# Conformal Field Theory

## Problem Set 1

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Th 11:00–12:30 C4 Wk 2.

$\boxed{1}$

- $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \frac{\lambda}{4!}(\phi_1^4 + \phi_2^4) + \frac{2\rho}{4!}\phi_1^2\phi_2^2$

$+2+2[\phi_i] - d = [S] = 0$ $\cancel{\Rightarrow [\phi] = \frac{2-d}{2}}$

$\Rightarrow [\phi_i] = \frac{d-2}{2}$ $\Rightarrow [\partial^m \phi_i^n] = m + n\frac{d-2}{2}$

$\therefore$ coefficient $[\alpha_{m,n}\partial^m \phi_i^n] = [\mathcal{L}] = d.$

$\Rightarrow [\alpha_{m,n}] = d - n\frac{d-2}{2} - m$

$\therefore$ $x' = xb$ (the rescaling)

$\therefore \int d^dx \, \frac{1}{2}(\partial_\mu \phi_i)^2 \to \int d^dx' \, b^{-d+2}(\partial_\mu' \phi_i)^2.$

$\therefore \int d^dx \, \frac{1}{2}(\partial_\mu \phi_i)^2 + \alpha_{m,n}\partial^m \phi^n \to \int d^dx' \frac{1}{2} b^{-d+2}(\partial_\mu' \phi_i)^2$
$+ b^{-d+m}\alpha_{m,n}\partial'^m \phi^n.$

redefine $\phi_i' = b^{\frac{-d+2}{2}}\phi_i$

so $S = \int d^dx' \frac{1}{2}(\partial_\mu' \phi_i')^2 + b^{\overbrace{\scriptstyle(-d+m+n\frac{d-2}{2})}}\alpha_{m,n}\partial'^m \phi'^n.$

$\equiv$ scaling dimension
$\equiv -\Delta$

$\therefore$ $\Delta = d - n\frac{d-2}{2} - m = [\alpha_{m,n}]$

For each operator :

$\Rightarrow$ $\phi_1^4$ : $m=0,\ n=4,\ \Delta = d - 4\frac{d-2}{2} = 4-d$

in $d=3$ $\quad \Delta = 1 > 0 \Rightarrow$ relevant operator

in $d=4$ $\quad \Delta = 0 \Rightarrow$ marginal operator

Similarly $\phi_2^4$ : $m=0$ $n=4$ $\Delta = 4-d$

in $\underline{d=3}$ $\underline{\Delta=1}$ $\Rightarrow$ relevant

in $\underline{d=4}$ $\underline{\Delta=0}$ $\Rightarrow$ marginal

and $\phi_1^2\phi_2^2$ : $m=0$, $n=4$ $\Delta = 4-d$

in $d=3$ $\Delta=1$ $\Rightarrow$ relevant.

in $d=4$ $\Delta=0$ $\Rightarrow$ marginal.

→ Dirac ~~Lag~~ Lagrangian : $\mathcal{L} = i\bar{\psi}\Gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$

the only operator is $\bar{\psi}\psi$.

$$S = \int d^dx \; i\bar{\psi}\Gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \underset{\substack{x'=xb \\ \not{\partial}' \equiv \Gamma^\mu\partial'_\mu}}{=} \int d^dx' b^{-d+1}(i\bar{\psi}\not{\partial}\psi) - m\bar{\psi}\psi b^{-d}$$

rescale $\psi' = b^{\frac{1-d}{2}}\psi$ , $\bar{\psi}' = b^{\frac{1-d}{2}}\bar{\psi}$

$$\Rightarrow \quad S = \int d^dx' \; i\bar{\psi}'\not{\partial}\psi' - m \underbrace{b^{-d} b^{\frac{d-1}{2}} b^{\frac{d-1}{2}}}_{\equiv b^{-\Delta}} \bar{\psi}'\psi'$$

$$\Rightarrow \quad \Delta = d - \frac{d-1}{2} - \frac{d-1}{2} = \underline{\underline{1}}$$

$\boxed{\bar{\psi}\psi \text{ is relevant operator}}$

→ Coleman-Weinberg model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^2 + m^2\phi^2 + \lambda\phi^4$$

$$= -\frac{1}{4} - \frac{1}{2}(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu) + \partial_\mu\phi\partial^\mu\phi$$
$$+ 2eA_\mu\phi\partial^\mu\phi + e^2A_\mu A^\mu\phi^2 + m^2\phi^2 + \lambda\phi^4$$

$\because$ propagator for $A_\mu$ is ~~$\partial A$~~ $\partial A \partial A$ and that for $\phi$ is in the form $\partial\phi\partial\phi$ $\therefore$ in terms of dimensions we can treat them as same fields, and we use the formula derived in the two-scalar Lagrangian case.

$$\Delta = d - n\frac{d-2}{2} - m \qquad (n = \Sigma \text{ powers on } A \text{ and } \phi)$$
$$= (d+n) - \frac{n}{2}d - m = (1-\frac{n}{2})d + n - m$$

operators : $(d=4)$

$A_\mu\phi\partial^\mu\phi \Rightarrow m=1 \quad n=3 \quad \Delta = \cancel{d-\frac{3}{2}d \ldots}$
$$= (1-\frac{3}{2})d + 3 - 1$$
$$= 2 - \frac{d}{2} = \underline{\underline{0}}$$

$\Rightarrow$ marginal

$A_\mu A^\mu \phi^2 \Rightarrow m=0 \quad n=4$
$$\Delta = (1-\frac{4}{2})d + 4 - 0 = 4 - d$$
$$= \underline{\underline{0}}$$

$\Rightarrow$ marginal

$\phi^2 \Rightarrow m=0 \quad n=2 \quad \Delta = (1-\frac{2}{2})d + 2 - 0$
$$= \underline{\underline{2}}$$

$\Rightarrow$ relevant

$\phi^4 \Rightarrow m=0 \quad n=4 \quad \Delta = \underline{\underline{0}}$

$\Rightarrow$ marginal

$\boxed{2}$ $\beta(g) = -\frac{1}{16\pi^2}\frac{1}{3}(11N_c - 2N_f)g^3 - \frac{1}{(16\pi^2)^2}\left(\frac{34}{3}N_c^2 - \frac{13}{3}N_fN_c\right)g^5$

$+ \cdots$

under perturbation theory.

$\rightarrow$ $\underline{g=0 \text{ is always a fixed point}}$ for all possible $N_c$, $N_f$.

if $g \neq 0$, fixed point $\beta(g) = 0$

$$\Rightarrow 0 = -\frac{1}{16\pi^2}\cdot\frac{1}{3}(11N_c - 2N_f) - \frac{1}{(16\pi^2)^2}\left(\frac{34}{3}N_c^2 - \frac{13}{3}N_fN_c\right)g^2$$

$$\therefore g^2 = \frac{\frac{1}{\cancel{16\pi^2}}\frac{1}{3}(11N_c - 2N_f)}{-\frac{1}{\cancel{16\pi^2}}\cdot\frac{1}{16\pi^2}\left(\frac{34}{3}N_c^2 - \frac{13}{3}N_fN_c\right)}$$

$$\cancel{= \frac{16\pi^2}{3}\,11N_c - 2N_f}$$

$$= -16\pi^2\frac{11N_c - 2N_f}{34N_c^2 - 13N_fN_c}$$

$$= 16\pi^2\frac{11N_c - 2N_f}{13N_f - 34N_c}\cdot\frac{1}{N_c}$$

$\because N_c > 0$, $N_f > 0$, $g^2 > 0$

~~$\therefore 11N_c = 2N_f$~~

∴ Need $\begin{cases} 11N_c - 2N_f > 0 \\ 13N_f - 34N_c > 0 \end{cases}$ $\Rightarrow$ $\frac{2}{11}N_f < N_c < \frac{13}{34}N_f.$

or $\begin{cases} 11N_c - 2N_f < 0 \\ 13N_f - 34N_c < 0 \end{cases}$

$\Rightarrow$ ~~$\frac{13}{34}N_f < N_c < \frac{2}{11}N_f$~~

impossible.

$\Rightarrow$ for $\frac{2}{11} < \frac{N_c}{N_f} < \frac{13}{34}$, we have

two more fixed points

$$g = 4\pi\sqrt{\frac{11N_c - 2N_f}{13N_fN_c - 34N_c^2}}$$

$\boxed{3}$ Translation : $x^\mu \to x'^\mu = x^\mu + a^\mu$

inversion : $x^\mu \to x'^\mu = \dfrac{x^\mu}{x^2}$

inversion → translation → inversion is then

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + a^\mu$$

$$\to \frac{\frac{x^\mu}{x^2} + a^\mu}{\left(\frac{x^\mu}{x^2} + a^\mu\right)\left(\frac{x_\mu}{x^2} + a_\mu\right)}$$

$$= \frac{\frac{x^\mu}{x^2} + a^\mu}{\frac{x^2}{(x^2)^2} + 2a_\mu x^\mu \frac{1}{x^2} + a^\mu a_\mu}$$

$$= \frac{x^\mu + a^\mu x^2}{1 + 2a_\mu x^\mu + a^\mu a_\mu x^2} = \frac{x^\mu + (-a^\mu)x^2}{1 - 2(-a)\cdot x + (-a)^2 x^2}$$

is a ~~spec~~ special conformal transformation

with $b^\mu = -a^\mu$

$\boxed{4}$

$$P_\mu = -i\partial_\mu \qquad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$$

$$D = -i x^\mu\partial_\mu \qquad K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$$

$$\rightarrow [D, P_\mu]f = -[x^\nu\partial_\nu, \partial_\mu]f = \partial_\mu(x^\nu\partial_\nu f) - x^\nu\partial_\nu\partial_\mu f.$$

$$= \underbrace{(\partial_\mu x^\nu)}_{\delta_\mu^\nu}\partial_\nu f + \cancel{x^\nu\partial_\mu\partial_\nu f} - \cancel{x^\nu\partial_\mu\partial_\nu f}$$

$$= \partial_\mu f = i(-i\partial_\mu)f = iP_\mu f$$

$$\rightarrow [D, K_\mu]f = -[x^\rho\partial_\rho, 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu]f$$

$$= (2x_\mu x^\nu\partial_\nu - x^2\partial_\nu)(x^\rho\partial_\rho f) - (x^\rho\partial_\rho)(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)f$$

$$= 2x_\mu x^\nu\partial_\nu(x^\rho\partial_\rho f) - x^2\partial_\mu(x^\rho\partial_\rho f) - x^\rho\partial_\rho(2x_\mu x^\nu\partial_\nu f) + x^\rho\partial_\rho(x^2\partial_\mu f)$$

$$= 2x_\mu x^\nu\cancel{(\partial_\nu x^\rho)}(\partial_\rho f) + 2x_\mu x^\nu x^\rho\partial_\nu\partial_\rho f - x^2 x^\rho\partial_\mu\partial_\rho f - x^2(\partial_\mu x^\rho)(\partial_\rho f) - 2x^\rho(\partial_\rho x_\mu)x^\nu\partial_\nu f - 2x^\rho x_\mu\cancel{(\partial_\rho x^\nu)}(\partial_\nu f) - 2x^\rho x_\mu x^\nu\partial_\rho\partial_\nu f + \cancel{2x^\rho\partial x^\nu\partial x_\nu} + 2x^\rho x^\nu(\partial_\rho x_\nu)(\partial_\mu f) + x^2 x^\rho\partial_\rho\partial_\mu f$$

$$= -2x^\rho\eta_{\rho\mu}x^\nu\partial_\nu f - x^2\eta_\mu^\rho\partial_\rho f + 2x^\rho x^\nu\eta_{\rho\nu}\partial_\mu f$$

$$= -2x_\mu x^\nu\partial_\nu f - x^2\partial_\mu f + 2x^2\partial_\mu f.$$

$$= (-i)[-i(2x_\mu x^\nu\partial_\nu - x^2\partial_\nu)]f = -iK_\mu f$$

$\rightarrow \ [L_{\mu\nu}, P_\rho]f = [x_\mu\partial_\nu - x_\nu\partial_\mu \ , \ \partial_\rho]f$

$$= (x_\mu\partial_\nu - x_\nu\partial_\mu)\partial_\rho f - \partial_\rho(x_\mu\partial_\nu f - x_\nu\partial_\mu f)$$

$$= x_\mu\partial_\nu\cancel{\partial_\rho} f - x_\nu\cancel{\partial_\mu}\partial_\rho f - x_\mu\cancel{\partial_\rho}\partial_\nu f - \eta_{\rho\mu}\partial_\nu f$$
$$+ x_\nu\cancel{\partial_\rho}\partial_\mu f + \eta_{\rho\nu}\partial_\mu f$$

$$= -i[\eta_{\rho\mu}(-i\partial_\nu) - \eta_{\nu\rho}(-i\partial_\mu)]f = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)f$$

$\rightarrow \ [L_{\mu\nu}, K_\rho]f = [x_\mu\partial_\nu - x_\nu\partial_\mu \ , \ 2x_\rho x^\sigma\partial_\sigma - x^2\partial_\rho]f$

$$= 2x_\mu\partial_\nu(x_\rho x^\sigma\partial_\sigma f) - x_\mu\partial_\nu(x^2\partial_\rho f)$$
$$-2x_\nu\partial_\mu(x_\rho x^\sigma\partial_\sigma f) + x_\nu\partial_\mu(x^2\partial_\rho f)$$
$$-2x_\rho x^\sigma\partial_\sigma(x_\mu\partial_\nu f) + x^2\partial_\rho(x_\mu\partial_\nu f)$$
$$+2x_\rho x^\sigma\partial_\sigma(x_\nu\partial_\mu f) - x^2\partial_\rho(x_\nu\partial_\mu f)$$

$$= 2x_\mu\eta_{\nu\rho}x^\sigma\partial_\sigma f + 2x_\mu x_\rho\eta^\sigma_\nu\partial_\sigma f + 2x_\mu x_\rho x^\sigma\partial_\nu\partial_\sigma f$$
$$-2x_\mu\cancel{x_\nu}\partial_\rho f - x_\mu x^2\cancel{\partial_\nu}\partial_\rho f$$
$$-2x_\nu\eta_{\mu\rho}x^\sigma\partial_\sigma f - 2x_\nu x_\rho\eta^\sigma_\mu\partial_\sigma f - 2x_\nu x_\rho x^\sigma\partial_\mu\partial_\sigma f$$
$$+2x_\nu\cancel{x_\mu}\partial_\rho f + x_\nu\cancel{x^2}\partial_\mu\partial_\rho f$$
$$-2x_\rho x^\sigma\cancel{x_\mu}\partial_\sigma\partial_\nu f - 2x_\rho x^\sigma\eta_{\sigma\mu}\partial_\nu f$$
$$+x^2x_\mu\cancel{\partial_\rho}\partial_\nu f + x^2\eta_{\rho\mu}\partial_\nu f$$
$$+2x_\rho x^\sigma\eta_{\sigma\nu}\partial_\mu f + 2x_\rho x^\sigma x_\nu\partial_\sigma\partial_\mu f$$
$$-x_\nu\cancel{x^2}\partial_\rho\partial_\mu f - x^2\eta_{\rho\nu}\partial_\mu f$$

$$= \cancel{2x_\mu}\ \eta_{\nu\rho}(\underbrace{2x_\mu x^\sigma\partial_\sigma f - x^2\partial_\mu f}_{iK_\mu})$$
$$- \eta_{\mu\rho}(\underbrace{2x_\nu x^\sigma\partial_\sigma f - x^2\partial_\nu f}_{iK_\nu})$$

$$= -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu)$$

$\rightarrow$ $$[D, L_{\mu\nu}]f = [x^\rho\partial_\rho, x_\mu\partial_\nu - x_\nu\partial_\mu]f$$
$$= x^\rho\partial_\rho(x_\mu\partial_\nu f) - x^\rho\partial_\rho(x_\nu\partial_\mu f)$$
$$- x_\mu\partial_\nu(x^\rho\partial_\rho f) + x_\nu\partial_\mu(x^\rho\partial_\rho f)$$
$$= \cancel{x^\rho x_\mu\partial_\rho\partial_\nu f} + \cancel{x_\mu\partial_\nu f}$$
$$- \cancel{x^\rho x_\nu\partial_\rho\partial_\mu f} - \cancel{x_\nu\partial_\mu f}$$
$$- \cancel{x_\mu x^\rho\partial_\nu\partial_\rho f} - \cancel{x_\mu\partial_\nu f}$$
$$+ \cancel{x_\nu x^\rho\partial_\mu\partial_\rho f} + \cancel{x_\nu\partial_\mu f} = 0$$

$\rightarrow$ $$[P_\mu, P_\nu]f = -[\partial_\mu, \partial_\nu]f = 0$$

$\rightarrow$ $$[K_\mu, K_\nu]f = -[2x_\mu x^\rho\partial_\rho - x^2\partial_\mu,\ 2x_\nu x^\sigma\partial_\sigma - x^2\partial_\nu]f$$
$$= -4x_\mu x^\rho\partial_\rho(x_\nu x^\sigma\partial_\sigma f) + 2x_\mu x^\rho\partial_\rho(x^2\partial_\nu f)$$
$$+ 2x^2\partial_\mu(x_\nu x^\sigma\partial_\sigma f) - x^2\partial_\mu(x^2\partial_\nu f)$$
$$+ 4x_\nu x^\sigma\partial_\sigma(x_\mu x^\rho\partial_\rho f) - 2x^2\partial_\nu(x_\mu x^\rho\partial_\rho f)$$
$$- 2x_\nu x^\sigma\partial_\sigma(x^2\partial_\mu f) + x^2\partial_\nu(x^2\partial_\mu f)$$

$$= -4x_\mu x^\rho x_\nu \cancel{x^\sigma} \partial_\rho \partial_\sigma f - 4x_{\cancel{\mu}} \cancel{x_\nu} x^\sigma \partial_\sigma f$$
$$- 4x_{\cancel{\mu}} \cancel{x_\nu} x^\sigma \partial_\sigma f$$
$$+ 4x_\mu \cancel{x^2} \partial_\nu f \quad + 2x_\mu \boxed{x^2 x^\rho} \partial_\rho \partial_\nu f$$
$$+ 2x^2 \cancel{\eta_{\mu\nu}} \cancel{x^\sigma} \partial_\sigma f \quad + 2x^2 \cancel{x_\nu} \partial_\mu f + 2x^2 x_\nu x^\sigma \partial_\sigma \partial_\mu f$$
$$- 2x^2 \cancel{x_\mu} \partial_\nu f \quad - x^4 \partial_\mu \partial_\nu f$$
$$+ 4x_\nu x^\sigma x_\mu \cancel{x^\rho} \partial_\sigma \partial_\rho f \quad + 4\cancel{x_\nu} \cancel{x_\mu} x^\rho \partial_\rho f$$
$$+ 4x_{\cancel{\nu}} \cancel{x_\mu} x^\rho \partial_\rho f$$
$$- 2x^2 \cancel{\eta_{\mu\nu}} \cancel{x^\rho} \partial_\rho f \quad - 2x^2 \cancel{x_\mu} \partial_\nu f \quad - 2x^2 x_\mu \boxed{x^\rho} \partial_\rho \partial_\nu f$$
$$- 4x_\nu \cancel{x^2} \partial_\mu f \quad - 2x_\nu x^2 x^\sigma \partial_\sigma \partial_\mu f$$
$$+ 2x^2 \cancel{x_\nu} \partial_\mu f \quad + x^4 \partial_\nu \partial_\mu f \quad = \underline{\underline{0}}$$

$$[D, D]f = -[x^\mu \partial_\mu, x^\nu \partial_\nu]f$$
$$= -x^\mu \partial_\mu (x^\nu \partial_\nu f) + x^\nu \partial_\nu (x^\mu \partial_\mu f)$$
$$= \underline{\underline{0}}$$

## [5] Quadratic Casimir

$$C^{(2)} = L_{\mu\nu}L^{\mu\nu} + \alpha P_\mu K^\mu + \beta K_\mu P^\mu + \gamma D^2$$

for any ~~arbitrary~~ operator $F$ in the conformal algebra, we require $[C^{(2)}, F] = 0$

$$\Rightarrow 0 = [L_{\mu\nu}L^{\mu\nu}, F] + \alpha[P_\mu K^\mu, F] + \beta[K_\mu P^\mu, F] + \gamma[D^2, F]$$

$$= [L_{\mu\nu}, F]L^{\mu\nu} + L_{\mu\nu}[L^{\mu\nu}, F]$$

$$+ \alpha[P_\mu, F]K^\mu + \alpha P_\mu[K^\mu, F] + \beta K_\mu[P^\mu, F] + \beta[K_\mu, F]P^\mu$$

$$+ \gamma D[D, F] + \gamma[D, F]D$$

$\Rightarrow$ $[C^{(2)}, D] = 0$ gives

$$0 = [\underbrace{L_{\mu\nu}, D}_{0}]L^{\mu\nu} + L_{\mu\nu}[\underbrace{L^{\mu\nu}, D}_{0}] + \alpha[P_\mu, D]K^\mu$$

$$+ \alpha P_\mu[K^\mu, D] + \beta K_\mu[P^\mu, D] + \beta[K_\mu, D]P^\mu$$

$$+ \gamma D[\underbrace{D, D}_{0}] + \gamma[\underbrace{D, D}_{0}]D$$

$$= \cancel{\alpha(-iP_\mu)K^\mu} + \cancel{\alpha P_\mu(iK_\mu)} + \cancel{\beta K_\mu(-iP^\mu)} + \cancel{\beta(iK_\mu)P^\mu}$$

$= 0$ always true.

$\Rightarrow$ doesn't fix anything.

$$\Rightarrow \quad 0 = [C^{(2)}, K_\rho]$$

$$= [L_{\mu\nu}, K_\rho] L^{\mu\nu} + L_{\mu\nu} [L^{\mu\nu}, K_\rho] + \alpha [P_\mu, K_\rho] K^\mu$$

$$+ \alpha P_\mu \cancel{[K^\mu, K_\rho]}_{=0} + \beta K_\mu [P^\mu, K_\rho] + \beta \cancel{[K_\mu, K_\rho]}_{=0} P^\mu$$

$$+ \gamma D [D, K_\rho] + \gamma [D, K_\rho] D$$

$$= -i (\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) L^{\mu\nu} - i L_{\mu\nu} (\eta^\mu{}_\rho K^\nu - \eta^\nu{}_\rho K^\mu)$$

$$+ \alpha (-2i) (\eta_{\mu\rho} D - L_{\rho\mu}) K^\mu$$

$$+ \beta K_\mu (-2i) (\eta^\mu{}_\rho D - L_\rho{}^\mu)$$

$$+ \gamma D (-i) (K_\rho) + \gamma (-i K_\rho D)$$

$$= \underline{-i K_\nu L_\rho{}^\nu} + \underline{i K_\mu L^\mu{}_\rho} - i L_{\rho\nu} K^\nu + i L_{\mu\rho} K^\mu$$

$$- 2\alpha i D K_\rho + 2\alpha i L_{\rho\mu} K^\mu$$

$$- 2\beta i K_\rho D + \underline{2\beta i K_\mu L_\rho{}^\mu}$$

$$- \gamma i D K_\rho - \gamma i K_\rho D$$

$$=$$

$\boxed{\text{use } L_{\mu\rho} = -L_{\rho\mu}}$ $$(-i)(2-2\beta) K^\mu L_{\rho\mu} + (-i)(2-2\alpha) L_{\rho\mu} K^\mu$$

$$+ (-i)(2\alpha+\gamma) D K_\rho + (-i)(2\beta+\gamma) K_\rho D$$

$$\Rightarrow \quad 0 = [C^{(2)}, P_\rho]$$

$$= [L_{\mu\nu}, P_\rho] L^{\mu\nu} + L_{\mu\nu}[L^{\mu\nu}, P_\rho] + \alpha \cancel{[P_\mu, P_\rho]} K^\mu$$

$$+ \alpha P_\mu [K^\mu, P_\rho] + \beta K_\mu \cancel{[P^\mu, P_\rho]} + \beta [K_\mu, P_\rho] P^\mu$$

$$+ \gamma D[D, P_\rho] + \gamma [D, P_\rho] D$$

$$= -i P_\nu L_\rho{}^\nu + i P_\mu L^\mu{}_\rho - i L_{\rho\nu} P^\nu + i L_{\mu\rho} P^\mu$$

$$+ \alpha (2i) P_\mu (\eta^\mu{}_\rho D - L^\mu{}_\rho) + \beta (2i)(\eta_{\mu\rho} D - L_{\mu\rho}) P^\mu$$

$$+ i\gamma D P_\rho + i\gamma P_\rho D$$

$$= (-i)(2 - 2\alpha) P^\mu L_{\rho\mu} + (-i)(2 - 2\beta) L_{\rho\mu} P^\mu$$

$$+ i(2\alpha + \gamma) P_\rho D + i(2\beta + \gamma) D P_\rho .$$

$$\Rightarrow \quad 0 = [C^{(2)}, L_{\rho\sigma}]$$

$$= [L_{\mu\nu}, L_{\rho\sigma}] L^{\mu\nu} + L_{\mu\nu} [L^{\mu\nu}, L_{\rho\sigma}]$$

$$+ \alpha [P_\mu, L_{\rho\sigma}] K^\mu + \alpha P_\mu [K^\mu, L_{\rho\sigma}]$$

$$+ \beta K_\mu [P^\mu, L_{\rho\sigma}] + \beta [K_\mu, L_{\rho\sigma}] P^\mu$$

$$+ \gamma D \cancel{[D, L_{\rho\sigma}]} + \gamma \cancel{[D, L_{\rho\sigma}]} D$$

$$= -i (L_{\mu\rho} \eta_{\nu\sigma} - L_{\mu\sigma} \eta_{\nu\rho} - L_{\nu\rho} \eta_{\mu\sigma} + L_{\nu\sigma} \eta_{\mu\rho}) L^{\mu\nu}$$

$$- i L^{\mu\nu} (L_{\mu\rho} \eta_{\nu\sigma} - L_{\mu\sigma} \eta_{\nu\rho} - L_{\nu\rho} \eta_{\mu\sigma} + L_{\nu\sigma} \eta_{\mu\rho})$$

$$+ \alpha i (\eta_{\rho\mu} P_\sigma - \eta_{\sigma\mu} P_\rho) K^\mu + \alpha i P_\mu (\eta_\rho{}^\mu K_\sigma - \eta_\sigma{}^\mu K_\rho)$$

$$+ \beta i K_\mu (\eta_\rho{}^\mu P_\sigma - \eta_\sigma{}^\mu P_\rho) + \beta i (\eta_{\rho\mu} K_\sigma - \eta_{\sigma\mu} K_\rho) P^\mu$$

$$= -i[ L_{\mu\rho} L^{\mu}{}_{\sigma} - L_{\mu\sigma} L^{\mu}{}_{\rho} - L_{\nu\rho} L_{\sigma}{}^{\nu} + L_{\nu\sigma} L_{\rho}{}^{\nu}$$

$$+ L^{\mu}{}_{\sigma} L_{\mu\rho} - L^{\mu}{}_{\rho} L_{\mu\sigma} - L_{\sigma}{}^{\nu} L_{\nu\rho} + L_{\rho}{}^{\nu} L_{\nu\sigma} )$$

$$= \underline{\underline{0}} \quad \text{always}$$

So overall ~~if~~ $[C^{(2)}, D] = [C^{(2)}, L_{\rho\sigma}] = 0$

always,.. and $[C^{(2)}, P_\mu] = [C^{(2)}, K_\mu] = 0$

if $2 - 2\beta = 0$ , $2 - 2\alpha = 0$ , $2\alpha + \gamma = 0$ , $2\beta + \gamma = 0$.

$\Rightarrow$ this can be satisfied if

$$\underline{\underline{\alpha = \beta = 1 \ , \quad \gamma = -2}}$$

So $\underline{\underline{C^{(2)} = L_{\mu\nu} L^{\mu\nu} + P_\mu K^\mu + K_\mu P^\mu - 2D^2}}$

$\boxed{6}$. $$[K_\mu, \phi_\alpha(x)] = K_\mu e^{-iPx} \phi_\alpha(0) e^{iPx} - e^{-iPx} \phi_\alpha(0) e^{iPx} K_\mu$$

$$= e^{-iPx} \left( e^{iPx} K_\mu e^{-iPx} \phi_\alpha(0) - \phi_\alpha(0) e^{iPx} K_\mu e^{-iPx} \right) e^{iPx}$$

$$= e^{-iPx} [\underbrace{e^{iPx} K_\mu e^{-iPx}}_{\equiv \hat{K}_\mu}, \phi_\alpha(0)] e^{iPx}$$

$$= e^{-iPx} [\hat{K}_\mu, \phi_\alpha(0)] e^{iPx}$$

where $\hat{K}_\mu = e^{iPx} K_\mu e^{-iPx}$

$$= \left(1 + ixP - \frac{(xP)^2}{2} + \cdots\right) K_\mu \left(1 - ixP - \frac{(xP)^2}{2} + \cdots\right)$$

use $e^{-A} B e^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \cdots$

in this case $A = -iP_\rho x^\rho$, $B = K_\mu$

$$\therefore \hat{K}_\mu = K_\mu + [K_\mu, P_\nu](-ix^\nu) + \frac{1}{2!}[[K_\mu, P_\nu], P_\rho](-ix^\nu)(-ix^\rho) + 0 + \underbrace{\cdots}_{0}$$

$$= K_\mu + (-ix^\nu)(2i)(\eta_{\mu\nu} D - L_{\mu\nu}) + \frac{1}{2}(-1) x^\nu x^\rho (2i)[\eta_{\mu\nu} D - L_{\mu\nu}, P_\rho]$$

$$= K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu}$$

$$\cancel{+} - i x^\nu x^\rho \left( \eta_{\mu\nu} \underbrace{[D, P_\rho]}_{iP_\rho} - \underbrace{[L_{\mu\nu}, P_\rho]}_{-i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)} \right)$$

$$= K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu}$$

$$+ x_\mu x^\rho P_\rho + x^\nu x_\mu P_\nu - x_\rho x^\rho P_\mu$$

$$\cancel{= K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu}}$$

$$\to [K_\mu, \phi_\alpha(x)] = e^{-iPx} [\hat{K}_\mu, \phi_\alpha(0)] e^{iPx}$$

$$[\hat{K}_\mu, \phi_\alpha(0)] = [K_\mu, \phi_\alpha(0)] + 2x_\mu [D, \phi_\alpha(0)] - 2x^\nu [L_{\mu\nu}, \phi_\alpha(0)]$$

$$+ x_\mu x^\rho [P_\rho, \phi_\alpha(0)] + x^\nu x_\mu [P_\nu, \phi_\alpha(0)] - x^2 [P_\mu, \phi_\alpha(0)]$$

$$= 0 + 2i x_\mu \Delta \phi_\alpha(0) - 2i x^\nu (S_{\mu\nu})_\alpha^{\ \beta} \phi_\beta(0) \quad (S_{\mu\nu} = -S_{\nu\mu})$$

$$+ i x_\mu x^\rho \partial_\rho \phi_\alpha(0) + i x_\mu x^\nu \partial_\nu \phi_\alpha(0) - i x^2 \partial_\mu \phi_\alpha(0)$$

$$= 2i x_\mu \Delta \phi_\alpha(0) + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \phi_\alpha(0)$$

$$+ 2i x^\nu (S_{\nu\mu})_\alpha^{\ \beta} \phi_\beta(0)$$

$\Rightarrow [K_\mu, \phi_\alpha(x)] =$ ~~$[K_\mu, \phi_\alpha(0)]$~~ expansion of $[\hat{K}_\mu, \phi_\alpha(0)]$ with $\phi_\alpha(0)$ replaced by $\phi_\alpha(x)$

$$= 2i x_\mu \Delta \phi_\alpha(x) + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \phi_\alpha(x) + 2i x^\nu (S_{\nu\mu})_\alpha^{\ \beta} \phi_\beta(x)$$

Note that $\because [[K, P], P]$ is a expansion ~~ab~~ only involving $P$ and $[P_\mu, P_\nu] = 0$

$\therefore [[[K, P], P], P]$ is indeed $0$