\mathrm{adj})}$

$$\therefore \cancel{\mathrm{Adj}} \quad (\mathrm{adj} A)_{ji} = \det A \, (A^{-1})_{ji}$$

$$\therefore \quad \frac{d}{d\alpha}\det A = (\det A)\, A^{-1}{}_{ji} \frac{dA_{ij}}{d\alpha}$$

$$= \det A \, \mathrm{tr}\left(A^{-1} \frac{dA}{d\alpha}\right) \qquad \text{(Jacobi's Formula)}$$

Now consider metric tensor g^{ab}. let ~~t~~ $|g| = \det(g)$

$$\frac{\partial |g|}{\partial x^c} = |g| \, \mathrm{tr}\left(g^{-1} \frac{\partial g}{\partial x^c}\right) \cancel{= |g| \, g^{ab} \partial g_{ab}} \qquad (g^{ab} g_{bc} = \delta^a{}_c)$$

$$\cancel{= |g| \, \mathrm{tr}(} \quad = |g| \, g^{ba} \frac{\partial g_{ab}}{\partial x^c} = |g| \, g^{ab} \frac{\partial g_{ab}}{\partial x^c}$$

$$\underline{\partial_c |g| = |g| \, g^{ab} \partial_c g_{ab}} \quad (*)$$

The Christoffel symbol

$$\cancel{\Gamma^a{}_{bc}} \quad \Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$$

~~Define~~ $\cancel{\Gamma_{abc} = g_{ad} \Gamma^d{}_{bc} = (\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc}) \times \frac{1}{2}}$

~~then clearly~~ $\cancel{\Gamma^a{}_{bc} = g^{ad}} \Gamma_{dbc}$

$\therefore \Gamma^a{}_{ac} = \frac{1}{2} g^{ad} (\partial_a g_{dc} + \partial_c g_{ad} - \partial_d g_{ac})$.

consider $g^{ad} (\partial_a g_{dc} - \partial_d g_{ac})$

g^{ad}: symmetric in a, d

$(\partial_a g_{dc} - \partial_d g_{ac})$: anti symmetric in a, d

$\rightarrow$ the product $= 0$

$\therefore \Gamma^a{}_{ac} = \frac{1}{2} g^{ad} \partial_c g_{ad} = \frac{1}{2} g^{ab} \partial_c g_{ab}$

substitute this into (*)

$\cancel{\partial_c |g| = |g|}$ $\partial_c |g| = |g| \times 2\Gamma^a{}_{ac}$

$\therefore \Gamma^a{}_{ac} = \frac{1}{2} \frac{1}{|g|} \partial_c |g| = \frac{1}{2} \frac{1}{(-|g|)} \partial_c (-|g|)$

in most cases $|g| = \det(g) < 0 \quad \therefore -|g| > 0$

$\therefore \Gamma^b{}_{ab} = \Gamma^b{}_{ba} = \frac{1}{2} \frac{1}{(-|g|)} \partial_a (-|g|) = \frac{1}{2} \partial_a \log(-|g|)$

$= \partial_a \frac{1}{2} \log(-|g|) = \underline{\partial_a \log \sqrt{-|g|}}$ $\square$ Good!

(3) * Here probably $|g| = |\det(g)|$ rather than $\det(g)$ as in (2). convention is a bit confusing...

Conventions in GR and QFT are a headache!

$\rightarrow 0 = \partial_a (\sqrt{|g|} F^{ab}) = \sqrt{|g|} \partial_a F^{ab} + F^{ab} \partial_a \sqrt{|g|}$

$= \sqrt{|g|} \partial_a F^{ab} + \frac{1}{2} \frac{1}{\sqrt{|g|}} \frac{\partial |g|}{\partial x^a} F^{ab}$

$\Leftrightarrow 0 = \partial_a F^{ab} + \underbrace{\frac{1}{2} \frac{1}{|g|} \partial_a |g|}_{\Gamma^c{}_{ac}} F^{ab} \Leftrightarrow 0 = \partial_a F^{ab} + \Gamma^c{}_{ac} F^{ab}$ ■

$\Leftrightarrow \partial_a F^{ab} + \Gamma^a{}_{ac} F^{cb} = 0 \Leftrightarrow \underline{\nabla_a F^{ab} = 0}$

$\uparrow$

swap a, c
and use $\Gamma^a{}_{ca} = \Gamma^a{}_{ac}$

$\Leftrightarrow$ The ~~first~~ First. Maxwell equation $\square$

(4) second equation

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0$$

$$\begin{aligned}
\Rightarrow \quad 0 = {} & \partial_a F_{bc} - \Gamma^\nu{}_{ba} F_{\nu c} - \Gamma^\nu{}_{ca} F_{b\nu} \\
& + \partial_b F_{ca} - \Gamma^\nu{}_{cb} F_{\nu a} - \Gamma^\nu{}_{ab} F_{c\nu} \\
& + \partial_c F_{ab} - \Gamma^\nu{}_{ac} F_{\nu b} - \Gamma^\nu{}_{bc} F_{a\nu} \\
= {} & (\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}) - \Gamma^\nu{}_{ab} (\underbrace{F_{\nu c} + F_{c\nu}}_{0}) \\
& - \Gamma^\nu{}_{bc} (\underbrace{F_{\nu a} + F_{a\nu}}_{0}) - \Gamma^\nu{}_{ca} (\underbrace{F_{b\nu} + F_{\nu b}}_{0})
\end{aligned}$$

$$\Rightarrow \quad 0 = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}$$

$\square$

Clear!