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C7.5 General Relativity I

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Problem Set 1

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Very well done!

perturbation Methods

1   1+2

$$X = X^{ab} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 2 \end{pmatrix} \qquad V = V^a = (-1\ 2\ 0\ -2)$$

$$\eta = \eta_{ab} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta^{ab}$$

$$\rightarrow X^a{}_b = X^{ac}\eta_{cb} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \cancel{X\cdot\eta}$$

$$= \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

Good!

$$\rightarrow X_a{}^b = \cancel{\eta_{ac}\eta^{bd}X^a} \ \eta_{ac}X^{cb} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$$

$$\rightarrow X^{(ab)} = \frac{1}{2}(X^{ab} + X^{ba}) = \begin{pmatrix} 2 & -\frac{1}{2} & 0 & -\frac{3}{2} \\ -\frac{1}{2} & 0 & 2 & \frac{3}{2} \\ 0 & 2 & 0 & \frac{1}{2} \\ -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & -2 \end{pmatrix}$$

Fantastic!

$$\cancel{X_{ab} = g_{ac}X^c{}_b = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \end{pmatrix}}$$

$$\cancel{X_{ab} = g_{ac}X^c{}_b = g_{ac}X^{cd}g_{db} =}$$

$$X_{ab} = \eta_{ac}X^{cd}\eta_{db} = \eta X \eta = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & +2 \end{pmatrix}$$

Great!

$$\rightarrow X_{[ab]} = \frac{1}{2}(X_{ab} - X_{ba}) = \begin{pmatrix} 0 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 1 & -1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

$$\rightarrow X^{\lambda}{}_{\lambda} = \mathrm{tr}(X^a{}_b) = -2 + 0 + 0 - 2 = \underline{\underline{-4}}$$

Careful!

$$\cancel{\rightarrow V^a V_a = V^a V_a = g_{ab} V^b =}$$

$$V_a = \eta_{ab} V^b = V^b \eta_{ba} = (-1\ 2\ 0\ -2)\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= (1\ \ 2\ \ 0\ \ -2)$$

$$\therefore V^a V_a = (-1)(1) + (2)(2) + (0)(0) + (-2)(-2)$$

$$= -1 + 4 + 4 = \underline{\underline{7}}$$

$$\rightarrow V_a X^{ab} \cancel{= V \cdot X = (0, 2, 3, 4)}$$

$$= (4, -2, 5, \cancel{-}1)^T$$

(2) (1+1)

1. $x'^a = L^{ab} x^b$ ~~one o~~

one of the ~~indice~~ index b's should be lower

$\rightarrow$ $\underline{x'^a = L^a{}_b x^b}$ Good!

2. $x'^a = L^b{}_c M^c{}_d x^d$

~~ind~~ $\rightarrow$ free index needs to match.

~~$x'^b = L$~~

$\underline{x'^b = L^b{}_c M^c{}_d x^d}$ ☺

3. $\delta^a_b = \delta^a_c \delta^c_d$

free indices don't match

$\underline{\delta^a_b = \delta^a_c \delta^c_b}$ ☺

4. $x'^a = L^a{}_c x^c + M^c{}_d x^d$

free indices don't match

$\underline{x'^a = L^a{}_c x^c + M^a{}_d x^d}$ Good!

5. $x'^a = L^a{}_c x^c + M^{ad} x^d$

one of the d's should be lower

$\underline{x'^a = L^a{}_c x^c + M^a{}_d x^d}$ Clear!

6. $\phi = (X^a A_a)(Y^a B_a)$

Same dummy indices can only appear twice

$$\phi = (X^a A_a)(Y^b B_b)$$

3

A (p,q) tensor looks like:

$$T^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q}$$

under change of coordinates $x^a \to x'^a$,

T transforms like

$$T'^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q} = \frac{\partial x'^{a_1}}{\partial x^{\mu_1}} \frac{\partial x'^{a_2}}{\partial x^{\mu_2}} \cdots \frac{\partial x'^{a_p}}{\partial x^{\mu_p}} \times$$

$$\frac{\partial x^{\nu_1}}{\partial x'^{b_1}} \frac{\partial x^{\nu_2}}{\partial x'^{b_2}} \cdots \frac{\partial x^{\nu_q}}{\partial x'^{b_q}} T^{\mu_1 \mu_2 \cdots \mu_p}{}_{\nu_1 \nu_2 \cdots \nu_q}$$

Good!

For Lorentz transformation

$$x'^a = \frac{\partial x'^a}{\partial x^b} x^b \qquad \frac{\partial x'^a}{\partial x^b} = \Lambda^a{}_b$$

$$\frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\beta = \frac{v}{c}$$

for boost along x-direction.

Great!

- To show something is a tensor, it is suffice to show that it has the correct transformation propert.

- Given $S'^{a}{}_{b} = \frac{\partial x'^{a}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{b}} S^{\mu}{}_{\nu}$, $T^{a}{}_{b} = \frac{\partial x'^{a}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{b}} T^{\mu}{}_{\nu}$

$$\rightarrow S'^{a}{}_{b} + T'^{a}{}_{b} = \underline{\left(\frac{\partial x'^{a}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{b}}\right)\left(S^{\mu}{}_{\nu} + T^{\mu}{}_{\nu}\right)}$$

Good!

$\rightarrow$ $S^{a}{}_{b} + T^{a}{}_{b}$ is a (1, 1) tensor.

- Given $S'^{a}{}_{b} = \frac{\partial x'^{a}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{b}} S^{\mu}{}_{\nu}$ $T'^{c} = \frac{\partial x'^{c}}{\partial x^{\rho}} T^{\rho}$

$$S'^{a}{}_{b} T'^{c} = \frac{\partial x'^{a}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{b}} S^{\mu}{}_{\nu} \frac{\partial x'^{c}}{\partial x^{\rho}} T^{\rho} = \underline{\frac{\partial x'^{a}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{b}} \frac{\partial x'^{c}}{\partial x^{\rho}} S^{\mu}{}_{\nu} T^{\rho}}$$

$\rightarrow$ $S^{a}{}_{b} T^{c}$ is a (2, 1) tensor ✓

- Given $S'^{ac}{}_{bd} = \frac{\partial x'^{a}}{\partial x^{\mu}} \frac{\partial x'^{c}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{b}} \frac{\partial x^{\sigma}}{\partial x'^{d}} S^{\mu\nu}{}_{\rho\sigma}$

contraction : change d to c

$$S'^{ac}{}_{bc} = \frac{\partial x'^{a}}{\partial x^{\mu}} \left(\frac{\partial x'^{c}}{\partial x^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{c}}\right) \frac{\partial x^{\rho}}{\partial x'^{b}} S^{\mu\nu}{}_{\rho\sigma}$$

$$= \frac{\partial x'^{a}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{b}} \delta^{\sigma}_{\nu} S^{\mu\nu}{}_{\rho\sigma}$$

$$= \underline{\frac{\partial x'^{a}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{b}} S^{\mu\sigma}{}_{\rho\sigma}}$$

Good!

$S'^{ac}{}_{bc}$ is a (1, 1) tensor

- Given $S'^b = \cancel{\frac{\partial x'^b}{\partial x^a}} \frac{\partial x'^b}{\partial x^\mu} S^\mu$

$$\partial'_a S'^b = \frac{\partial}{\partial x'^a}\left(\frac{\partial x'^b}{\partial x^\mu} S^\mu\right) = \frac{\partial x'^b}{\partial x^\mu}\frac{\partial S^\mu}{\partial x'^a} + S^\mu \frac{\partial^2 x'^b}{\partial x'^a \partial x^\mu}$$

$$= \frac{\partial x'^b}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^a}\frac{\partial S^\mu}{\partial x^\nu} + S^\mu \frac{\partial^2 x'^b}{\partial x'^a \partial x^\mu}$$

$$= \underline{\frac{\partial x'^b}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^a}\partial_\nu S^\mu} + S^\mu \frac{\partial^2 x'^b}{\partial x'^a \partial x^\mu}$$

① ②

term ② $= 0$ if $\frac{\partial x'^b}{\partial x^\mu}$ is ~~a~~ constant

For Lorentz ~~constant~~ transformation this is the case

$\therefore$ $\partial_a S^b$ is a (1,0) tenser

For general transformation ② may $\neq 0$

$\therefore$ ~~$\partial_a b$~~ $\partial_a S^b$ is not a tenser in general

Fantastic!

④ (1+2)

An completely anti-symmetric tensor ((0, m) tensor)

$A_{a_1 \cdots a_m}$ satisfies that any swap of indices generates a minus sign

i.e. $\cancel{A_{a_1 \cdots a_i a_j \cdots a_m} = -A_{a_1 \cdots a_j a}}$

$$-A_{a_1 \cdots a_i \cdots a_j \cdots a_m} = A_{a_1 \cdots a_j \cdots a_i \cdots A_m}$$

~~completely~~

Tensor in dimension n means that

$A_{a_1 \ldots a_m}$, $\forall\, a_i,\ i \in \{1, 2, \cdots, m\}$, $a_i \in \mathbb{Z}$ and $1 \leq a_i \leq n$

If ~~$m \leq n$~~ $m > n$:

at least two of $\{a_1, a_2, \cdots, a_m\}$ must take the same value, suppose $a_i = a_j$ for some $i, j \in \{1, 2, \cdots, m\}$

$$A_{a_1 \cdots a_i \cdots a_j \cdots a_m} \underset{\text{complete antisymmetry}}{=} -A_{a_1 \cdots a_j \cdots a_i \cdots a_m} \underset{a_i = a_j}{=} -A_{a_1 \cdots a_i \cdots a_j \cdots a_m}$$

$= 0$ // Good!

But If $m \leq n$, all a_i's can be distinct.

Hence ~~A (0, m)~~ (0, m) completely antisymmetric tensor A vanishes unless $m \leq n$

- Suppose $m \le n$, then ~~once~~ $\{a_1, a_2, \cdots, a_m\}$ is a subset of $\{1, 2, \cdots, n\}$

If we can independently choose the value of component $A_{a_1 \cdots a_m}$, ~~then~~ $\{a_1, \cdots, a_m\} \subseteq \{1, 2, \cdots, n\}$, then the values of $A_{\sigma(a_1)\sigma(a_2)\cdots\sigma(a_m)}$ ~~determined~~ are fixed ~~for~~ all permutations σ of set $\{a_1, \cdots, a_m\}$ because any swap of indices simply generates a minus sign.

- All components with at least 2 equal indices are 0 as proven before

$\therefore$ the problem reduces to how many subsets of m ~~ele~~ distinct elements $\{a_1, \cdots, a_m\}$ are there in $\{1, \cdots, n\}$

The answer is $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$

Great work!

In an inertial frame the metric

$g^{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \quad \therefore \det(g) = g = -1$

$\therefore \epsilon_{0123} = \sqrt{-(-1)} = 1$

Fantastic!

In 4D ϵ_{abcd} has $\binom{4}{4} = 1$ independent coordinate, and is set to $\sqrt{-g}$

$\therefore$ the tensor is completely determined.

In general coordinate transformations

$$x \to x'(x)$$

we look at the form of

$$\frac{\partial x^\mu}{\partial x'^a} \frac{\partial x^\nu}{\partial x'^b} \frac{\partial x^\rho}{\partial x'^c} \frac{\partial x^\sigma}{\partial x'^d} \epsilon_{\mu\nu\rho\sigma} = E_{abcd}$$

consider swapping a, b

$$E_{bacd} = \frac{\partial x^\mu}{\partial x'^b} \frac{\partial x^\nu}{\partial x'^a} \epsilon_{\mu\nu\rho\sigma} \frac{\partial x^\rho}{\partial x'^c} \frac{\partial x^\sigma}{\partial x'^d}$$

$$= \frac{\partial x^\nu}{\partial x'^b} \frac{\partial x^\mu}{\partial x'^a} \epsilon_{\nu\mu\rho\sigma} \frac{\partial x^\rho}{\partial x'^c} \frac{\partial x^\sigma}{\partial x'^d}$$

exchange dummies μ, ν

$$= \frac{\partial x^\mu}{\partial x'^a} \frac{\partial x^\nu}{\partial x'^b} \underbrace{\epsilon_{\nu\mu\rho\sigma}}_{= -\epsilon_{\mu\nu\rho\sigma}} \frac{\partial x^\rho}{\partial x'^c} \frac{\partial x^\sigma}{\partial x'^d}$$

$$= - E_{abcd}$$

Fantastic!

$\to$ complete anti-symmetry is perserved. ①

Now consider

$$E_{0123} = \frac{\partial x^\mu}{\partial x'^0} \frac{\partial x^\nu}{\partial x'^1} \frac{\partial x^\rho}{\partial x'^2} \frac{\partial x^\sigma}{\partial x'^3} \epsilon_{\mu\nu\rho\sigma}$$

~~this is precisely the definition of~~

we normalise ϵ to get $\tilde{\epsilon}_{\mu\nu\rho\sigma} = \dfrac{\epsilon_{\mu\nu\rho\sigma}}{\sqrt{-g}} = \begin{cases} 1 & 0123 \ldots \\ -1 & 1023 \ldots \\ 0 & 0023 \ldots \end{cases}$

$$\therefore \quad \frac{E_{0123}}{\sqrt{-g}} = \cancel{\epsilon_{0123}} \; \tilde{\epsilon}_{\mu\nu\rho\sigma} \frac{\partial x^\mu}{\partial x'^0} \frac{\partial x^\nu}{\partial x'^1} \frac{\partial x^\rho}{\partial x'^2} \frac{\partial x^\sigma}{\partial x'^3}$$

this is precisely the definition of the determinant of the 4×4 matrix $\frac{\partial x^\alpha}{\partial x'^\beta} = \left(\frac{\partial x}{\partial x'}\right)^\alpha{}_\beta = \frac{\partial x}{\partial x'}$

consider the transformation :

$$g'_{ab} = \frac{\partial x^\mu}{\partial x'^a} \frac{\partial x^\nu}{\partial x'^b} g_{\mu\nu}$$

Good!

$$= \frac{\partial x^\mu}{\partial x'^a} g_{\mu\nu} \frac{\partial x^\nu}{\partial x'^b} = \left(\frac{\partial x}{\partial x'}\right)^T g \frac{\partial x}{\partial x'}$$

$$g' = \det(g'_{ab}) = \underbrace{\det\left(\left(\frac{\partial x}{\partial x'}\right)^T\right)}_{= \det\left(\frac{\partial x}{\partial x'}\right)} \det(g) \det\left(\frac{\partial x}{\partial x'}\right)$$

$$= \det{}^2\left(\frac{\partial x}{\partial x'}\right) \det(g)$$

$$= g$$

$$\therefore \det{}^2\left(\frac{\partial x}{\partial x'}\right) = \frac{g'}{g} \Rightarrow \det\left(\frac{\partial x}{\partial x'}\right) = \sqrt{\frac{g'}{g}}$$

$$\because \frac{E_{0123}}{\sqrt{-g}} = \det\left(\frac{\partial x}{\partial x'}\right) = \sqrt{\frac{-g'}{-g}}$$

$$\Rightarrow E_{0123} = \sqrt{-g'} \quad ②$$

$\because$ ①, ② $\therefore$ $E_{0123} = \epsilon'_{0123}$, $\cancel{\epsilon_{abcd}}$ $E_{abcd} = \epsilon'_{abcd}$

$\therefore$ ϵ'_{abcd} and ϵ_{abcd} are related by tensor transformation

$$t'_{abcd} = \frac{\partial x^{\mu}}{\partial x'^{a}} \frac{\partial x^{\nu}}{\partial x'^{b}} \frac{\partial x^{\rho}}{\partial x'^{c}} \frac{\partial x^{\sigma}}{\partial x'^{d}} t_{\mu\nu\rho\sigma}$$

$\therefore$ t_{abcd} is a general $(0,4)$ tensor.

Fantastic work!

5 (1+2)

$$\partial_{[a}F_{bc]} \equiv \frac{1}{3!}\left(\partial_a F_{bc} - \partial_a F_{cb} + \partial_b F_{ac} - \partial_b F_{ca} + \partial_c F_{ab} - \partial_c F_{ba}\right) = 0$$

$$\Longleftrightarrow \partial_a F_{bc} - \partial_a F_{cb} + \partial_b F_{ac} - \partial_b F_{ca} + \partial_c F_{ab} - \partial_c F_{ba} = 0$$

$\because F_{ab} = -F_{ba}$

$$\Longleftrightarrow \cancel{2\partial_a F} \; 2\partial_a F_{bc} + 2\partial_b F_{ca} + 2\partial_c F_{ab} = 0$$

$$\Longleftrightarrow \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \quad \square$$

Great!

$\rightarrow$ $E_a = F_{ab}V^b$ $\quad \therefore \; E_a V^a = F_{ab}V^b V^a$

$$= F_{ba}V^a V^b$$

swap a, b $= \cancel{F_{ba}} \; F_{ba}V^b V^a$

$$= -F_{ab}V^b V^a = -E_a V^a = 0$$

$\therefore E_a V^a = 0$ ① Good!

E_a has 4 components but 1 constraint ①, so it has 3 independent components. $\square$

$\rightarrow$ Similarly $B_a = -\frac{1}{2}\epsilon_{abcd}F^{bc}V^d$

$$B_a V^a = -\frac{1}{2}\epsilon_{abcd}F^{bc}V^d V^a$$

swap a, d $\rightarrow$ $= -\frac{1}{2}\epsilon_{dbca}F^{bc}V^a V^d = -\frac{1}{2}\underbrace{\epsilon_{dbca}}_{-\epsilon_{abcd}}F^{bc}V^d V^a$ Fantastic!

$$= -\left[-\frac{1}{2}\epsilon_{abcd}F^{bc}V^d V^a\right] = -B_a V^a = 0$$

$\therefore$ $B_a V^a = 0$ $\therefore$ similarly B_a has only 3 independent components $\circ$

$\rightarrow$ ~~inco~~ rest observer in inertial frame $\circ$

$V^a = (1,0,0,0)$

$\because$ $E_a = F_{ab} V^b$ $\cancel{\therefore E_0 = F_{00}V^0 + F_{01}V^1 + F_{02}V^2 + F_{03}V^3}$

$\therefore E_a = F_{a0}V^0 + F_{a1}V^1 + F_{a2}V^2 + F_{a3}V^3$

$\because$ $V^1 = V^2 = V^3 = 0$ $\therefore$ $E_a = F_{a0}V^0$

$\therefore E_0 = F_{00}V^0$ $\because$ $F_{ab} = -F_{ba}$ $\therefore$ when $a=b$, F_{aa} (No sum) $= 0 \rightarrow F_{00} = 0$

$\therefore$ $E_0 = 0$ $\Rightarrow$ $E_a = (0, \vec{E})$ $\circ$ Great!

$B_a = -\frac{1}{2}\epsilon_{abcd}F^{bc}V^d$ similarly $\because V^1 = V^2 = V^3 = 0$

$\therefore$ $\cancel{B_a = -\frac{1}{2}\epsilon_{abcd}F^{bc}}$ $B_a = -\frac{1}{2}\epsilon_{abc0}F^{bc}V^0$

$\because$ ϵ_{abcd} is completely anti-symmetric

$\therefore$ $\epsilon_{0bc0} = 0$

$\therefore$ $B_0 = -\frac{1}{2}\epsilon_{0bc0}F^{bc}V^0 = 0$

$\therefore$ $B_a = (0, \vec{B})$ $\circ$ Good!

For ~~E~~ E_i $(i = 1,2,3)$:

$$E_i = F_{i0} \underbrace{V^0}_{=1} = F_{i0}$$

For B_i $(i = 1,2,3)$:

$$B_i = \cancel{F} - \frac{1}{2} \epsilon_{ibc0} F^{bc} V^0 = \cancel{-\frac{1}{2}\epsilon_{0i}} = -\frac{1}{2} \epsilon_{ijk0} F^{jk} \underbrace{V^0}_{=1}$$

the sign of Levi-Civita tensor depends on the number of swaps from ~~sorting~~ the increasing order.

Good!

$$\epsilon_{0123} = \epsilon_{123} = 1 \quad \text{for } g_{ab} = (-1,1,1,1)$$

$\therefore$ $\epsilon_{0ijk} = \epsilon_{ijk}$ for they have gone through same number of swaps from 0123 and 123 respectively.

$$\therefore \quad \epsilon_{ijk} = \epsilon_{0ijk} = -\epsilon_{i0jk} = \epsilon_{ij0k} = -\epsilon_{ijk0}$$

$$\therefore \quad B_i = -\frac{1}{2} \epsilon_{ijk0} F^{jk}$$

$$= \frac{1}{2} \epsilon_{ijk} F^{jk}$$

$\rightarrow$ $\partial_a F^{ab} = \cancel{4\pi J^b}$

~~$b=0$: $\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} =$~~

~~$=0$ $\quad \frac{\partial}{\partial x}$~~

~~consider F~~

$$\partial_a F^{ab} = g_{ac}\partial^c F^{ab} \cancel{= g_{ac} g^{\nu a}\partial^c F_a{}^b}$$

$$= \underbrace{g_{ac} g^{a\nu}}_{\delta^\nu_c}\partial^c F_\nu{}^b = \delta^\nu_c \partial^c F_\nu{}^b = \partial^\nu F_\nu{}^b$$

$$= \partial^a F_a{}^b$$

$\therefore$ $\partial^a F_a{}^b = -4\pi J^b$ lower b on both sides.

$$g_{bc}\partial^a F_a{}^c = g_{bc}(-4\pi J^c) \Rightarrow \underline{\partial^a F_{ab} = -4\pi J_b}$$

$$J_b = g_{bc}J^c = \cancel{\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \rho \\ \vec{J} \end{pmatrix}} = (\rho\ \ \vec{J})\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

$$= (-\rho, \vec{J})$$

$$\partial^a = g^{ab}\partial_b = \left(+\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \left(\frac{-1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$\therefore$ $\partial^a F_{ab} = -4\pi J_b$:

If $b=0$ $\quad \partial^0 F_{00} + \partial^1 F_{10} + \partial^2 F_{20} + \partial^3 F_{30} = -4\pi J_0$

$$\rightarrow \quad 0 + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -4\pi(-\rho)$$

$$\rightarrow \quad \underline{\nabla \cdot E = 4\pi\rho} \quad (M1)$$

Great!

If $b = i$ $(i=1,2,3)$

$$\partial^a F_{ai} = -4\pi J_i$$

$$\partial^0 F_{0i} + \partial^1 F_{1i} + \partial^2 F_{2i} + \partial^3 F_{3i} = -4\pi J_i$$

$$\partial^0 F_{0i} = -\frac{1}{c}\frac{\partial}{\partial t} F_{0i} = \frac{1}{c}\frac{\partial E_i}{\partial t}$$

$$\downarrow$$

$$= -F_{i0} = -E_i$$

~~$B_k = \frac{1}{2}\epsilon_{kji} F^{ji}$~~

~~For $k=1$~~ $\quad$ ~~$B_1 = \frac{1}{2}(\epsilon_{123} F^{23} + \epsilon_{132} F^{32})$~~

~~$= \frac{1}{2}(F^{23} - F^{32})$~~

$$B_i = \frac{1}{2}\epsilon_{ijk} F^{jk} \quad \therefore \quad B^i = \frac{1}{2}\epsilon^{ijk} F_{jk}$$

($\because$ in 3D subspace metric is $(1,1,1)$, up/down doesn't matter)

Good! Keep it up.

consider :

$$\epsilon_{lmi} B^i = \frac{1}{2}\epsilon_{lmi}\epsilon^{ijk} F_{jk}$$

(ϵ, δ identity) $\longrightarrow$ $= \frac{1}{2}(\delta_l^j \delta_m^k - \delta_l^k \delta_m^j) F_{jk} = \frac{1}{2}(F_{lm} - F_{ml})$

F ~~symmetric~~ antisymmetric $\longrightarrow$ $= -F_{ml}$ $\qquad \Rightarrow (F_{ij} = -\epsilon_{jik} B^k = \epsilon_{ijk} B^k$

$$\therefore \quad \epsilon_{lmi}\partial^m B^i = -\partial^m F_{ml} \qquad (m, l = 1,2,3)$$

$$\rightarrow -\epsilon_{ijk}\partial^j B^k = +\partial^j F_{ji} = \partial^1 F_{1i} + \partial^2 F_{2i} + \partial^3 F_{3i}$$

$$= -(\vec{\nabla} \times \vec{B})_i \quad ✓$$

$\therefore$ We have

$$\frac{1}{c}\frac{\partial E_i}{\partial t} - (\vec{\nabla}\times\vec{B})_i = -4\pi J_i$$

$$\therefore \quad \vec{\nabla}\times\vec{B} = 4\pi\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \quad (M4)$$

Fantastic!

$$\rightarrow \partial_{[a}F_{bc]} = 0 \iff \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0$$

From previous page ~~we~~ we know that

$$\cancel{F_{ab}\ \partial_a F_{bc}} \Rightarrow \partial_i F_{jk} = \partial_i[+\epsilon_{jkl}B^l]$$
$$= +\epsilon_{jkl}\partial_i B^l$$

For $a,b,c = i,j,k = \{1,2,3\}$:

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$$

$$\therefore +\epsilon_{jkl}\partial_i B^l + \epsilon_{kim}\partial_j B^m + \epsilon_{ijn}\partial_k B^n = 0$$

* contract with ϵ^{ijk}

$$\therefore \epsilon^{ijk}\epsilon_{jkl}\partial_i B^l + \epsilon^{ijk}\epsilon_{kim}\partial_j B^m + \epsilon^{ijk}\epsilon_{ijn}\partial_k B^n = 0$$

$\cancel{\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}}$

$$\rightarrow 2\delta^i_l\partial_i B^l + 2\delta^j_m\partial_j B^m + 2\delta^k_n\partial_k B^n = 0$$

$\epsilon^{ijk}\epsilon_{ijm} = \delta^k_m$

$$\therefore \partial_l B^l + \partial_m B^m + \partial_n B^n = 0$$

$$\rightarrow 3\partial_l B^l = 0 \rightarrow \partial_l B^l = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0 \quad (M2)$$

Great work!

For $a, b, c = 0, i, j$ $(i, j = \{1, 2, 3\})$

if $a = 0$, $b, c = i, j$

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0$$

$$F_{ij} = \epsilon_{ijk} B^k \quad \therefore \partial_0 F_{ij} = \epsilon_{ijk} \partial_0 B^k = \epsilon_{ijk} \left(\frac{1}{c} \frac{\partial B^k}{\partial t} \right)$$

$$\underset{\swarrow}{\partial_i F_{j0}} + \underset{\swarrow}{\partial_j F_{0i}} = \partial_i E_j - \partial_j E_i = \epsilon_{ijk} (\underline{\nabla} \times \underline{E})^k$$

$E_j \quad = -F_{i0} = -E_i$

$$\therefore \epsilon_{ijk} \left[\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} \right]_k = 0$$

Good!

$$\underbrace{\epsilon^{ijl} \epsilon_{ijk}}_{2\delta^l_k} [\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t}]^k = 0$$

$$\therefore (\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t})^l = 0$$

$(l = 1, 2, 3)$

$$\cancel{\therefore \underline{\nabla} \times \underline{E} \quad \frac{\partial \underline{B}}{\partial t}} \quad \cancel{\underline{\nabla} \times \underline{E}} \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (M3)$$

$$\because \partial_a F^{ab} = -4\pi J^b \quad \therefore \partial_b F^{ba} = -4\pi J^a$$

$$\therefore \partial_a \partial_b F^{ba} = -4\pi \partial_a J^a$$

$$\partial_a \partial_b F^{ba} \underbrace{=}_{\text{swap } a, b} \partial_b \partial_a F^{ab} \underbrace{=}_{\text{partial derivatives commute}} \partial_a \partial_b F^{ab} \underbrace{=}_{F_{ab} = -F_{ba}} -\partial_a \partial_b F^{ba}$$

$$= 0 \quad \Rightarrow \partial_a J^a = 0$$

Good!

$\partial_a J^a = 0 \rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

The continuity equation, represents charge conservation

✓

Energy - momentum tensor

$$T^{ab} = \frac{1}{4\pi}[F^{ac}F^b{}_c - \frac{1}{4}(F^{cd}F_{cd})\eta^{ab}]$$

$$\partial_a T^{ab} = \frac{1}{4\pi}\left[\partial_a(F^{ac}F^b{}_c) - \frac{1}{4}\eta^{ab}\partial_a(F^{cd}F_{cd})\right]$$

$$\cancel{= \frac{1}{4\pi}[\eta_{c\nu}\partial_a F^{ac} F^{b\nu}}$$

$$= \frac{1}{4\pi}\Big[(\partial_a F^{ac})F^b{}_c + F^{ac}\partial_a F^b{}_c$$

$$- \frac{1}{4}\eta^{ab}(\underbrace{(\partial_a F^{cd})F_{cd} + F^{cd}(\partial_a F_{cd})}_{\text{they are equal}})\Big]$$

$\because \partial_a F^{ac} = -4\pi J^c$ they are equal $= 2F^{cd}\partial_a F_{cd}$

$\therefore \frac{1}{4\pi}(\partial_a F^{ac})F^b{}_c = -J^c F^b{}_c = -F^b{}_c J^c$

consider the rest of the terms : $\frac{1}{4\pi}M^b$

$$M^b = F^{ac}\partial_a F^b{}_c - \frac{1}{2}\eta^{ab}F^{cd}\partial_a F_{cd}$$

$$= F^{ac}\eta^{b\beta}(\partial_a F_{\beta c}) - \frac{1}{2}\eta^{ab}F^{\beta c}(\partial_a F_{\beta c})$$

(by simple index relabelling)

$$\because \quad M^b = -F^{ac}\eta^{b\beta}(\partial_a F_{c\beta}) + \frac{1}{2}\eta^{ab}F^{\beta c}(\partial_\beta F_{ca} + \partial_c F_{a\beta})$$

$F_{\beta c} = -F_{c\beta}$

$\partial_a F_{\beta c} + \partial_\beta F_{ca} + \partial_c F_{a\beta} = 0$

$$= \frac{1}{2}\Big(-F^{ac}\eta^{b\beta}\underbrace{\partial_a F_{c\beta} + \eta^{ab}F^{\beta c}}_{①}\partial_\beta F_{ca}\Big)$$

$$+ \frac{1}{2}\Big(-F^{ac}\eta^{b\beta}\underbrace{\partial_a F_{c\beta} + \eta^{ab}F^{\beta c}}_{②}\partial_c F_{a\beta}\Big)$$

$$\because \quad \eta^{ab}F^{\beta c}\underbrace{\partial_\beta F_{ca} = F^{ac}}_{\text{swap } a,\ \beta}\eta^{b\beta}\partial_a F_{c\beta} \quad \therefore \Rightarrow ① = 0$$

$\eta^{ab} = \eta^{ba}$

$$\because \quad \eta^{ab}F^{\beta c}\partial_c F_{a\beta} = F^{\beta c}\partial_c \overset{\substack{\beta\to c \\ c\to a}}{F^b{}_\beta} = \underline{F^{ca}\partial_a F^b{}_c}$$

Good!

$\cancel{F^{ac}\eta^{b\beta}\partial_a F_{c\beta} = F^{ac}\partial_a F^b{}_c = F^{ac}}$

$\cancel{= F^{ac}}$

$$F^{ac}\eta^{b\beta}\partial_a F_{c\beta} = F^{ac}\partial_a F_c{}^b = (-F^{ca})\partial_a(-F^b{}_c)$$

$$= \underline{F^{ca}\partial_a F^b{}_c}$$

$\therefore \Rightarrow ② = 0$

$\because ① = ② = 0 \quad \therefore \underline{M^b = 0}$ Great!

$$\therefore \partial_a T^{ab} = -F^b{}_c J^c$$

- If ~~J^a~~ $J^c = 0$, then $\partial_a T^{ab} = 0$ conserved

If $J^c \neq 0$, the energy and momentum are not conserved because the presence of charge and current densities.

~~may~~ electromagnetic fields ~~does~~ do work ~~~~ and provides impulse on charged particles and ~~loose~~ lose energy ~~energy~~ and momentum (strictly speaking only $\vec{E}$ field does work)

Good!

Although, the energy and momentum are conserved when we consider

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_{MATTER} + \mathcal{L}_{coupling}$$

i.e., the energy/momentum lost by the EM field is transferred to the matter system.

6 (1+5) For $x^a(s)$ to be time-like,

$$g_{ab}\dot{x}^a\dot{x}^b < 0 \quad \forall s \in \quad s_1 \le s \le s_2$$

where $\dot{x}^a = \frac{dx^a}{ds}$ $\quad g_{ab} = \eta_{ab} = \mathrm{diag}(-1,1,1,1)$

good! in Minkowski spacetime.

proper time

$$\Delta\tau = \int_{s_1}^{s_2} (-\eta_{ab}\dot{x}^a\dot{x}^b)^{\frac{1}{2}}\, ds$$

variational approach :

$$\Delta\tau + \delta(\Delta\tau) = \int_{s_1}^{s_2} (-\eta_{ab}(\dot{x}^a + \delta\dot{x}^a)(\dot{x}^b + \delta\dot{x}^b))^{\frac{1}{2}}\, ds$$

$$= \int_{s_1}^{s_2} (-\eta_{ab}\dot{x}^a\dot{x}^b \underbrace{- \eta_{ab}\dot{x}^a\delta\dot{x}^b - \eta_{ab}\delta\dot{x}^a\dot{x}^b} \cancel{- \eta_{ab}\underbrace{\delta\dot{x}^a\delta\dot{x}^b}_{2^{nd}\text{ order too small}}})^{\frac{1}{2}}\, ds$$

$$\eta_{ab}\dot{x}^a\delta\dot{x}^b = \underbrace{\eta_{ba}}_{\text{swap } a,b}\dot{x}^b\delta\dot{x}^a = \underbrace{\eta_{ab}}_{\eta_{ab}=\eta_{ba}}\delta\dot{x}^a\dot{x}^b$$

Good!

$$= \int_{s_1}^{s_2} (-\eta_{ab}\dot{x}^a\dot{x}^b - 2\eta_{ab}\dot{x}^a\delta\dot{x}^b)^{\frac{1}{2}}\, ds$$

$$= \int_{s_1}^{s_2} ds\,(-\eta_{ab}\dot{x}^a\dot{x}^b)^{\frac{1}{2}} \left(1 - \frac{2\eta_{ab}\dot{x}^a\delta\dot{x}^b}{(-\eta_{ab}\dot{x}^a\dot{x}^b)}\right)^{\frac{1}{2}}$$

$$= \underbrace{\int_{s_1}^{s_2} ds\,(-\eta_{ab}\dot{x}^a\dot{x}^b)^{\frac{1}{2}}}_{\Delta\tau} \left[1 - \frac{\eta_{ab}\dot{x}^a\delta\dot{x}^b}{(\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b})^2}\right]$$

$$\rightarrow \delta(\Delta\tau) = -\int_{s_1}^{s_2} ds \frac{\eta_{ab}\dot{x}^a \delta\dot{x}^b}{\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b}} = -\int_{s_1}^{s_2} ds \frac{\eta_{ab}\dot{x}^a \frac{d\delta}{ds}}{\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b}}$$

$$= -\underbrace{\left[\frac{\eta_{ab}\dot{x}^a\delta x^b}{\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b}}\right]_{\delta x^b(s_1)}^{\delta x^b(s_2)}}_{=0} + \int_{s_1}^{s_2} ds \frac{d}{ds}\left(\frac{\eta_{ab}\dot{x}^a}{\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b}}\right)\delta x^b$$

$\because$ end points fixed

Good!

for any $\delta x^b(s)$, extremal proper time occurs at $\delta(\Delta\tau) = 0$

$$\therefore \frac{d}{ds}\left(\frac{\eta_{ab}\dot{x}^a}{\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b}}\right) = 0$$

$$\therefore \cancel{\eta_{ab}\dot{x}^a\dot{x}} \quad \frac{\dot{x}^a}{\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b}} = c^a \quad \swarrow \text{ constant vector}$$

Clear! $(\cancel{\eta_{ab}\dot{x}^a\dot{x}^b}$

such that $\cancel{\eta_{ab}\dot{x}\ \eta_{ab}\dot{c}^a c}$ $\eta_{ab}c^a c^b = -1$

the tangent of the ~~curve~~ curve $\dot{x}^a(s)$ is always pointing to the direction of constant vector c^a

$\rightarrow$ the path is a straight line. Good!

Good!

By $\boxed{\text{reverse triangle inequality}}$ in Minkowski spacetime, straight line has the longest "distance" between two events in M_4.

$\rightarrow$ this is ~~an~~ a maximum. Good!

we may reparametrize the curve using s'

$$\cancel{\dot{x}'^a(s') =} \quad \frac{dx^a(s')}{ds'} = \frac{\partial s}{\partial s'} \frac{dx^a(s)}{ds}$$

$$\frac{dx^a(s')}{ds'} = \dot{x}^a(s'), \qquad \frac{dx^a(s)}{ds} = \dot{x}^a(s)$$

so $$\sqrt{-\eta_{ab}\dot{x}^a(s')\dot{x}^b(s')} = \sqrt{-\eta_{ab}\frac{\partial s}{\partial s'}\dot{x}^a(s)\frac{\partial s}{\partial s'}\dot{x}^b(s)}$$

$$= \frac{\partial s}{\partial s'}\sqrt{-\eta_{ab}\dot{x}^a(s)\dot{x}^b(s)}$$

We can freely choose the function $s(s')$ so that we can freely choose $\frac{\partial s}{\partial s'}$ to make $\sqrt{-\eta_{ab}\dot{x}^a(s')\dot{x}^b(s')}$ constant. s' is affine parameter.

Consider the function $$S = -\frac{1}{2}\int_{s_1}^{s_2} ds\, \eta_{ab}\dot{x}^a\dot{x}^b$$

for an affine parameter s.

Good!

Variational principle

$$S + \delta S = -\frac{1}{2}\int_{s_1}^{s_2} ds\, \eta_{ab}(\dot{x}^a + \delta\dot{x}^a)(\dot{x}^b + \delta\dot{x}^b)$$

$$= -\frac{1}{2}\int_{s_1}^{s_2} ds\, \eta_{ab}\dot{x}^a\dot{x}^b - \int_{s_1}^{s_2} ds\, \eta_{ab}\dot{x}^a\delta\dot{x}^b$$

$$+ O(\delta x^2) \cancel{= S - \delta S}$$

$$= S - \int_{s_1}^{s_2} ds\, \eta_{ab} \dot{x}^a \delta\dot{x}^b = S + \delta S \,.$$

$$\rightarrow \quad 0 = \delta S = - \int_{s_1}^{s_2} ds\, \eta_{ab} \dot{x}^a \delta\dot{x}^b$$

$$= - \underbrace{\left[\eta_{ab} \dot{x}^a \delta x^b \right]_{\delta x^b(s_1)}^{\delta x^b(s_2)}}_{=0} + \int_{s_1}^{s_2} ds \frac{d}{ds} (\eta_{ab} \dot{x}^a) \delta x^b$$

For arbitrary δx^b, this is 0 Fantastic!

$$\therefore \frac{d}{ds} (\eta_{ab} \dot{x}^a) = 0 \quad \therefore \eta_{ab} \dot{x}^a = d^a = \text{const.}$$

$$\because \sqrt{-\eta_{ab} \dot{x}^a \dot{x}^b} = \text{const} \quad \text{for affine parameter } s$$

$$\therefore \frac{\eta_{ab} \dot{x}^a}{\sqrt{-\eta_{ab} \dot{x}^a \dot{x}^b}} = C^a = \text{const}.$$

$\Rightarrow$ equivalent to extremising proper time.

If s, s' are both affine parameters, then they are restricted by

Great!

$$\underline{s' = as + b} \quad \text{so that} \quad \frac{\partial s}{\partial s'} = \text{constant} = a.$$

$$S = -\int_{s_1}^{s_2} ds \, [\underbrace{\frac{m}{2}\eta_{ab}\dot{x}^a\dot{x}^b - qA_a\dot{x}^a}_{=-L}]$$

$= -L$ where L is the Lagrangian

general variational approach :

Euler - Lagrange equation : $\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^\nu}\right) = \frac{\partial L}{\partial x^\nu}$

$$\therefore \frac{d}{ds}\left(\frac{1}{2}m\left(\eta_{ab}\frac{\partial \dot{x}^a}{\partial \dot{x}^\nu}\dot{x}^b + \eta_{ab}\dot{x}^a\frac{\partial \dot{x}^b}{\partial \dot{x}^\nu}\right) - q\frac{\partial}{\partial \dot{x}^\nu}(A_a\dot{x}^a)\right) = \frac{\partial L}{\partial x^\nu}$$

$$\therefore \frac{d}{ds}\left(\frac{1}{2}m\eta_{ab}(\delta^a_\nu\dot{x}^b + \delta^b_\nu\dot{x}^a) - qA_a\delta^a_\nu\right) = q\dot{x}^a\frac{\partial A_a}{\partial x^\nu}$$

$$\therefore \quad m\ddot{x}_\nu - q\frac{dA_\nu}{ds} = q\dot{x}^a\partial_\nu A_a$$

$$\frac{dA_\nu}{ds} = \frac{\partial A_\nu}{\partial x^\lambda}\frac{dx^\lambda}{ds} = \dot{x}^\lambda\partial_\lambda A_\nu = \dot{x}^a\partial_a A_\nu$$

$$\therefore \quad m\ddot{x}_\nu - q\dot{x}^a\partial_a A_\nu = q\dot{x}^a\partial_\nu A_a$$

$$\times\ \eta^{b\nu} \Rightarrow \quad m\ddot{x}^b - q\dot{x}^a\partial_a A^b = q\dot{x}^a\partial^b A_a$$

$$\therefore \quad m\ddot{x}^b = q(\partial^b A_a - \partial_a A^b)\dot{x}^a$$

swap $a \leftrightarrow b \Rightarrow$ $\quad m\ddot{x}^a = q(\partial^a A_b - \partial_b A^a)\dot{x}^b$

$$\therefore \quad \ddot{x} = \frac{q}{m}F^a{}_b\dot{x}^b \quad \text{for} \quad F^a{}_b = \partial^a A_b - \partial_b A^a$$

Great!

$$\ddot{x}^a = \frac{q}{m} F^a{}_b \dot{x}^b$$, lower a gives

$$\ddot{x}_a = \frac{q}{m} F_{ab} \dot{x}^b$$

contract with $\dot{x}^a$ gives $\dot{x}^a \ddot{x}_a = \frac{q}{m} F_{ab} \dot{x}^a \dot{x}^b$

$$F_{ab}\dot{x}^a\dot{x}^b = \underbrace{F_{ba}\dot{x}^b\dot{x}^a}_{\text{swap } a,b} = \underbrace{F_{ba}\dot{x}^a\dot{x}^b = -F_{ab}\dot{x}^a\dot{x}^b}_{\substack{F \text{ antisymmetric} \\ \because F_{ab} = \partial_a A_b - \partial_b A_a}} = 0$$

✓

$\therefore \quad \dot{x}^a \ddot{x}_a = 0$

$\therefore \cancel{\frac{d}{ds}\left(-\eta_{ab}\dot{x}\dot{x}\right)}$

$\therefore \frac{d}{ds}(\dot{x}^a \dot{x}_a) = \dot{x}^a \ddot{x}_a + \ddot{x}^a \dot{x}_a = 2\underbrace{\dot{x}^a \ddot{x}_a}_{=0} = 0$

$\therefore \quad \dot{x}^a \dot{x}_a = \text{const}$

$\therefore \sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b} = \sqrt{-\dot{x}^a\dot{x}_a} = \text{const.}$

Great! Keep it up.

(7) (1+5)

4-momentum density ~~$J^a = -T^{ab}V_b$~~ $J^a = -T^{ab}V_b$

directional pressure

(5)

$$P_x = T_{ab}X^a X^b$$ Good!

= pressure in direction of X^a

(X_a is such that $X^a X_a = 1$
$U^a X_a = 0$).

perfect fluid :

$$T^{ab} = (\rho + P) U^a U^b + P\eta^{ab}$$

observer also moving with U^a

~~∴ energy density~~

∴ 4 momentum density

$$J^a = -T^{ab}U_b = -(\rho+P)U^a \underbrace{U^b U_b}_{(-1)} + P \underbrace{\eta^{ab}U_b}_{U^a}$$

$$= +(\rho+P)U^a - PU^a = \rho U^a$$

∴ ρ = mass density Clear!

pressure in X^a ~~$P_x = T_{ab}X^a X^b = P$~~

~~$= (\rho+P)U^a U^b$~~

Good!

$$P_x = \cancel{T_{ab}} T_{ab}X^a X^b = (\rho+P)\underbrace{U_a U_b X^a X^b}_{=0 \because U_a X^a = U_b X^b = 0} + P\eta_{ab}X^a X^b = P \underbrace{X_a X^a}_{=1} = P$$

normal

$\therefore P =$ pressure.

the tensor $h^a{}_b = \delta^a{}_b + U^a U_b$

1. $h^a{}_b U^b = \delta^a{}_b U^b + U^a \underbrace{U_b U^b}_{-1}$

$= U^a - U^a = 0$

2. $h^a{}_b h^b{}_c = (\delta^a{}_b + U^a U_b)(\delta^b{}_c + U^b U_c)$

$= \delta^a{}_b \delta^b{}_c + \delta^a{}_b U^b U_c + \delta^b{}_c U^a U_b + U^a \underbrace{U_b U^b}_{-1} U_c$

$= \delta^a{}_c + U^a U_c + \cancel{U^a U_c} - \cancel{U^a U_c}$

$= \delta^a{}_c + U^a U_c = h^a{}_c$

3. $h^a{}_a = \underbrace{\delta^a{}_a}_{4} + \underbrace{U^a U_a}_{-1} = 3$

$h^a{}_b$ is a ~~proj~~ projector onto hypersurface $\perp$ U^a

this is evident $\because$ we observe that

$h^a{}_b U^b = 0 \Rightarrow U^b$ projected to surface $\perp$ U^b is of course 0

$h^a{}_b h^b{}_c = h^a{}_c \Rightarrow$ Projecting once and twice are essentially the same operation.

$P^2 = P \Rightarrow P$ is projection

$h^a{}_a = 3 \Rightarrow$ this tensor is correctly normalised to the dimension of hypersurface

Eamer, projects onto 3-dim subspace

$$h_{ab} = \eta_{ac} h^c{}_b = \eta_{ac}(\delta^c{}_b + U^c U_b)$$
$$= \cancel{\eta_{ab} U} \quad \underline{\underline{\eta_{ab} + U_a U_b}}$$

Induced metric on orthogonal hypersurface

$$\partial_a T^{ab} = 0 \Rightarrow 0 = \cancel{\partial_a f} \, (\partial_a \rho + \partial_a P) U^a U^b + (\rho + P) U^a \partial_a U^b$$
$$+ (\rho + P) U^b \partial_a U^a + \eta^{ab} \partial_a P \Rightarrow ①^b$$

Now, project $①^b \perp U^b$:

$$h^c{}_b ①^b \Rightarrow \cancel{\partial_a f} \, (\partial_a \rho + \partial_a P) U^a \underbrace{h^c{}_b U^b}_{0}$$
$$+ (\rho + P) \underbrace{h^c{}_b U^b}_{0} \partial_a U^a$$
$$+ (\rho + P) h^c{}_b U^a \partial_a U^b + \eta^{ab} h^c{}_b \partial_a P = 0$$

$$\Rightarrow (\rho + P)(\delta^c_b + U^c U_b)(\partial_a U^b) U^a \cancel{+ \eta^{ac} \partial_a P = 0}$$
$$+ h^{ca} \partial_a P = 0$$

$$\therefore (\rho + P) \underbrace{U^a \partial_a U^c}_{= \frac{dU^c}{d\tau}} + (\rho + P) U^a U^c \underbrace{U_b \partial_a U^b}_{=0} ①$$
$$+ h^{ca} \partial_a P = 0$$

Good!

$$- \frac{dU^c}{d\tau} = \frac{dx^a}{d\tau} \frac{\partial U^c}{\partial x^a} = U^a \partial_a U^c \qquad \left(U^a = \frac{dx^a}{d\tau} \right)$$

$$- 0 = \cancel{\partial} \, \partial_a (\underbrace{U^b U_b}_{=-1}) = U^b \partial_a U_b + U_b \partial_a U^b = 2 U_b \partial_a U^b$$

Good!

$$\rightarrow U_b \partial_a U^b = 0$$

$$\therefore \; (\rho+P)\frac{\partial U^c}{\partial \tau} + h^{ca}\partial_a P = 0$$

$$\therefore \; (\rho+P)\frac{\partial U^a}{\partial \tau} + h^{ab}\partial_b P = 0 \qquad (2)$$

Fantastic!

Project $(1)^b \; // \; U^b$

$(1)^b - h^b{}_a (1)^a$:

$$0 = (\partial_a \rho + \partial_a P) U^a U^b + (\rho+P)\cancel{U^a}\partial_a U^b$$

$$\cancel{+(\rho+P)U^a\partial_b U^a} + (\rho+P) U^b \partial_a U^a + \eta^{ab}\partial_a P$$

$$-(\rho+P)\cancel{U^a}\partial_a U^b$$

$$-h^{ab}\partial_a P$$

Now $(\eta^{ab} - h^{ab})\partial_a P = (\cancel{\eta^{ab}} - \cancel{\eta^{ab}} - U^a U^b)\partial_a P$

$$= -U^a U^b \partial_a P$$

$$\therefore \; 0 = U^a U^b \partial_a \rho + U^a \cancel{U^b} \partial_a P + \cancel{(\rho+P)}$$

$$+\rho U^b \partial_a U^a + P U^b \partial_a U^a - \cancel{U^a U^b}\partial_a P$$

$$\Rightarrow \; 0 = \cancel{U^a U^b \partial_a P} \; U^b U^a \partial_a \rho + U^b \rho \partial_a U^a$$

$$+ U^b P \partial_a U^a$$

$$= U^b (U^a \partial_a \rho + \rho \partial_a U^a + P \partial_a U^a)$$

$\therefore \quad 0 = U^b (\partial_a (\rho U^a) + P \partial_a U^a)$

contract with U_b, use $U^b U_b = -1$ gives

$$\partial_a (\rho U^a) + P \partial_a U^a = 0 \quad (3)$$

Good!

In (2), if we set $P = 0 \Rightarrow \rho \frac{dU^a}{d\tau} = 0$

$\Rightarrow \frac{dU^a}{d\tau} = 0 \Rightarrow$ following geodesics.

Great!.

Non-relativistic approximations:

1 $U^a = (1, \vec{u}) \quad |\vec{u}| \ll 1 \Rightarrow$ speed of fluid flow much less than the speed of light. ($|\vec{u}| \ll c$)

so $\gamma \simeq 1 \quad U^a = (\gamma, \gamma\vec{u}) \simeq (1, \vec{u})$ ✓

2 $P \ll \rho \Rightarrow$ kinetic energy much less than rest mass energy ($P \ll \rho c^2$) $\frac{P}{c^2} \ll \rho$ ✓

3. $|\vec{u}| \partial_t P \ll |\vec{\nabla} P|$ $\quad \left(\frac{|\vec{u}|}{c^2} \partial_t P \ll |\vec{\nabla} P| \right)$

two parts: $\frac{|\vec{u}|}{c} \ll 1$ (~~proven~~ obvious)

And $\frac{1}{c} \frac{\partial}{\partial t} P \ll |\vec{\nabla} P| = \left| \frac{\partial P}{\partial \vec{x}} \right|$. this is because fluid non-relativistic $\therefore \quad c\,dt \gg |d\vec{x}|$

$\Rightarrow \partial_t \ll |\vec{\nabla}|$

(consistent).

Good!

Scalar equation :

$\partial^a = (-\partial_t, \vec{\nabla})$

$\partial_a = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}) = (\partial_t, \vec{\nabla})$

$U^a = (1, \vec{u})$

$\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$

$$\partial_a(\rho U^a) + P\partial_a U^a = 0$$

$$\therefore \partial_t(\rho \cdot 1) + \vec{\nabla}\cdot(\rho\vec{u})$$

$$+ P(\cancel{\partial_t 1} + \vec{\nabla}\cdot\vec{u}) = 0$$

$$\therefore \partial_t\rho + \vec{\nabla}\cdot(\rho\vec{u}) + P(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\because P \ll \rho \quad \therefore |P(\vec{\nabla}\cdot\vec{u})| \ll |\vec{\nabla}\cdot(\rho\vec{u})|$$

$$\therefore \partial_t\rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$ Great!

~~Vector equation~~ (also compare $|\partial_t\rho|$ and $\cancel{|\vec{\nabla}\cdot(\rho\vec{u})|}$ $|P(\vec{\nabla}\cdot\vec{u})|$

let's see that $\partial_t \ll \vec{\nabla}$, $P \ll \rho$, $|\vec{u}| \ll 1$

and $\partial_t \ll |\vec{\nabla}|$ by the same order as $|\vec{u}| \ll 1$

they both represent non-relativistic speed of fluid

$\therefore$ since $P \ll \rho$

$\therefore |\partial_t\rho| \gg |P(\vec{\nabla}\cdot\vec{u})|$)

Vector equation :

$$(\rho + P)\frac{dU^a}{d\tau} + h^{ab}\partial_b P = 0 \qquad h^{ab} = \eta^{ab} + U^a U^b$$

$$\therefore (\rho + P)U^b\partial_b U^a + \eta^{ab}\partial_b P + U^a U^b \partial_b P = 0$$

$$\because \rho \gg P \quad \therefore \rho + P \simeq \rho$$ Good!

$$\therefore \rho U^b\partial_b U^a + \partial^a P + U^a U^b \partial_b P = 0$$

$$U^b \partial_b = (\partial_t + \vec{u}\cdot\vec{\nabla})$$

- consider $a = 0$:

$\because U^0 = 1$

$$\therefore U^b \partial_b U^0 = 0$$

$$\partial^0 p + (1)\, U^b \partial_b p = 0$$

$$\therefore -\partial_t p + (\partial_t + \vec{u}\cdot\nabla) p = 0 \Rightarrow \vec{u}\cdot\vec{\nabla} p = 0 \quad \times$$

not true, this degree of approximation doesn't work for this one.

- consider $a = i$ $\quad i \in \{1, 2, 3\}$

$$\rho U^b \partial_b U^i + \partial^i p + U^i U^b \partial_b p = 0$$

$$\therefore \underbrace{\rho(\partial_t + \vec{u}\cdot\vec{\nabla})\vec{u}}_{①} + \underbrace{\vec{\nabla} p}_{②} + \underbrace{\cancel{\vec{u}(\partial_t + \vec{u}\cdot\vec{\nabla})p}}_{③} = 0$$

$\because |\vec{u}\,\partial_t p| \ll |\vec{\nabla} p|$, $\vec{u}(\vec{u}\cdot\vec{\nabla} p) \ll |\vec{u}|$

$\because |\vec{u}| \ll 1$

$\therefore ③ \ll ②$

$|\rho \partial_t \vec{u}| \gg |\vec{u}\,\partial_t p| \quad \because \rho \gg p \quad \therefore ① \gg ③$

$\because ③$ neglected

Good!

$$\therefore \rho(\partial_t + \vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla} p$$