α Good job! Excellent work.

String Theory I

Problem Set 2

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Th 11.00 wk 4.

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$\boxed{1}$ $[\alpha_m^\mu, \alpha_n^\nu]_{PB} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{PB} = i m \eta^{\mu\nu} \delta_{m+n,0}$

$[p^\mu, x^\nu]_{PB} = \eta^{\mu\nu}$, all other $= 0$

(a)

$P^\mu = p^\mu$

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu}{n} - i \sum_{n=1}^{\infty} \frac{\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu}{n}$$

$\rightarrow$ $[P^\mu, P^\nu] = [p^\mu, p^\nu] = 0$ ✓

$$\rightarrow [P^\mu, M^{\nu\rho}] = \underbrace{[p^\mu, x^\nu]}_{\eta^{\mu\nu}} p^\rho - \underbrace{[p^\mu, x^\rho]}_{\eta^{\mu\rho}} p^\nu$$ ✓

$$+ \underbrace{\left[p^\mu, -i \sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^\nu \alpha_n^\rho - \alpha_{-n}^\rho \alpha_n^\nu) - i \sum_{n=1}^{\infty} \frac{1}{n}(\tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\rho - \tilde{\alpha}_{-n}^\rho \tilde{\alpha}_n^\nu)\right]}_{=0}$$

$$= \eta^{\mu\nu} p^\rho - \eta^{\mu\rho} p^\nu$$ ✓

$\rightarrow$ $[M^{\mu\nu}, M^{\rho\lambda}] =$

$$\Big[x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu) - i \sum_{n=1}^{\infty} \frac{1}{n}(\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu),$$
$$x^\rho p^\lambda - x^\lambda p^\rho - i \sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^\rho \alpha_n^\lambda - \alpha_{-n}^\lambda \alpha_n^\rho) - i \sum_{n=1}^{\infty} \frac{1}{n}(\tilde{\alpha}_{-n}^\rho \tilde{\alpha}_n^\lambda - \tilde{\alpha}_{-n}^\lambda \tilde{\alpha}_n^\rho)\Big]$$

$$= \underbrace{[x^\mu p^\nu - x^\nu p^\mu, x^\rho p^\lambda - x^\lambda p^\rho]}_{①} +$$

$$\underbrace{\left[-i \sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu), -i \sum_{m=1}^{\infty} \frac{1}{m}(\alpha_{-m}^\rho \alpha_m^\lambda - \alpha_{-m}^\lambda \alpha_m^\rho)\right]}_{②}$$

$$+\left[-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\tilde{\alpha}_{-n}^{\mu}\tilde{\alpha}_{n}^{\nu}-\tilde{\alpha}_{-n}^{\nu}\tilde{\alpha}_{n}^{\mu}\right),\ -i\sum_{m=1}^{\infty}\frac{1}{m}\left(\tilde{\alpha}_{-m}^{\rho}\tilde{\alpha}_{m}^{\lambda}-\tilde{\alpha}_{-m}^{\lambda}\tilde{\alpha}_{m}^{\rho}\right)\right]$$

③

~~① Note $[AB, CD] = AC[B,D] + AD[B,C] + BC[A,D] + BD[A,C]$~~

Use:

① $[AB, CD] = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]B$

$$① = [x^\mu p^\nu, x^\rho p^\lambda] - [x^\nu p^\mu, x^\rho p^\lambda] - [x^\mu p^\nu, x^\lambda p^\rho] + [x^\nu p^\mu, x^\lambda p^\rho]$$

$$= \cancel{x^\mu[p^\nu, x^\rho]p^\lambda + x^\rho[x^\mu, p^\lambda]}$$

$$= \eta^{\nu\rho}x^\mu p^\lambda - \eta^{\mu\lambda}x^\rho p^\nu - \eta^{\mu\rho}x^\nu p^\lambda + \eta^{\nu\lambda}x^\rho p^\mu - \eta^{\nu\lambda}x^\mu p^\rho + \eta^{\mu\rho}x^\lambda p^\nu + \eta^{\mu\lambda}x^\nu p^\rho - \eta^{\nu\rho}x^\lambda p^\mu$$

$$= \eta^{\nu\rho}(x^\mu p^\lambda - x^\lambda p^\mu) - \eta^{\mu\rho}(x^\nu p^\lambda - x^\lambda p^\nu) - \eta^{\nu\lambda}(x^\mu p^\rho - x^\rho p^\mu) + \eta^{\mu\lambda}(x^\nu p^\rho - x^\rho p^\nu)$$

$$② = (-i)^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}[\alpha_{-n}^{\mu}\alpha_{n}^{\nu}-\alpha_{-n}^{\nu}\alpha_{n}^{\mu},\ \alpha_{-m}^{\rho}\alpha_{m}^{\lambda}-\alpha_{-m}^{\lambda}\alpha_{m}^{\rho}]$$

$\because m \geq 1$, $n \geq 1$ $\therefore m+n \neq 0$ $\therefore \delta_{m+n,0} = 0$

$$= (-i)^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\Big\{[\alpha_{-n}^{\mu}\alpha_{n}^{\nu}, \alpha_{-m}^{\rho}\alpha_{m}^{\lambda}] - [\alpha_{-n}^{\nu}\alpha_{n}^{\mu}, \alpha_{-m}^{\rho}\alpha_{m}^{\lambda}] - [\alpha_{-n}^{\mu}\alpha_{n}^{\nu}, \alpha_{-m}^{\lambda}\alpha_{m}^{\rho}] + [\alpha_{-n}^{\nu}\alpha_{n}^{\mu}, \alpha_{-m}^{\lambda}\alpha_{m}^{\rho}]\Big\}$$

$\delta_{n-m,0} = \delta_{m-n,0} = \delta_{mn}$

$$= (-i)^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\Big\{n\eta^{\nu\rho}\delta_{mn}\alpha_{-n}^{\mu}\alpha_{m}^{\lambda} - m\eta^{\mu\lambda}\delta_{mn}\alpha_{-m}^{\rho}\alpha_{n}^{\nu}$$
$$- n\eta^{\mu\rho}\delta_{mn}\alpha_{-n}^{\nu}\alpha_{m}^{\lambda} + m\eta^{\nu\lambda}\alpha_{-m}^{\rho}\alpha_{n}^{\mu}\delta_{mn} - n\eta^{\nu\lambda}\alpha_{-n}^{\mu}\alpha_{m}^{\rho}\delta_{mn}$$
$$+ m\eta^{\mu\rho}\alpha_{-m}^{\lambda}\alpha_{n}^{\nu}\delta_{mn} + n\eta^{\mu\lambda}\alpha_{-n}^{\nu}\alpha_{m}^{\rho}\delta_{mn} - m\eta^{\nu\rho}\delta_{mn}\alpha_{-m}^{\lambda}\alpha_{n}^{\mu}\Big\}$$

$$= -i\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\times n\right)\{\eta^{\mu\rho}(\alpha_{-n}^{\nu}\alpha_n^{\lambda}-\alpha_{-n}^{\lambda}\alpha_n^{\nu}).$$

$$-\eta^{\nu\rho}(\alpha_{-n}^{\mu}\alpha_n^{\lambda}-\alpha_{-n}^{\lambda}\alpha_n^{\mu}) - \eta^{\mu\lambda}(\alpha_{-n}^{\nu}\alpha_n^{\rho}-\alpha_{-n}^{\rho}\alpha_n^{\nu}).$$

$$+\eta^{\nu\lambda}(\alpha_{-n}^{\mu}\alpha_n^{\rho}-\alpha_{-n}^{\rho}\alpha_n^{\mu}).\}.$$

$$= \cancel{\eta}\ \eta^{\mu\rho}\left(-i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^{\nu}\alpha_n^{\lambda}-\alpha_{-n}^{\lambda}\alpha_n^{\nu})\right)$$

$$-\eta^{\nu\rho}\left(-i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^{\mu}\alpha_n^{\lambda}-\alpha_{-n}^{\lambda}\alpha_n^{\mu})\right).$$

$$-\eta^{\mu\lambda}\left(-i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^{\nu}\alpha_n^{\rho}-\alpha_{-n}^{\rho}\alpha_n^{\nu})\right).$$

$$+\eta^{\nu\lambda}\left(-i\sum_{n=1}^{\infty}(\alpha_{-n}^{\mu}\alpha_n^{\rho}-\alpha_{-n}^{\rho}\alpha_n^{\mu})\right).$$

Similar to ②, ③ ⇒

$$③ = \eta^{\mu\rho}\left(-i\sum_{n=1}^{\infty}\frac{1}{n}(\tilde{\alpha}_{-n}^{\nu}\tilde{\alpha}_n^{\lambda}-\tilde{\alpha}_{-n}^{\lambda}\tilde{\alpha}_n^{\nu})\right).$$

$$-\eta^{\nu\rho}\left(-i\sum_{n=1}^{\infty}\frac{1}{n}(\tilde{\alpha}_{-n}^{\mu}\tilde{\alpha}_n^{\lambda}-\tilde{\alpha}_{-n}^{\lambda}\tilde{\alpha}_n^{\mu})\right).$$

$$-\eta^{\mu\lambda}\left(-i\sum_{n=1}^{\infty}\frac{1}{n}(\tilde{\alpha}_{-n}^{\nu}\tilde{\alpha}_n^{\rho}-\tilde{\alpha}_{-n}^{\rho}\tilde{\alpha}_n^{\nu})\right).$$

$$+\eta^{\nu\lambda}\left(-i\sum_{n=1}^{\infty}\frac{1}{n}(\tilde{\alpha}_{-n}^{\mu}\tilde{\alpha}_n^{\rho}-\tilde{\alpha}_{-n}^{\rho}\tilde{\alpha}_n^{\mu})\right).$$

① + ② + ③ ⇒

$$[M^{\mu\nu}, M^{\rho\lambda}] = \eta^{\nu\rho}M^{\mu\lambda} - \eta^{\mu\rho}M^{\nu\lambda} - \eta^{\nu\lambda}M^{\mu\rho} + \eta^{\mu\lambda}M^{\nu\rho}$$

Clear work! What is the name of this algebra?

On sorry didn't see this! $\square$.

$\rightarrow$ recovers the Poincaré algebra

(b) in light-cone coordinates

$$X^\mu(\sigma,\tau) = \cancel{X_R^\mu(\sigma^-)} \rightarrow X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)$$

where $\sigma^- = \tau - \sigma$, $\sigma^+ = \tau + \sigma$

~~at~~ Stress tensor $T_{++} = \cancel{x}\,\partial_+ X \cdot \partial_+ X = \partial_+ X_L \cdot \partial_+ X_L = \dot{X}_L^2.$

$$T_{--} = \partial_- X \cdot \partial_- X = \partial_- X_R \cdot \partial_- X_R = \dot{X}_R^2.$$

$$L_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma}\, T_{--}\, d\sigma \Big|_{\tau=0}$$

Why can we evaluate at $\tau = 0$?

$$= \frac{T}{2}\int_0^\pi e^{-2im\sigma}\, \dot{X}_R^2\, d\sigma \Big|_{\tau=0}$$

$\rightarrow$ Closed strings:

$$X_R = \underbrace{\tfrac{1}{2}x^\mu}_{\text{constant.}} + \tfrac{1}{2}\underbrace{l^2 p^\mu}_{\alpha_0^\mu}(\tau-\sigma) + \frac{i}{2}\, l \sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)}, \quad \boxed{\alpha_0^\mu = \tfrac{1}{2} l p^\mu}$$

$$\partial_- X_R = \dot{X}_R = l\,\alpha_0^\mu + l\sum_{n\neq 0}\cancel{\tfrac{i}{2}(-2i)\tfrac{1}{n}n}\, e^{-2in(\tau-\sigma)}\alpha_n^\mu$$

$$= l\sum_{n=-\infty}^{\infty}\alpha_n^\mu e^{-2in(\tau-\sigma)},$$

$$L_m = \frac{Tl^2}{2}\int_0^\pi e^{-2im\sigma}\sum_{n=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\alpha_n^\mu \alpha_{r\mu}\, e^{-2i(n+r)(\tau-\sigma)}\, d\sigma.$$

$$\cancel{= \frac{Tl^2}{2}\sum_n\sum_r\int_0^\pi d\sigma}$$

$$L_m = \frac{T l^2}{2} \sum_n \sum_r \alpha_n \cdot \alpha_r \int_0^\pi d\sigma\, e^{2i(n+r-m)\sigma}$$

the integral $\int_0^\pi d\sigma\, e^{2i(n+r-m)\sigma} = \pi$ when $n+r-m=0$

and it $= \frac{1}{2i(n+r-m)} e^{2i(n+r-m)\sigma}\Big|_0^\pi = 0$ if $n+r-m \neq 0$.

integer $\neq 0$

$\therefore$

$$L_m = \frac{\pi T l^2}{2} \sum_n \sum_r \alpha_n \cdot \alpha_r \, \delta_{n+r,m}$$

$$= \frac{\pi T l^2}{2} \sum_n \alpha_{m-n} \cdot \alpha_n$$

$\because$ $l = \frac{1}{\sqrt{\pi T}}$ $\therefore$ $\pi T l^2 = 1$

$$\therefore L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n$$ with $\alpha_0^\mu = \frac{1}{2} l p^\mu$.

similarly

$$\tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n$$ with $\tilde{\alpha}_0^\mu = \frac{1}{2} l p^\mu$

Great!

$$[L_m, L_n] = \frac{1}{4} \sum_{k,l} [\alpha_{m-k} \cdot \alpha_k, \alpha_{n-l} \cdot \alpha_l]$$

consider $a_{m-k} \cdot a_k = a^\mu_{m-k} a_{k\mu}$

$a_{n-l} \cdot a_l = a^\nu_{n-l} a_{l\nu}$

$$[L_m, L_n] = \frac{1}{4}\sum_{k,l} [\alpha^\mu_{m-k}\alpha_{k\mu}, \alpha^\nu_{n-l}\alpha_{l\nu}].$$

$$= \frac{1}{4}\sum_{k,l} \alpha^\mu_{m-k}[\underbrace{\alpha_{k\mu}, \alpha^\nu_{n-l}}_{ik\eta_{\mu\nu}\delta_{n+k-l,0}}]\alpha_{l\nu} + \alpha^\mu_{m-k}\alpha^\nu_{n-l}[\underbrace{\alpha_{k\mu}, \alpha_{l\nu}}_{ik\eta_{\mu\nu}\delta_{k+l,0}}]$$

$$+ [\underbrace{\alpha^\mu_{m-k}, \alpha^\nu_{n-l}}_{i(m-k)\eta^{\mu\nu}\delta_{m-k+n-l,0}}]\alpha_{k\mu}\alpha_{l\nu} + \alpha^\nu_{n-l}[\underbrace{\alpha_{m-k\mu}, \alpha_{l\nu}}_{i(m-k)\eta_{\mu\nu}\delta_{m-k+l,0}}]\alpha^\mu_k$$

$$= \frac{i}{4}\sum_{k,l}\big(k\,\alpha_{m-k}\cdot\alpha_l\,\delta_{k+n-l,0} + k\,\alpha_{m-k}\cdot\alpha_{n-l}\,\delta_{k+l,0}$$

$$+ (m-k)\,\alpha_l\cdot\alpha_k\,\delta_{m-k+n-l} + (m-k)\,\alpha_{n-l}\cdot\alpha_k\,\delta_{m-k+l}\big)$$

$$= \frac{i}{2}\sum_k k\,\alpha_{m-k}\cdot\alpha_{k+n} + \frac{i}{2}\sum_k (m-k)\,\alpha_{m-k+n}\cdot\alpha_k$$

① ②

in first term, change variable $k \to k' = k+n$

$$\Rightarrow [L_m, L_n] = \left(\frac{i}{2}\sum_{k'}(k'-n)\,\alpha_{m-k'+n}\cdot\alpha_{k'}\right)$$

$$+\left(\frac{i}{2}\sum_{k'}(m-k')\,\alpha_{m-k'+n}\cdot\alpha_{k'}\right).$$

$$= \frac{i}{2}\sum_k (m-n)\,\alpha_{m+n-k}\cdot\alpha_k$$

$$= i(m-n)\,L_{m+n}$$

Fantastic!

□

$$L_m = \frac{1}{2}\sum_k \alpha_{m-k}\cdot\alpha_k = \frac{1}{2}\sum_n \alpha_{m-k}\alpha_{n\nu}$$

$$[L_m, \alpha_n^\mu] = \frac{1}{2}\sum_n [\alpha_{m-k}\alpha_{k\nu}, \alpha_n^\mu].$$

$$= \frac{1}{2}\sum_n \alpha_{m-k,\nu}\underbrace{[\alpha_k^\nu, \alpha_n^\mu]}_{i\eta^{\mu\nu}k\,\delta_{k+n,0}} + \underbrace{[\alpha_{m-k}^\nu, \alpha_n^\mu]}_{i(m-k)\eta^{\mu\nu}\delta_{m-k+n,0}}\alpha_{k,\nu}$$

$$= \frac{1}{2}\sum_n -ni\,\alpha^\mu_{m+n} - ni\,\alpha^\mu_{m+n}$$

$$= -in\,\alpha^\mu_{m+n}$$

Good!

$\square$

$$[L_m, \tilde{\alpha}_n^\mu] = \sum_k [\alpha_{m-k}\cdot\alpha_k, \tilde{\alpha}_n^\mu] = 0$$

$\square$

$$\cancel{[\tilde{\alpha}_k^\mu, \tilde{\alpha}_n^\nu]}$$

$$[\alpha_k^\mu, \tilde{\alpha}_n^\nu] = 0$$

similarly $[\tilde{L}_m, \alpha_n^\mu] = 0$

$\square$

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \alpha_0^\mu l\sigma^- + \frac{i}{2}l\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-}$$

$\cancel{X_L^\mu(\sigma^+)}$

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \tilde{\alpha}_0^\mu l\sigma^+ + \frac{i}{2}l\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+}$$

$[\alpha_0^\mu, \alpha_n^\mu] = 0$

$$X^\mu(\tau,\sigma) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)$$

Good!

$$\therefore [L_m, X^\mu] = [L_m, X_R^\mu(\sigma^-)] + \frac{1}{2}[L_m, x^\mu]$$

$\alpha_0^\mu = \frac{1}{2}lp^\mu$

$$= \frac{i}{2}l\sum_{n\neq 0}\frac{1}{n}e^{-2in\sigma^-}[L_m, \alpha_n^\mu] + \frac{1}{2}[\alpha_m\alpha_0 + \alpha_0\alpha_m, x^\mu]$$

$$= \frac{i}{2}l\sum_{n\neq 0}\frac{1}{n}e^{-2in\sigma^-}[L_m, \alpha_n^\mu] + \frac{l}{2}\alpha_{m,\nu}\underbrace{[p^\nu, x^\mu]}_{\eta^{\mu\nu}}$$

$$= \frac{1}{2} l \sum_{n\neq 0} \frac{1}{\cancel{n}} e^{-2in\sigma^-} (\cancel{-i}) \cancel{n}\, \alpha^\mu_{m+n} + \frac{1}{2}\alpha^\mu_m$$

$$= \frac{1}{2} l \sum_{n\neq 0} e^{-2in\sigma^-} \alpha^\mu_{m+n} + \frac{1}{2}\alpha_m^\mu = \underline{\frac{1}{2} \sum_n e^{-2in\sigma^-} \alpha^\mu_{m+n}}$$

D.

~~Previously we've calculated that.~~

$$\cancel{\partial_- X_R^\mu = l \sum_n \alpha_n^\mu e^{-2in\sigma^-}}$$

$$\cancel{\therefore V_m X^\mu = \frac{1}{2} e^{-2im\sigma^-}(\partial_- X_R^\mu + \underbrace{\partial_- X_L^\mu}_{=0})}$$

$$\cancel{= \frac{1}{2} e^{-2im\sigma^-} \partial_- X_R^\mu = \frac{l}{2} e^{-2im\sigma^-} \sum_n \alpha_n^\mu e^{-2in\sigma^-}}$$

$$\cancel{= \frac{1}{2} \sum_n \alpha_n^\mu e^{-2i(n+m)\sigma^-}}$$

~~$k = n+m$~~

~~$n = k-m$~~

$$\cancel{= \frac{1}{2} \sum_k \alpha^\mu_{k-m} e^{-2ik\sigma^-}}$$

Previously we've calculated that

$$\partial_- X_R^\mu = l \sum_n \alpha_n^\mu e^{-2in\sigma^-}$$

$$\therefore V_m^- X^\mu = \frac{1}{2} e^{2im\sigma^-} (\partial_- X_R^\mu + \underbrace{\partial_- X_L^\mu}_{=0}) = \frac{1}{2} e^{2im\sigma^-} \partial_- X_R^\mu$$

$$= \frac{l}{2} e^{2im\sigma^-} \sum_n \alpha_n^\mu e^{-2in\sigma^-} = \frac{1}{2}\sum_n \alpha_n^\mu e^{-2i(n-m)\sigma^-}$$

$k = n-m$

$n = k+m$

$$= \underline{\underline{\frac{1}{2} \sum_k \alpha^\mu_{k+m} e^{-2ik\sigma^-}}} = [L_m, X^\mu]$$

D

agrees

Similarly $[\tilde{L}_m, X^\mu] = [\tilde{L}_m, X_L^\mu(\sigma^+)] + \frac{1}{2}[\tilde{L}_m, X^\mu]$

$$= \frac{i}{2} l \sum_{n\neq 0} \frac{1}{n} e^{-2in\sigma^+} \underbrace{[\tilde{L}_m, \tilde{\alpha}_n^\mu]}_{-in\tilde{\alpha}^\mu_{m+n}} + \underbrace{[\tilde{\alpha}_m \cdot \tilde{\alpha}_0, x^\mu]}_{=\tilde{\alpha}_{m\,\nu}\eta^{\mu\nu}\frac{l}{2}}.$$

$$= \frac{l}{2} \sum_{n\neq 0} e^{-2in\sigma^+} \tilde{\alpha}^\mu_{m+n} + \frac{l}{2}\tilde{\alpha}_m^\mu$$

$$= \frac{l}{2} \sum_n \tilde{\alpha}^\mu_{m+n} e^{-2in\sigma^+}$$

$$V_m^+ X^\mu = V_m^+ X_L^\mu = \frac{1}{2} e^{2im\sigma^+} \partial_+ X_L^\mu$$

$$= \frac{1}{2} e^{2im\sigma^+} l \sum_n \tilde{\alpha}_n^\mu e^{-2in\sigma^+}$$

$$= \frac{l}{2} \sum_n \tilde{\alpha}_n^\mu e^{-2i(n-m)\sigma^+} = \frac{l}{2}\sum_k \tilde{\alpha}^\mu_{k+m} e^{-2ik\sigma^+}$$

$$= [\tilde{L}_m, X^\mu]$$

$\square$

$\boxed{2}$ we promote $x^\mu, p^\mu, \alpha_n^\mu$ to operators.

↑↑

(a) we promote also $p^\mu = p^\mu$ and

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu)$$
$$- i\sum_{n=1}^{\infty} \frac{1}{n}(\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu)$$

So they are the generators of Poincaré symmetry.

The commutators are now

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad [p^\mu, x^\nu] = -i\eta^{\mu\nu}$$

which is obtained by $[\ ]_{PB} \to -i[\ ]$

Note that there is no ordering issue in $M^{\mu\nu}$ when quantising it, since Σ is from 1 to ∞.

Good, but there could be an issue when $\mu = \nu$, but $M_{\mu\nu}$ is antisymmetric so there is nothing to worry about.

So we simply multiply the result in $\boxed{I}$ by $-i$ to get

$$[p^\mu, p^\nu] = 0$$

$$[p^\mu, M^{\nu\rho}] = -i\eta^{\mu\nu}p^\rho + i\eta^{\mu\rho}p^\nu$$

$$[M^{\mu\nu}, M^{\rho\lambda}] = -i\eta^{\nu\rho}M^{\mu\lambda} + i\eta^{\mu\rho}M^{\nu\lambda} + i\eta^{\nu\lambda}M^{\mu\rho} - i\eta^{\mu\lambda}M^{\nu\rho}$$

which are the Poincaré Algebra

Not quite! There are definitely ordering ambiguities here

(b) In $\boxed{I}$ when calculating $[L_m, L_n]$ we arrived at the following intermediate step:

(*) $[L_m, L_n] = \frac{1}{2}\sum_k k\alpha_{m-k}\cdot\alpha_{k+n} + \frac{1}{2}\sum_k (m-k)\alpha_{m-k+n}\cdot\alpha_k$.

(we've multiplied $(-i)$ to make $[\ ]_{PB}$ to be $[\ ]$)

Before this step everything is the same in the quantum case as in the classical case (except factor $-i$)) because we simply used commutation relations.

Now from (*), if $m+n \neq 0$ we will arrive at

$[L_m, L_n] = (m-n)L_{m+n}$ just as the classical case

because

$[\alpha^\mu_{m-k}, \alpha^\nu_{k+n}] = [\alpha^\mu_{m-k+n}, \alpha^\nu_k] = 0$ iff $m+n \neq 0$

and that means normal ordering doesn't change the value of $[L_m, L_n]$.

However, if ~~not~~ $m+n=0$, then

$[L_m, L_n] = \frac{1}{2}\sum_k k\,\alpha_{m-k}\cdot\alpha_{k-m} + \frac{1}{2}\sum_k (m-k)\,\alpha_{-k}\cdot\alpha_k$.

$\because$ α^μ_n and α^ν_{-n} ~~does~~ do not commute

$\therefore$ ~~we~~ we do normal ordering to this case an complex-number should be introduced (because commutator of α_{-n} and α_n is a number)

Fantastic!

$\therefore$ For $m=-n$, ~~then~~ the value of $[L_m, L_n]$ should differ from the classical result by a ~~complex~~ c-number.

Hence $[L_m, L_n] = (m-n)L_{m+n} + A(m)\,\delta_{m+n,0}$ ①

where $A(m)$ is that ~~complex~~ c-number, it only depends on m because ~~A(m)~~ A only has effect when $m=-n$.

(i)

$\because$ ① $\therefore$ $[L_m, L_{-m}] = 2mL_0 + A(m)$

and $[L_{-m}, L_m] = -2mL_0 + A(-m)$

Add the above two equations:

$$0 = 0 + A(m) + A(-m) \Rightarrow -A(m) = A(-m)$$

(ii) for $k+m+n=0$,

Jacobi identity $[L_k, [L_n, L_m]] + [L_n, [L_m, L_k]] + [L_m, [L_k, L_n]] = 0$

use $[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m,-n}$ this gives

$$\Rightarrow [L_k, (n-m)L_{n+m} + A(n)\delta_{m+n,0}] + [L_n, (m-k)L_{m+k} + A(m)\delta_{m+k,0}] + [L_m, (k-n)L_{k+n} + A(k)\delta_{k+n,0}] = 0.$$

$$\Rightarrow (n-m)(k-n-m)L_{k+n+m} + A(k)\delta_{k+n+m,0}(n-m) + (m-k)(n-m-k)L_{k+n+m} + A(n)\delta_{k+n+m,0}(m-k) + (k-n)(m-k-n)L_{k+n+m} + A(m)\delta_{k+n+m,0}(k-n) = 0$$

use $k+n+m=0$

$$\Rightarrow 2(\underbrace{(n-m)k + (m-k)n + (k-n)m}_{=0})L_0 + (n-m)A(k) + (m-k)A(n) + (k-n)A(m) = 0$$

$$\Rightarrow (n-m)A(k) + (m-k)A(n) + (k-n)A(m) = 0$$

(iii) setting $k=1$ and $m=-n-1$ $(m+n+k=0)$

then use (ii),

$$(2n+1)A(1) - (n+2)A(n) + (1-n)\underbrace{A(-n-1)}_{=(n-1)A(n+1)} = 0$$

$$\therefore \quad A(n+1) = \frac{(n+2)(A(n)) - (2n+1)A(1)}{n-1}$$

So the above ~~equati~~ recursive equation shows that knowing $A(1)$ and $A(2)$ is enough for one to ~~determining~~ determine all $A(n)$'s.

So in the expansion

$$A(m) = c_0 + c_1 m + c_2 m^2 + c_3 m^3 + \cdots$$

there should be only 2 unknowns among $\{c_i\}$ $(i=0,1,2,\ldots)$

first note that $-A(m) = A(-m) \Rightarrow A(0)=0 \Rightarrow c_0 = 0$

And $A(m) = -A(-m)$ itself $\Rightarrow$ ~~c_1, c_3, c_5~~ $A(m)$ is odd

$\Rightarrow \quad c_2, c_4, c_6, \ldots = 0.$

only. $c_1, c_3, c_5, \ldots,$ can be non-zero

try ~~c_1~~ $\{c_1, c_3$ to be determined, $c_5 = c_7 = c_9 = \cdots = 0\}$

then $A(m) = c_1 m + c_3 m^3$

for $n+m+k=0$

$$(n-m)A(k)+(m-k)A(n)+(k-n)A(m)$$

$$=(n-m)(c_1k+c_3k^3)+(m-k)(c_1n+c_3n^3)+(k-n)(c_1m+c_3m^3)$$

$$=c_1(\underbrace{k(n-m)+n(m-k)+m(k-n)}_{=0})$$

$$+c_3(k^3(n-m)+n^3(m-k)+m^3(k-n)).$$

$$=c_3(k^3n-k^3m+n^3m-n^3k+m^3k-m^3n.)$$

$$=c_3(kn(k^2-n^2)+mn(n^2-m^2)+mk(m^2-k^2)).$$

$$=c_3(kn(k+n)(k-n)+mn(m+n)(n-m)+mk(m+k)(m-k))$$

$$=-c_3(mkn)(k-n+n-m+m-k)=0$$

So $A(m)=c_1m+c_3m^3$ clearly solves the equation in (ii), and requires 2 equations to determine two ~~the~~ unknowns c_1 and c_3, corresponding to the fact that if we known $A(2)$ and $A(1)$ we know the whole expression of $A(m)$.

In other words, $A(m)=c_1m+c_3m^3$ is a solution, but there ~~is~~ is no solution that is more general than ~~th~~ it. ~~So this~~ because more general solution would require more than 2 ~~un~~ unknown coefficients which is impossible. Hence,

$A(m)=c_1m+c_3m^3$ is the most general solution
Great!

□

(iv) use (iii), $\langle 0;0|[L_m, L_{-m}]|0,0\rangle$

$$= \langle 0;0|\, 2mL_0 + A(m)\, |0,0\rangle$$
$$= 2m\langle 0;0|L_0|0;0\rangle + A(m)\underbrace{\langle 0;0|0;0\rangle}_{1}$$

$\because (L_0 - a)|0,0\rangle = 0$ for some undetermined constant ce

$$\therefore \langle 0;0|[L_m, L_{-m}]|0,0\rangle = 2ma + A(m)$$
$$= (c_1 + 2a)m + c_3 m^3$$

so, effectively we can shift the definition of L_0 by a constant to make $L_0|0,0\rangle = 0$ ✓

This ~~operat~~ operation otherwise does not disturb the Virasoro algebra because a constant commutes with everything. ✓

then we absorb the $+2a$ into the definition of c_1 so $\langle 0;0|[L_m, L_{-m}]|0;0\rangle = A(m) = c_1 m + c_3 m^3$ ✓

$\therefore$ ~~$c_1 + c_3 = A(1) = \langle 0;0|[L_1, L_{-1}]|0;0\rangle$~~

OR we simply use the fact that in

$|0;0\rangle$, $p^\mu = 0$ $\because \alpha_0^\mu = \frac{1}{2}\ell p^\mu$ $\therefore$ ~~α_0^μ~~ $\alpha_0^\mu|0;0\rangle = 0$

$\because L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n$, we know $\alpha_0^\mu|0;0\rangle = 0$ and α_n^μ annihilates all $|0;p^\mu\rangle$ states if $n \geq 1$

$\therefore L_0|0;0\rangle = 0 \Rightarrow \langle 0;0|[L_m, L_{-m}]|0;0\rangle = A(m)$ ✓
$$= c_1 m + c_3 m^3$$

So we know that $\cancel{\alpha}\ \alpha^{\mu}_{n}|0;0\rangle = 0$ if $n \geq 0$

$\therefore c_1 + c_3 = A(1) = \langle 0;0|[L_1, L_{-1}]|0;0\rangle$

$= \langle 0;0|L_1 L_{-1}|0;0\rangle - \langle 0;0|L_{-1}L_1|0;0\rangle$

$L_1 = \frac{1}{2}\sum_n \alpha_{1-n}\alpha_n \qquad L_{-1} = \frac{1}{2}\sum_n \alpha_{-1-n}\alpha_n$

$\therefore L_1 = \frac{1}{2}(\cdots + \alpha_{-2}\alpha_3 + \alpha_{-1}\alpha_2 + \alpha_0\alpha_1 + \alpha_1\alpha_0 + \alpha_2\alpha_{-1} + \cdots)$

$L_{-1} = \frac{1}{2}(\cdots + \alpha_{-3}\alpha_2 + \alpha_{-2}\alpha_1 + \alpha_{-1}\alpha_0 + \alpha_0\alpha_{-1} + \alpha_1\alpha_{-2} + \cdots)$

and $[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = 0$ if $m+n \neq 0$

$\therefore L_1|0;0\rangle = 0 \quad L_{-1}|0;0\rangle = 0$ because we can always put the α^{μ}_{n} $(n \geq 0)$ terms on the right and annihilate $|0;0\rangle$

$\Rightarrow \quad c_1 + c_3 = 0$

$8c_3 + 2c_1 = A(2) = \langle 0;0|[L_2, L_{-2}]|0,0\rangle$

$= \langle 0;0|L_2 L_{-2}|0;0\rangle - \langle 0,0|L_{-2}L_2|0,0\rangle$

$L_2 = \frac{1}{2}(\cdots + \alpha_{-2}\alpha_4 + \alpha_{-1}\alpha_3 + \alpha_0\alpha_2 + \alpha_1\alpha_1 + \alpha_2\alpha_0 + \alpha_3\alpha_{-1} + \cdots)$

$L_{-2} = \frac{1}{2}(\cdots + \alpha_{-4}\alpha_2 + \alpha_{-3}\alpha_1 + \alpha_{-2}\alpha_0 + \alpha_{-1}\alpha_{-1} + \alpha_0\alpha_{-2} + \alpha_1\alpha_{-3} + \cdots)$

$\cancel{\therefore}$ in L_2, each term has at least α_n $(n \geq 0)$

$\therefore L_2|0;0\rangle = 0$ (this is obvious $\because$ $|0;0\rangle$ is a physical state)

But $L_{-2}|0;0\rangle \neq 0$ $\because$ the presence of $\alpha_{-1}\cdot\alpha_{-1}$ term.

similarly $\because$ $\alpha_{-n}^{\mu} = (\alpha_n^{\mu})^{\dagger}$ $\therefore$ $\langle 0;0|\alpha_n^{\mu} = 0$ iff $n \leq 0$.

and so all terms in $\langle 0;0|L_2$ are 0 except the ~~oth~~ $\alpha_1\alpha_1$ term which have all positive n.

$$\therefore \quad 8C_3 + 2C_1 = \frac{1}{4}\langle 0;0|\alpha_1\cdot\alpha_1\,\alpha_{-1}\cdot\alpha_{-1}|0;0\rangle$$

$$= \frac{1}{4}\langle 0;0|\alpha_1^{\mu}\alpha_{1\mu}\,\alpha_{-1}^{\nu}\alpha_{-1\nu}|0;0\rangle$$

$$= \frac{1}{4}\langle 0;0|\alpha_{1\mu}\underbrace{[\alpha_1^{\mu},\alpha_{-1}^{\nu}]}_{\eta^{\mu\nu}}\alpha_{-1\nu}|0;0\rangle$$

$$+ \frac{1}{4}\langle 0;0|\alpha_{1\mu}\alpha_{-1}^{\nu}\alpha_1^{\mu}\alpha_{-1\nu}|0;0\rangle$$

$$= \frac{1}{4}\eta_{\mu\nu}\langle 0;0|\alpha_1^{\mu}\alpha_{-1}^{\nu}|0;0\rangle$$

$$+ \frac{1}{4}\langle 0;0|\alpha_{1\mu}\alpha_{-1\nu}\underbrace{[\alpha_1^{\mu},\alpha_{-1}^{\nu}]}_{\eta^{\mu\nu}}|0;0\rangle.$$

$$+ \frac{1}{4}\langle 0;0|\alpha_{1\mu}\alpha_{-1\nu}\alpha_{-1}^{\nu}\underbrace{\alpha_1^{\mu}|0;0\rangle}_{=0}$$

$$= \frac{1}{2}\eta_{\mu\nu}\langle 0;0|\alpha_1^{\mu}\alpha_{-1}^{\nu}|0;0\rangle = \frac{1}{2}\eta_{\mu\nu}\langle 0;0|\underbrace{[\alpha_1^{\mu},\alpha_{-1}^{\nu}]}_{\eta^{\mu\nu}}|0;0\rangle$$

$$+ \frac{1}{2}\eta_{\mu\nu}\langle 0;0|\alpha_{-1}^{\nu}\underbrace{\alpha_1^{\mu}|0;0\rangle}_{=0}$$

$$= \frac{1}{2}\underbrace{\eta_{\mu\nu}\eta^{\mu\nu}}_{D} = \frac{1}{2}D$$ Fantastic!

$\Rightarrow \begin{cases} c_1 + c_3 = 0 \\ 2c_1 + 8c_3 = \frac{1}{2}D \end{cases} \Rightarrow 6c_3 = \frac{1}{2}D \Rightarrow \begin{cases} c_3 = \frac{1}{12}D \\ c_1 = -\frac{1}{12}D \end{cases}$

$$\Rightarrow A(m) = \frac{D}{12}(m^3 - m) \quad ✓$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\,\delta_{m+n,0}$$

□

(iv) $\boxed{3}$ mass-shell condition:

$\because$ ~~$\alpha_0 = \ell p$~~ $\alpha_0^\mu = \ell p^\mu$, $\ell^2 = 2\alpha'$

~~phys physical state $(L_0 - a)|0;p\rangle = 0$~~

~~$\Rightarrow \frac{1}{2}\alpha_0^2|0;p\rangle = a|0;p\rangle \Rightarrow \frac{1}{2}\ell^2 p\cdot p = a.$~~ ✓

~~$\Rightarrow \alpha' p\cdot p = a$~~

states at level one: $|\xi.p\rangle = \xi\cdot\alpha_{-1}|0;p\rangle$, $\xi \in \mathbb{R}^{1,D-1}$

a polarisation vector.

mass shell condition.

$$(L_0 - a)|\xi.p\rangle = 0$$

~~$L_{+1}|\xi.p\rangle = 0$~~

$$\Rightarrow \left(\frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n\right)|\xi.p\rangle = a|\xi.p\rangle.$$

$$\therefore \left(\frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n\right)\xi\cdot\alpha_{-1}|0;p\rangle = a(\xi\cdot\alpha_{-1}|0;p\rangle)$$

$$\Rightarrow \left(\alpha' p\cdot p + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n\mu}\right)(\xi^\nu \alpha_{-1\nu})|0;p\rangle = a\,\xi^\nu \alpha_{-1\nu}|0;p\rangle$$

$\Rightarrow$ use $[\alpha_n, \alpha_m] = 0$ for $n \neq m$ to, ξ constant

$$\Rightarrow \cancel{(\alpha' p\cdot p} \; \cancel{(\alpha' p\cdot p} \; (\alpha' p\cdot p\, \zeta^\nu \alpha_{-1\nu} + \alpha_{-1}^\mu \alpha_{1\mu} \zeta^\nu \alpha_{-1\nu})|0;p\rangle$$
$$= a\, \zeta^\nu \alpha_{-1\nu} |0;p\rangle.$$

$$\alpha_{-1}^\mu \alpha_{1\mu} \zeta^\nu \alpha_{-1\nu} = \zeta^\nu \alpha_{-1}^\mu \alpha_{-1\nu} \alpha_{1\mu} + \zeta^\nu \alpha_{-1}^\mu \underbrace{[\alpha_{1\mu}, \alpha_{-1\nu}]}_{\eta^{\mu\nu}}$$

$$= \zeta^\nu \alpha_{-1\nu} + \zeta^\nu \alpha_{-1}^\mu \alpha_{-1\nu} \alpha_{1\mu}$$

$$\cancel{\Rightarrow \alpha' p\cdot p + \zeta^\mu \alpha_{-1}^\mu \alpha_{-1\nu}\alpha_{1\mu} + \zeta^\nu}$$

$$\cancel{\alpha' p\cdot p\, \zeta^\nu} \; ((\alpha' p\cdot p + 1)\zeta^\nu \alpha_{-1\nu} + \zeta^\nu \alpha_{-1}^\mu \underbrace{\cancel{\alpha_{-1\nu}\alpha_{1\mu})|0;p\rangle}}_{=0}.$$
$$= a\, \zeta^\nu \alpha_{-1\nu} |0;p\rangle$$
$$\Rightarrow \quad \alpha' p\cdot p + 1 = a \Rightarrow \underline{\underline{\alpha' p\cdot p = a - 1}}$$

the condition $L_1|\zeta,p\rangle = 0$ gives:

$$0 \overset{!}{=} L_1|\zeta,p\rangle = \left(\frac{1}{2}\sum_n \alpha_{1-n}\alpha_n\right)\zeta\cdot\alpha_{-1}|0;p\rangle.$$

$$= \frac{1}{2}(\alpha_1\alpha_0 + \alpha_0\alpha_1)\,\zeta\cdot\alpha_{-1}|0;p\rangle.$$

$$= \frac{1}{2}(\underbrace{\alpha_1^\mu \alpha_{0\mu}}_{\text{commute}} \zeta^\nu \alpha_{-1\nu} + \underbrace{\alpha_0^\mu \alpha_{1\mu}}_{\text{commute}} \zeta^\nu \alpha_{-1\nu})|0;p\rangle.$$

$$= \zeta^\nu \alpha_0^\mu \alpha_{1\mu} \alpha_{-1\nu} |0;p\rangle$$

$$= \zeta^\nu \alpha_0^\mu \underbrace{[\alpha_{1\mu}, \alpha_{-1\nu}]}_{\eta_{\mu\nu}} |0;p\rangle + \zeta^\nu \alpha_0^\mu \alpha_{-1\nu} \underbrace{\alpha_{1\mu}|0;p\rangle}_{=0}$$

$$= \zeta^\nu \alpha_{0\nu}|0;p\rangle = \zeta\cdot(\tfrac{1}{2}\ell p)|0;p\rangle$$

$$\Rightarrow \quad \underline{\underline{\zeta\cdot p = 0}}$$

The norm:

$$\| |\xi, p\rangle \|^2 = \| \xi\cdot\alpha_{-1} |0, p\rangle \| \qquad (\xi \in \mathbb{R}^{1,D-1})$$

$$= \cancel{\langle \xi, p | (\xi\cdot\alpha_{-1})^\dagger (\xi\cdot\alpha_{-1}) |\xi, p\rangle}$$

$$= \langle 0; p | (\xi\cdot\alpha_{-1})^\dagger (\xi\cdot\alpha_{-1}) | 0; p\rangle.$$

$$= \langle 0; p | \alpha_1\cdot\xi\ \xi\cdot\alpha_{-1} | 0; p\rangle.$$

$$= \langle 0; p | \alpha_1^\mu \xi_\mu\ \xi_\nu \alpha_{-1}^\nu | 0; p\rangle.$$

$$= \langle 0; p | \xi_\mu \xi_\nu \alpha_{+1}^\mu \alpha_{-1}^\nu | 0; p\rangle$$

$$= \xi_\mu \xi_\nu \Big(\langle 0; p | \underbrace{[\alpha_1^\mu, \alpha_{-1}^\nu]}_{\eta^{\mu\nu}} | 0; p\rangle + \cancel{\langle 0; p | \alpha_{-1}^\nu \underbrace{\alpha_1^\mu | 0; p\rangle}_{=0}} \Big)$$

$$= \eta^{\mu\nu} \xi_\mu \xi_\nu \underbrace{\langle 0; p | 0; p\rangle}_{=1} = \underline{\underline{\xi\cdot\xi}} \quad \checkmark$$

metric $\eta = \text{diag}(-1, 1, 1, \ldots)$

<u>If negative norm</u> $\Rightarrow \xi\cdot\xi < 0 \quad \therefore \xi$ is time like

$\Rightarrow \xi$ can be choosen (in some frame) to be $\xi = (1, 0, 0, \ldots)$

In that case since physical states require

$\xi\cdot p = 0 \quad \therefore \quad p = (0, p_1, p_2, \ldots) \Rightarrow p$ is spacelike.

Hence to make spectrum of one-level states without ghosts, we need

$$\underline{\underline{a \leq 1}} \quad \square$$

Good!

String Theory I (Class 2)

(1) Lot $\tilde{L}_0$ is the Hamiltonian on the worldsheet.
(assertion)

Hamiltonian generates time evolution via P.B

$[L_0, L_m] = -imL_m \neq 0$

Review of Hamiltonian mechanics.

given a function. $f(p, q; t)$

then $\frac{df}{dt} = \{H, f\}_{P.B} + \frac{\partial f}{\partial t}$.

$$\mathcal{L} = \frac{T}{2}(\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X)$$

$$\Pi = \frac{\delta \mathcal{L}}{\delta \partial_\tau X} = T \partial_\tau X.$$

$$L_m = \frac{T}{2}\int_0^\pi d\sigma \left(e^{2im(\tau-\sigma)} \partial_- X \cdot \partial_- X\right)$$

$$= \frac{e^{2im\tau}\, T}{2}\int_0^\pi d\sigma \left(\frac{e^{-2im\sigma}}{4}\left(\frac{\Pi}{T} - \partial_\sigma X\right)\left(\frac{\Pi}{T} - \partial_\sigma X\right)\right)$$

$L_m = L_m(X, \Pi; \tau)$

$$\frac{\partial L_m}{\partial \tau} = \{H, L_m\} + 2im L_m$$

$H = 2(L_0 + \tilde{L}_0)$

then $\{H, L_m\} = -im L_m$.

$$\Rightarrow \frac{dL_m}{d\tau} = 0.$$

idea is

$$\frac{dL_m}{d\tau} = \{H, L_m\} + \frac{\partial L_m}{\partial \tau}.$$

they sum to 0

$$\frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-2i(m+n)\sigma} = \frac{1}{2} e^{2im\sigma^-} \partial_- X^\mu(\sigma^+, \sigma^-)$$

this is the naive transformation of $X^\mu(\sigma^-)$ under the change of coordinates

$$\sigma^- \to (\sigma^-)' = \sigma^- - \frac{1}{2} e^{2in\sigma^-} \epsilon$$

small parameter

$$V_m^- = -\frac{1}{2} e^{2in\sigma^-} \frac{\partial}{\partial \sigma^-}.$$

vector fields that satisfy the Witt algebra.